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Comparative statics with linear objectives: normal demand, monotone marginal costs, and ranking multi-prior beliefs

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Key words: parallelogram property, increasing differences, ambiguity, first order stochastic dominance, normal demand, marginal costs

Comparative statics with linear objectives: normality, complementarity, and ranking multi-prior beliefs

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Abstract

We formulate an order over constraint sets $A \subseteq \mathbb{R}^\ell$, called the *parallelogram order*, which guarantees that $\operatorname{argmin}\left\{p \cdot x : x \in A\right\}$ increases in the product order as A increases in the set order, for vectors $p \in \mathbb{R}^\ell$. Using this result, we characterize the utility/production functions that lead to normal demand as well as the closely related class of production functions with marginal costs that increase with factor prices. By generalizing the concept of supermodularity, we also characterize the class of production functions for which factors are complements. In the context of decision-making under uncertainty, our new set order leads to natural generalizations of first order stochastic dominance in multi-prior models.

Keywords: parallelogram order, increasing differences, complementarity, ambiguity, first order stochastic dominance, normal demand, marginal costs

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1 Introduction

This paper studies the monotone comparative statics of optimization problems with linear objectives. We pose the following question. Given two nonempty subsets A, A' of the Euclidean space $\mathbb{R}^\ell$, what relation between them guarantees that $\Phi' = \operatorname{argmin}\left\{p \cdot x : x \in A'\right\}$ is higher (in an appropriate sense) than $\Phi = \operatorname{argmin}\left\{p \cdot x : x \in A\right\}$, for all $p \in \mathbb{R}^\ell$?

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When Φ' and Φ are singletons consisting of x' and x, respectively, it is clear that 'higher' means $x' \geq x$. More generally, if Φ' and Φ are nonempty sets, a minimal requirement for Φ' to be higher than Φ, is for the former to dominate by the latter in the *weak order*: for any $x \in \Phi$, $x' \in \Phi'$, there is $y \in \Phi$, $y' \in \Phi'$ such that $x' \geq y$ and $y' \geq x$.

A well-known result in monotone comparative statics states that if A' dominates A in the *strong set order*, then argmin $\{F(x) : x \in A'\}$ dominates argmin $\{F(x) : x \in A\}$ in the same sense, so long as F is submodular or (more generally) quasisubmodular (see Topkis (1978) and Milgrom and Shannon (1994)). By definition, A' dominates A by the strong set order if for any $x \in A$, $x' \in A'$, we have $x \wedge x' \in A$ and $x \vee x' \in A'$.[1] The strong set order implies the weak order since we can choose $y' = x \vee x' \geq x$ and $y = x \wedge x' \leq x'$. Given that linear functions are also submodular, this result gives a possible solution to the problem we pose. However, since we consider a class of objective functions than is narrower than the class of submodular functions, we could potentially allow for more general comparisons between constraint sets. This would be particularly desirable in those applications where the strong set order is too restrictive.

We find that, for the question we pose, the relevant relationship between A' and A is the *parallelogram order*. Set A' dominates A by the parallelogram order if for any $x \in A$, $x' \in A'$, there is $y \in A$, $y' \in A'$ such that $x' \geq y$, $y' \geq x$, and $x + x' = y + y'$. This ordering is stronger than the weak order, but weaker than the strong set order, which requires $y = x \wedge x'$ and $y' = x \vee x'$. We show that, if A' dominates A by the parallelogram order, then $\Phi' = \text{argmin}\,\{p \cdot x : x \in A'\}$ dominates $\Phi = \text{argmin}\,\{p \cdot x : x \in A\}$ in the same sense, for any $p \in \mathbb{R}^\ell$. Furthermore, the values of the optimization problem satisfy *increasing differences* in the sense that, whenever $p' \geq p$, then

$$\min\{p' \cdot y : y \in A'\} - \min\{p \cdot y : y \in A'\} \;\geq\; \min\{p' \cdot y : y \in A\} - \min\{p \cdot y : y \in A\}.$$

Furthermore, if sets A and A' convex, then the following three statements are equivalent: (1) A' dominates A by the parallelogram order; (2) Φ' dominates Φ by the *weak order* for all $p \in \mathbb{R}^\ell$; (3) the values obey increasing differences.

Many basic decision problems in economics belong to the class of linear optimization problems. We give three broad applications of our results.

[1] Here, $x \wedge x'$ denotes the greatest lower bound (meet) of x and x', while $x \vee x'$ denotes the least upper bound (join). The function F is *submodular* if $F(x) + F(x') \geq F(x \vee x') + F(x \wedge x')$ for all $x, x' \in \mathbb{R}^\ell$, and is *supermodular* if the inequality is reversed.

Application 1: Factor Demand and Marginal Cost An obvious application is to the study of normality in a firm's conditional factor demand or (what is formally similar) a consumer's Marshallian demand. Suppose a firm produces with ℓ factors and has the production function $F : \mathbb{R}_+^\ell \to \mathbb{R}_+$. We denote the upper contour sets by $U(q)$, i.e., $U(q) = \{x \in \mathbb{R}_+^\ell : F(x) \geq q\}$. If $p \in \mathbb{R}_{++}^\ell$ is the price vector of the ℓ factors, then the firm's factor demand at output q is $H(p, q) = \text{argmin}\,\{p \cdot \tilde{x} : \tilde{x} \in U(q)\}$. It is natural to ask what conditions on F will guarantee that factor demand is *normal*, in the sense that $H(p, q)$ rises with q, at least with respect to the weak order.

Normality will hold whenever $U(q')$ dominates $U(q)$ by the strong set order, for $q' \geq q$. However, this condition is very stringent and fails for any strictly increasing production function F. Indeed, suppose $x \in U(q)$ with $F(x) = q$ and $x' \in U(q')$ with $x' \ngeq x$. Then $x > x \wedge x'$ which implies that $F(x) > F(x \wedge x')$, and so $x \wedge x'$ cannot be in $U(q)$.

There are well-known examples of production functions that generate normal demand, including homothetic technologies, or functions that are supermodular and concave in x_{-i} for all $i = 1, \ldots, \ell$ (see Quah, 2007). We show that these two classes have upper contour sets that are ranked by the parallelogram order. However, the parallelogram order also covers functions not captured by these two classes. For example, suppose a firm has two plants/teams, each of which has a homothetic production function. Then the firm's overall production function that maps a given bundle of factors x to the greatest output possible among all possible allocations of x (between the two plants/teams), will satisfy the parallelogram property, even though it is not necessarily homothetic.

When upper contour sets are ranked by the parallelogram order, our basic result states that the resulting value function has increasing differences. In this context, it means that the firm's marginal cost of raising output increases with factor prices. This property is sufficient (and, in a sense, necessary) to guarantee that a firm's profit-maximizing response to higher factor prices is to produce less (and raise the price of its output).

Application 2: Complementary demand A firm with technology F chooses a bundle of factors $x \in \mathbb{R}_+^\ell$ to maximize profit $F(x) - p \cdot x$, for input prices $p \in \mathbb{R}_{++}^\ell$. The factors are said to be *complements* if lowering the price of a factor i raises the (unconditional) demand for *all* factors. A well-known condition to guarantee that complementarity holds is for F to be a supermodular function, but this property is not necessary for complementarity. We deploy our basic theorem to give a sharper analysis of this problem.

The firm's profit maximization problem could be thought of as a linear optimization problem, subject to a constraint which is the firm's production possibility set. We show that a change in the price of a factor (say, a fall in the price of some factor i) could be modeled as a change in the firm's production possibility set. Complementarity holds if the production possibility set before and after the drop in the price of i can be ranked by the parallelogram order. This approach yields a weaker property on F that guarantees complementarity, which we call *super*modularity*.[2] In addition, we show that whenever F is concave, super*modularity is also necessary for factors to be complements.

The function F is super*modular if, for any x and x', there is $y \leq x \wedge x'$ and $y' \geq x \vee x'$ and such that $x + x' = y + y'$ and $F(x) + F(x') \leq F(y) + F(y')$. This property generalizes supermodularity, which requires $y' = x \vee x'$ and $y = x \wedge x'$ and is, in fact, strictly weaker. For example, consider again a firm with two plants/ teams, each of which has a supermodular production function. The overall production function of this firm may still violate supermodularity, but we show that it satisfies super*modularity, which is preserved under such aggregation (unlike supermodularity).

Moreover, our approach to complementarity allows us to develop conditions for *partial* complementarity, under which a subset of factors could be complements of each other, but not of factors outside that set. Related to this, we show that complementarity is always a *symmetric* property, i.e., if F is such that a fall in the price of i always raises the demand for j, then a fall in the price of j always raises the demand for i.

Application 3: First order stochastic dominance in multi-prior models Imagine an agent who has to take an action under uncertainty. The agent's payoff is $g(x, s)$, where $x \in X \subseteq \mathbb{R}$ denotes the agent's action and $s \in S \subseteq \mathbb{R}$ is the realized state of the world. If the agent maximizes expected utility, with λ being the cumulative distribution function on S, then the utility of action x is $f(x, \lambda) = \int g(x, s) d\lambda(s)$.

Now, suppose that g satisfies increasing differences, which means that the marginal payoff of a higher action increases with s; formally, for any $x' \geq x$, the difference $g(x', s) - g(x, s)$ is increasing in s. It seems reasonable that this would imply that the *expected* marginal payoff of a higher action would also be greater when higher states are

[2] We suggest reading 'super*modular' as 'super-star-modular.'

more likely. This intuition is correct: if λ' first order stochastically dominates λ, then f satisfies increasing differences, i.e., $f(x', \lambda') - f(x, \lambda') \geq f(x', \lambda) - f(x, \lambda)$, for any $x' \geq x$. This in turn is sufficient to guarantee that $\text{argmax}\left\{f(x, \lambda') : x \in X\right\}$ dominates $\text{argmax}\left\{f(x, \lambda) : x \in X\right\}$ by the strong set order.[3]

Suppose that instead of maximizing expected utility, the agent is ambiguity averse and has maxmin preferences à la Gilboa and Schmeidler (1989), so that the ex-ante utility of action x is $f(x, \Lambda) = \min\left\{\int_S g(x, s) d\lambda(s) : \lambda \in \Lambda\right\}$, where Λ denotes a set of cumulative distribution functions over S. Now, suppose that there is a shift in the set of distributions from Λ to Λ'. Assuming that g has increasing differences, what relationship between Λ' and Λ guarantees that this is also true of f, i.e., for any $x' \geq x$,

$$f(x', \Lambda') - f(x, \Lambda') \;\geq\; f(x', \Lambda) - f(x, \Lambda)$$

We show that the Nature's problem of choosing a distribution in Λ (or Λ') that minimizes expected utility can be formulated as a constrained linear optimization problem. Our basic result then tells us that f satisfies increasing differences if the constraint sets are ranked by the parallelogram order; in this context, the constraint sets are Λ' and Λ, and Λ' dominates Λ by the parallelogram order if, for any cumulative distribution functions $\lambda \in \Lambda$ and $\lambda' \in \Lambda'$, there is $\mu \in \Lambda$ and $\mu' \in \Lambda'$ such that μ is first order stochastically dominated by λ', μ' first order stochastically dominates λ, and $\lambda + \lambda' = \mu + \mu'$.[4]

Organization of the paper. Section 2 is devoted to the basic results and a discussion on the related literature, in particular, Topkis (1978), Milgrom and Shannon (1994), and Quah (2007). Conditions for normal factor demand and the closely related (but weaker) conditions needed for marginal cost to increase with factor prices are formulated in Section 3. This section also discusses the normality of Marshallian demand and the normality of the set of efficient bundles. Section 4 formulates conditions for factors to be complements

[3] As a simple example, let x be the agent's current consumption in a two-date model where s is the uncertain income of tomorrow. Assuming that tomorrow's consumption has diminishing marginal utility, a first order shift in tomorrow's income distribution will increase today's consumption (see Example 15).

[4] Another natural set generalization of first order stochastic dominance is the relationship between Λ and Λ' which guarantees that $\min\left\{\int_S \phi(s) d\lambda : \lambda \in \Lambda\right\} \leq \min\left\{\int_S \phi(s) d\lambda : \lambda \in \Lambda'\right\}$, for any increasing function ϕ. This leads to an order between Λ and Λ' that is weaker than the parallelogram order (see Section S.6 of the Online Supplement); thus, in multi-prior models, there is a distinction between stochastic dominance that guarantees monotone utility and stochastic dominance that guarantees monotone comparative statics.

and introduces the concept of super*modularity. In Section 5 we formulate first order stochastic dominance for multi-prior models; it covers the maxmin model as well as the variational and multiplier preference models. There is an Appendix containing the more elaborate proofs and also an Online Supplement. In particular, the Online Supplement contains results on monotone decision rules for ambiguity averse agents in a dynamic setting, generalizing known results for agents maximizing discounted expected utility (see Hopenhayn and Prescott, 1992).

2 The parallelogram order

A *partial order* $\geq_X$ over a set X is a reflexive, transitive, and antisymmetric binary relation. A partially ordered set, or a *poset*, is a pair $(X, \geq_X)$ consisting of a set X and a partial order $\geq_X$. Whenever it causes no confusion, we denote $(X, \geq_X)$ with X. A poset is a *lattice* if, for any x and x' in X, their meet (the greatest lower bound) $x \wedge x'$ and their join (the least upper bound) $x \vee x'$ both belong to X. A subset Y of a lattice X is a *sublattice* if for any x, $x' \in Y$, $x \vee x'$ and $x \wedge x'$ are also in Y.

Most of our analysis is carried out in the Euclidean space $\mathbb{R}^\ell$. For any vector $x \in \mathbb{R}^\ell$, we denote its i'th entry by x_i; for any set $K \subseteq \{1, 2, \ldots, \ell\}$, let $x_K := (x_i)_{i \in K}$ be the sub-vector of entries in x that belong to K. Thus, we can write x as (x_K, x_{-K}), where $x_{-K} := (x_i)_{i \notin K}$. The product order $\geq$ on $\mathbb{R}^\ell$ is defined as follows: for any x, $x' \in \mathbb{R}^\ell$, $x' \geq x$ if $x'_i \geq x_i$ for all $i = 1, 2, \ldots, \ell$. The relation is said to be *strict*, and denoted by $x' > x$, whenever $x' \geq x$ and $x' \neq x$. It is straightforward to check that $(\mathbb{R}^\ell, \geq)$ constitutes a lattice, with $(x \wedge x')_i = \min\{x_i, x'_i\}$ and $(x \vee x')_i = \max\{x_i, x'_i\}$, for $i = 1, 2, \ldots, \ell$.

Definition (Parallelogram order)**.** Let A, $A' \subseteq \mathbb{R}^\ell$ and $K \subseteq \{1, 2, \ldots, \ell\}$. The set A' *dominates* A *in* K *by the parallelogram order* if for any $x \in A$, $x' \in A'$, there is $y \in A$, $y' \in A'$ such that $x + x' = y + y'$ and $x'_K \geq y_K$, $y'_K \geq x_K$. Whenever we refer to the parallelogram order without mentioning K, our default is $K = \{1, 2, \ldots, \ell\}$.

Given two nonempty sets $A, A' \subseteq \mathbb{R}^\ell$, we say that A' dominates A in K by the *weak order* if, for any $x \in A$ there $y' \in A'$ such that $x_K \leq y'_K$, and for any $x' \in A'$ there is $y \in A$ such that $y_K \leq x'_K$. Clearly, whenever A' dominates A in K by the parallelogram order, then A' also dominates A in K by the weak order.

A widely-used property in monotone comparative statics is the *strong set order* (see Topkis, 1978). Given a lattice X and subsets A, A' of X, the set A' dominates A by the strong set order if, for any $x \in A$, $x' \in A'$, we have $x \wedge x' \in A$ and $x \vee x' \in A'$. If X is a sublattice of $\mathbb{R}^\ell$, then when A' dominates A by the strong set order, A' also dominates A by the parallelogram order, since we can choose $y = x \wedge x'$ and $y' = x \vee x'$.

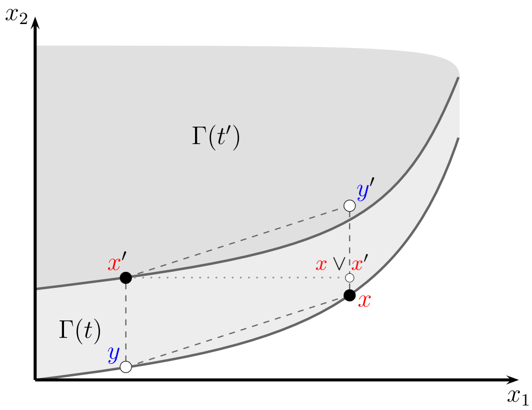
Our comparative statics results are typically formulated in a setting where there is a collection of sets related by the parallelogram order. This is formally captured through a correspondence Γ from a poset $(T, \geq_T)$ to $\mathbb{R}^\ell$; we say that Γ is *$\mathcal{P}$-increasing in K* if, for any $t' \geq_T t$, $\Gamma(t')$ dominates $\Gamma(t)$ in K by the parallelogram order. Clearly, if Γ is nonempty-valued and $\mathcal{P}$-increasing in K then it is also *$\mathcal{W}$-increasing in K*, in the sense that, if $t' \geq_T t$, then the set $\Gamma(t')$ dominates $\Gamma(t)$ in K by the weak order.[5]

Example 1. Figure 1a depicts values of a correspondence Γ for $t' >_T t$. The mapping is $\mathcal{P}$-increasing for $K = \{1, 2\}$ since, given $x \in \Gamma(t)$ and $x' \in \Gamma(t')$, we can find $y \in \Gamma(t)$ and $y' \in \Gamma(t')$ such that $x' \geq y$ and $y' \geq x$, and the four points form a parallelogram. This holds because the boundary of the set $\Gamma(\tilde{t})$ becomes flatter as $\tilde{t}$ increases. Formally, if $x_2 = \bar{x}_2(x_1, \tilde{t})$ is the equation of the boundary of $\Gamma(\tilde{t})$ (for x_1 in an interval X_1), then Γ is $\mathcal{P}$-increasing if $\bar{x}_2$ is increasing in $(x_1, \tilde{t})$ and $d\bar{x}_2/dx_1$ is decreasing in $\tilde{t}$. (See the Appendix for a proof of this claim and the converse.) Note that Γ is *not* increasing in the strong set order. Indeed, in the figure, vector $x \vee x'$ is not in $\Gamma(t')$.

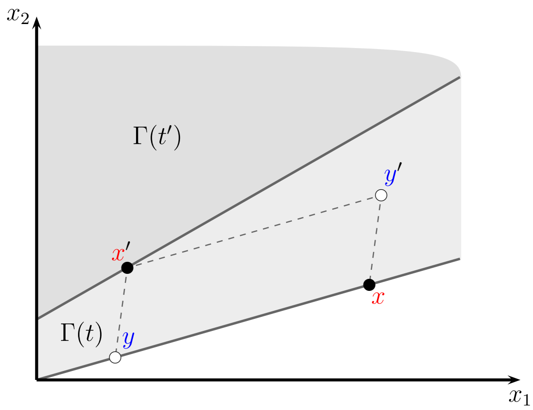
In contrast, in Figure 1b, the boundary of $\Gamma(t')$ is steeper rather than flatter than that of $\Gamma(t)$. The figure depicts $x \in \Gamma(t)$ and $x' \in \Gamma(t')$ with $x'_1 < x_1$ but it is impossible to find $y \in \Gamma(t)$ and $y' \in \Gamma(t')$ such that $x'_1 \geq y_1$, $y'_1 \geq x_1$, and $x + x' = y + y'$. For the 'unsuccessful' choice of y and y' shown in the figure, $x'_1 \geq y_1$, $y \in \Gamma(t)$, and $x + x' = y + y'$, but $y' \notin \Gamma(t')$. Thus Γ is not $\mathcal{P}$-increasing in $\{1\}$ or in $\{1, 2\}$. We leave the reader to check that Γ *is* $\mathcal{P}$-increasing in $\{2\}$.

The parallelogram order is closed under scalar multiplication and addition; these features turn out to be important in certain applications (see Examples 6 and 14). In contrast, the strong set order is closed under scalar multiplication, but not under addition. For example, although the set $A' = \{1\}$ dominates $A = \{0\}$, and $B = B' = \{0, 2\}$ are

[5] Similarly, we could speak of Γ being $\mathcal{P}$-decreasing or $\mathcal{W}$-decreasing in K.



(a) $\Gamma(t')$ dominates $\Gamma(t)$ by parallelogram order for $K = \{1, 2\}$.



(b) $\Gamma(t')$ fails to dominate $\Gamma(t)$ for $K = \{1\}$ and $K = \{1, 2\}$.

Figure 1: The parallelogram order

(trivially) ranked in the strong set order, the set $A + B = \{0, 2\}$ is not dominated by $A' + B' = \{1, 3\}$ in this sense. We omit the obvious proof of the next result.

Proposition 1. *Let the correspondences* Γ, $\Gamma' : T \to \mathbb{R}^\ell$ *be nonempty-valued and* $\mathcal{P}$*-increasing in* $K \subseteq \{1, \ldots, \ell\}$*. Then the correspondence* $\alpha\Gamma + \Gamma'$ *is* $\mathcal{P}$*-increasing in* K*, for any* $\alpha \geq 0$*.*

Our first main result justifies the attention we give to the parallelogram order. It states that when a family of constraint sets are ordered in this sense, then so are the solutions to an optimization problem with a linear objective.

Theorem 1. *Suppose the correspondence* $\Gamma : T \to \mathbb{R}^\ell$ *is* $\mathcal{P}$*-increasing in* K*. Then for any* $p \in \mathbb{R}^\ell$*, the correspondence* $\Phi : T \to \mathbb{R}^\ell$*, given by*

$$\Phi(t) \;:=\; \operatorname{argmin}\Big\{p \cdot y : y \in \Gamma(t)\Big\}, \tag{1}$$

is $\mathcal{P}$*-increasing in* K*. Furthermore, if the values of* Γ *are nonempty, compact, and convex, then* Γ *is* $\mathcal{P}$*-increasing in* K *if* Φ *is* $\mathcal{W}$*-increasing in* K *for every* $p \in \mathbb{R}^\ell$*.*

Remark 2.1. The first part of this result makes no ancillary assumptions on Γ and, thus, $\Phi(t)$ may be empty for some values of t. When $\Phi(t)$ is nonempty (as it will be if Γ is nonempty and compact) then Φ will also be $\mathcal{W}$-increasing in K if it is $\mathcal{P}$-increasing in K. The second part of this result states that, under ancillary assumptions on Γ which (in particular) guarantee that Φ has nonempty, compact, and convex values, then Γ must be $\mathcal{P}$-increasing in K if we require Φ to be $\mathcal{W}$-increasing in K.

Remark 2.2. While Theorem 1 focuses on minimization problems, it is clear that, for any $p \in \mathbb{R}^\ell$, the correspondence $\Psi(t) := \operatorname{argmax}\{p \cdot y : y \in \Gamma(t)\}$ inherits the parallelogram order from Γ, since $y \in \Gamma(t)$ maximizes $p \cdot y$ if, and only if, it minimizes $(-p) \cdot y$.

Remark 2.3. Consider three nonempty compact and convex sets where A^3 dominates A^2, and A^2 dominates A^1 in K by the parallelogram order. Define Γ by $\Gamma(t) = A^t$. By Theorem 1, for every $p \in \mathbb{R}^\ell$, $\Phi(3)$ dominates $\Phi(2)$ in K and $\Phi(2)$ dominates $\Phi(1)$ in K. This clearly implies that $\Phi(3)$ dominates $\Phi(1)$ in K. By the converse part of this theorem, we conclude that $\Gamma(3) = A^3$ dominates $\Gamma(1) = A^1$ in K by the parallelogram order. We conclude that domination in K by the parallelogram order is a transitive relation on the family of nonempty, compact, and convex subsets of $\mathbb{R}^\ell$. Obviously, this relation is also reflexive, and thus it constitutes a preorder. We show in Section S.1 of the Online Supplement that this relation is also anti-symmetric if $K = \{1, 2, \dots, \ell\}$.

We postpone discussion of the second claim in Theorem 1, which is an immediate consequence of Theorem 2 (see Remark 2.4). The proof of the first claim is straightforward and useful for building intuition.

Proof of the first part of Theorem 1. Take any $p \in \mathbb{R}^\ell$, $t' \geq_T t$, and $x \in \Phi(t)$, $x' \in \Phi(t')$. Since $x \in \Gamma(t)$, $x' \in \Gamma(t')$, and Γ is $\mathcal{P}$-increasing, there is $y \in \Gamma(t)$, $y' \in \Gamma(t')$ such that $x + x' = y + y'$ and $x'_K \geq y_K$, $y'_K \geq x_K$. We claim that $y \in \Phi(t)$ and $y' \in \Phi(t')$. Since $y \in \Gamma(t)$ and $x \in \Phi(t)$, it must be that $p \cdot y \geq p \cdot x$. Similarly, $p \cdot y' \geq p \cdot x'$. Thus,

$$p \cdot (y + y') \;\geq\; p \cdot (x + x') \;=\; p \cdot (y + y'),$$

which holds only if $p \cdot y = p \cdot x$ and $p \cdot y' = p \cdot x'$, and so $y \in \Phi(t)$, $y' \in \Phi(t')$. $\square$

The following example illustrates the use of Theorem 1.

Example 2. A firm hires an employee with a utility u that depends on the effort level $e \geq 0$ the employee exerts, and the payment $c \geq 0$ to the employee. The employee has an outside opportunity that yields utility t. The firm transforms e into revenue re, where $r > 0$. Thus, it chooses (e, c) to maximize $re - c$ subject to $\Gamma(t) = \{(e, c) \in X : u(e, c) \geq t\}$, where X is the domain of u. Assuming that $u(e, c)$ is strictly decreasing in e and strictly increasing in c, the indifference curves are upward sloping, as depicted in

Figure 1(a) (with effort on the horizontal axis). If Γ is $\mathcal{P}$-increasing, then an improvement in the outside opportunity t will lead to the firm paying more and requiring higher effort. This holds if the indifference curves $c = \bar{c}(e,t)$ become flatter with higher t, i.e., for each e, the derivative $d\bar{c}/de$ increases with t (see Example 1). A quick check with implicit differentiation will confirm that this occurs if u is supermodular in (e,c) and convex in c.

Theorem 1 guarantees the pairwise comparability of the optimal solutions, in the sense that $\Phi(t)$ and $\Phi(t')$ are ordered whenever t and t' are ordered. In fact, under mild assumptions, we could reach a stronger conclusion. The next result (proved in the Appendix) states that there is an increasing selection from Φ.

Proposition 2. *Suppose that the correspondence $\Phi : T \to \mathbb{R}^\ell$ is $\mathcal{P}$-increasing in K and has nonempty and compact values. Then there is a function $\phi : T \to \mathbb{R}^\ell$ such that $\phi(t) \in \Phi(t)$ for all $t \in T$, and $\phi_K(t') \geq \phi_K(t)$ whenever $t' \geq_T t$.*[6]

Value Functions Given the constraint sets $\Gamma(t)$ and $p \in \mathbb{R}^\ell$, we may define the *value function* $f : \mathbb{R}^\ell \times T \to \mathbb{R}$, where $f(p,t) := \min\{p \cdot y : y \in \Gamma(t)\}$. This function has *increasing differences in* (p_K, t) if, for any $p'_K \geq p_K$ and $t' \geq_T t$, and p_{-K},

$$f\big((p'_K, p_{-K}), t'\big) - f\big((p_K, p_{-K}), t'\big) \;\geq\; f\big((p'_K, p_{-K}), t\big) - f\big((p_K, p_{-K}), t\big). \tag{2}$$

For certain comparative statics applications (see Propositions 3 and 7), this property on f plays a crucial role and the next theorem gives a characterization of this property. The condition on Γ needed to guarantee that f satisfies increasing differences is close to, but not identical with, what is needed to guarantee that Φ is $\mathcal{P}$-increasing in K. The required property is that the convex hull of $\Gamma(t')$, which we denote by $\operatorname{co}\Gamma(t')$, dominates $\operatorname{co}\Gamma(t)$ in K by the parallelogram order whenever $t' \geq_T t$; in other words, the correspondence $\operatorname{co}\Gamma$ is $\mathcal{P}$-increasing in K. This property is weaker than requiring Γ to be $\mathcal{P}$-increasing in K. Indeed, one could check that $\operatorname{co}\Gamma(t')$ dominates $\operatorname{co}\Gamma(t)$ in K by the parallelogram order if (and, obviously, only if) for any $x \in \Gamma(t)$, $x' \in \Gamma(t')$, there is $y \in \operatorname{co}\Gamma(t)$, $y' \in \operatorname{co}\Gamma(t')$ such that $x + x' = y + y'$ and $x'_K \geq y_K$, $y'_K \geq x_K$.[7] It immediately follows from this

[6] If Φ is merely $\mathcal{W}$-increasing in K (rather than $\mathcal{P}$-increasing in K) then there is no guarantee that it would have an increasing selection. See Example 3.4 in Kukushkin (2013).

[7] Indeed, for any $x \in \operatorname{co}\Gamma(t)$ and $x' \in \operatorname{co}\Gamma(t')$, we can find $\alpha^j \geq 0$, x^j and x'^j such that $\sum_{j=1}^J \alpha^j x^j = x$ and $\sum_{j=1}^J \alpha^j x'^j = x'$. For each pair, x^j and x'^j, there is $y^j \in \operatorname{co}\Gamma(t)$ and $y'^j \in \operatorname{co}\Gamma(t')$ such that $y^j \leq x'^j$, $x^j \leq y'^j$, and $x^j + x'^j = y^j + y'^j$. The required condition holds with $y = \sum_{j=1}^J \alpha^j y^j$ and $y' = \sum_{j=1}^J \alpha^j y'^j$.

observation that Γ is $\mathcal{P}$-increasing in K only if the correspondence $\operatorname{co}\Gamma$ is $\mathcal{P}$-increasing in K.

Theorem 2. *Let T be a poset and $\Gamma : T \to \mathbb{R}^\ell$ be a correspondence with nonempty and compact values. For any $K \subseteq \{1, 2, \dots, \ell\}$, the following statements are equivalent.*

(i) $\operatorname{co}\Gamma$ *is $\mathcal{P}$-increasing in K.*

(ii) *For any $p \in \mathbb{R}^\ell$, $\operatorname{co}\Phi$ is $\mathcal{P}$-increasing in K, where Φ is defined as in (1).*

(iii) *For any $p \in \mathbb{R}^\ell$, Φ is $\mathcal{W}$-increasing in $\{i\}$, for each $i \in K$.*

(iv) *The value function $f : \mathbb{R}^\ell \times T \to \mathbb{R}$ has increasing differences in (p_K, t).*

Remark 2.4. The claim in Theorem 1 that, when Γ has nonempty, compact, and convex values then it is $\mathcal{P}$-increasing for K only if Φ is $\mathcal{P}$-increasing in K (for all $p \in \mathbb{R}^\ell$) follows from the equivalence of (i) and (iii) in Theorem 2 since, when Γ is convex-valued, $\Gamma = \operatorname{co}\Gamma$. Indeed, the equivalence of (i) and (iii) tells us something stronger: if (for all $p \in \mathbb{R}^\ell$) we require Φ to be $\mathcal{W}$-increasing in $\{i\}$ for each $i \in K$, then $\Gamma = \operatorname{co}\Gamma$ must be $\mathcal{P}$-increasing in K. Thus the $\mathcal{P}$-increasing property on Γ is necessary even for a very minimal requirement on comparative statics.

Remark 2.5. Comparing this result with Theorem 1, we see that the $\mathcal{P}$-increasing property on $\operatorname{co}\Gamma$ guarantees that Φ is increasing in $\{i\}$ for each $i \in K$, but not necessarily in K *jointly*. This phenomenon is illustrated in Example 4 in the following section.

Proof of Theorem 2. By Theorem 1, if (i) holds then $\Psi(t) := \operatorname{argmin}\big\{p \cdot y : y \in \operatorname{co}\Gamma(t)\big\}$ is $\mathcal{P}$-increasing in K; (ii) follows immediately since $\Psi(t) = \operatorname{co}\Phi(t)$.

To show that (ii) implies (iii), let $\operatorname{co}\Phi$ be $\mathcal{P}$-increasing for K. Then, $t' \geq_T t$ and $x \in \Phi(t)$ imply $y'_K \geq x_K$, for some $y' \in \operatorname{co}\Phi(t')$. Thus, there are vectors $z^j \in \Phi(t')$ and numbers $\alpha^j \geq 0$, for $j = 1, \dots, m$, such that $y' = \sum_{j=1}^m \alpha^j z^j$ and $\sum_{j=1}^m \alpha^j = 1$. Since $y'_K \geq x_K$, there is j with $z_i^j \geq x_i$. Analogously, for any $x' \in \Phi(t')$ and $i \in K$, there is some $z \in \Phi(t)$ satisfying $x'_i \geq z_i$.

We prove that (iii) implies (iv). It is well-known that f is a concave function. In particular, the map from $z \in [p_i, p'_i]$ to $f(z, p_{-i}, t)$ is concave and continuous over the interval $[p_i, p'_i]$. Hence, it is absolutely continuous and, thus, almost everywhere differentiable (see Theorem 25.5 in Rockafellar, 1970), with

$$f\big((p'_i, p_{-i}), t\big) - f\big((p_i, p_{-i}), t\big) \;=\; \int_{p_i}^{p'_i} \frac{\partial f}{\partial p_i}\big((z, p_{-i}), t\big) dz.$$

By Theorem 25.1 in Rockafellar (1970) (Shephard's Lemma), if $\partial f/\partial p_i\big((z,p_{-i}),t\big)$ exists then $y_i = y_i' = \partial f/\partial p_i\big((z,p_{-i}),t\big)$, for any $y, y' \in \Phi(t)$, where $p = (z, p_{-i})$. Since Φ is $\mathcal{W}$-increasing $\{i\}$ (for $i \in K$), for any $t' \geq_T t$, we have $\partial f/\partial p_i(z,p_{-i},t) \leq \partial f/\partial p_i(z,p_{-i},t')$, for almost all $z \in [p_i, p_i']$. This leads to $f\big((p_i',p_{-i}),t\big) - f\big((p_i,p_{-i}),t\big) \;\leq\; f\big((p_i',p_{-i}),t'\big) - f\big((p_i,p_{-i}),t'\big)$. Thus, we have shown that the function f has increasing differences in (p_i,t), for any $i \in K$, which implies that f has increasing differences in (p_K,t).

The claim that (iv) implies (i) is harder to prove and we leave that to the Appendix. Our argument employs the separating hyperplane theorem to show that if Γ fails to be $\mathcal{P}$-pseudoincreasing, then f violates increasing differences. $\square$

Theorem 2 determines the comparative statics of minimization problems with an arbitrary linear objective. In some applications we require comparative statics over the narrower class of *strictly increasing* linear objectives. The next theorem (which we prove in the Appendix) covers that case. Instead of requiring Γ to be compact-valued, we impose the following assumption: $\Gamma(t)$ is closed and upward comprehensive, and its asymptotic cone, denoted by $\mathbf{A}\Gamma(t)$, is equal to $\mathbb{R}^\ell_+$, for all $t \in T$. These conditions guarantee that $\operatorname{co}\Gamma(t)$ is a closed set and $\Phi(t)$ is nonempty for all $p \in \mathbb{R}^\ell_{++}$.[8] A sufficient (but by no means necessary) condition for $\mathbf{A}\Gamma(t) = \mathbb{R}^\ell_+$ is for $\Gamma(t)$ to be *bounded from below*, i.e., there is $\underline{x}^t$ such that $y \geq \underline{x}^t$ for all $y \in \Gamma(t)$.

Theorem 3. *Let T be a poset and $\Gamma : T \to \mathbb{R}^\ell$ be a correspondence such that $\Gamma(t)$ is nonempty, closed, upward comprehensive, and satisfies $\mathbf{A}\Gamma(t) \subseteq \mathbb{R}^\ell_+$, for all $t \in T$. For any $K \subseteq \{1,2,\ldots,\ell\}$, the following statements are equivalent.*

(i) $\operatorname{co}\Gamma$ *is $\mathcal{P}$-increasing in K.*

(ii) *For any $p \in \mathbb{R}^\ell_{++}$, $\operatorname{co}\Phi$ is $\mathcal{P}$-increasing in K, where Φ is given in (1).*

(iii) *For any $p \in \mathbb{R}^\ell_{++}$, Φ is $\mathcal{W}$-increasing in $\{i\}$, for each $i \in K$.*

(iv) *The value function $f : \mathbb{R}^\ell_{++} \times T \to \mathbb{R}$ has increasing differences in (p_K, t).*

Remark 2.6. Suppose that, in addition to the properties listed in this theorem, Γ is convex-valued. Then, the equivalence of (i) and (iii) means that Γ is $\mathcal{P}$-increasing in K if, for every $p \in \mathbb{R}^\ell_{++}$, Φ is $\mathcal{W}$-increasing in $\{i\}$, for each $i \in K$.

[8] This claim is related to fairly standard results in convex analysis; for completeness we provide a proof in Section S.2 (Proposition S.2) of the Online Supplement. A set $S \subseteq \mathbb{R}^\ell$ is *upward comprehensive* if $x \in S$ and $x' \geq x$ implies $x' \in S$. The *asymptotic cone of S* consists of all limits $\tilde{x} \in \mathbb{R}^\ell$ of sequences $\{\lambda_n x_n\}$, for some $x_n \in S$ and positive scalars λ_n that converge to 0.

Remark 2.7. There is an analogous version of Theorem 3 for maximization problems. If Γ has closed, downward comprehensive, and bounded from above values, the following are equivalent: (i) $\operatorname{co}\Gamma$ is $\mathcal{P}$-increasing in K; (ii) $\operatorname{co}\tilde{\Phi}$ is $\mathcal{P}$-increasing in K, where $\tilde{\Phi}(t) := \operatorname{argmax}\{p \cdot y : y \in \Gamma(t)\}$, for any $p \in \mathbb{R}^{\ell}_{++}$; (iii) $\tilde{\Phi}$ is $\mathcal{W}$-increasing in $\{i\}$, for each $i \in K$ and for any $p \in \mathbb{R}^{\ell}_{++}$; (iv) the value $\tilde{f}(p,t) := \max\{p \cdot y : y \in \Gamma(t)\}$ has increasing differences in (p_K, t). This result can be obtained by applying Theorem 3 to the correspondence $\Gamma^* : T \to \mathbb{R}^{\ell}$, where $\Gamma^*(t) = -\Gamma(t)$; note that $\operatorname{co}\Gamma^*$ is $\mathcal{P}$-decreasing in K if, and only if, $\operatorname{co}\Gamma$ is $\mathcal{P}$-increasing in K.

We now discuss the relationship between our results and the monotone comparative statics results by Topkis (1978), Milgrom and Shannon (1994), and Quah (2007).

Let $(X, \geq_X)$ be a lattice and $\Gamma : T \to X$ be a correspondence that is increasing in the strong set order. Topkis (1978) showed that the correspondence of optimal points $\Phi(t) := \operatorname{argmin}\{\phi(y) : y \in \Gamma(t)\}$ is also increasing in the strong set order if the objective function $\phi : X \to \mathbb{R}$ is *submodular*, i.e., satisfies $\phi(x) + \phi(x') \geq \phi(x \wedge x') + \phi(x \vee x')$, for any $x, x' \in X$. Observing that any comparative statics result on Φ must be independent of strictly increasing transformations of the objective function, Milgrom and Shannon (1994) generalize Topkis' result by showing that it suffices for ϕ to satisfy the ordinal counterpart of submodularity, called *quasisubmodularity*; this property requires that $\phi(x \wedge x') \geq (>)\, \phi(x)$ implies $\phi(x') \geq (>)\, \phi(x \vee x')$, for any $x, x' \in X$.

Quah (2007) observes that for certain economic problems, the strong set order on Γ is an overly strong assumption. He develops a comparative statics result that requires an ordinal condition on the objective function ϕ that is stronger than quasisubmodularity (called $\mathcal{C}$-quasisubmodularity[9]), while relaxing the strong set order requirement on Γ. Specifically, the correspondence Γ is required to be increasing in the *$\mathcal{C}$-flexible set order for $K \subseteq \{1, 2, \ldots, \ell\}$*, which means that, for any $t' \geq_T t$, $x \in \Gamma(t)$, and $x' \in \Gamma(t')$ with $x'_K \not\geq x_K$, there is some $\lambda \in [0,1]$ such that $[\lambda x' + (1-\lambda)(x \wedge x')] \in \Gamma(t)$ and $[\lambda x + (1-\lambda)(x \vee x')] \in \Gamma(t')$. Obviously, this order is weaker than the strong set order, which corresponds to the case where $\lambda = 0$. It is shown that if Γ increases in the $\mathcal{C}$-flexible set order for K, then so does Φ, for any $\mathcal{C}$-quasisubmodular function ϕ.

In this paper, we push the approach in Quah (2007) even further, by requiring the

[9]A sufficient condition for this property is that ϕ is submodular and convex.

objective ϕ to be *linear*, in order to obtain the most permissive conditions on Γ needed for monotone comparative statics. Notice that the $\mathcal{C}$-flexible set order is a special case of the parallelogram order since, if we let $y = \lambda x' + (1-\lambda)(x \wedge x')$ and $y' = \lambda x + (1-\lambda)(x \vee x')$, then $y' \geq x$, $x' \geq y$, and $x + x' = y + y'$. In some applications, the parallelogram order holds but the $\mathcal{C}$-flexible set order does not. For example, we know that $\Gamma(t')$ dominates $\Gamma(t)$ by the parallelogram order in Figure 1a, yet the $\mathcal{C}$-flexible set order *cannot* hold between these sets. This is because the boundaries of $\Gamma(t)$ and $\Gamma(t')$ are strictly upward sloping and we could always choose x and x' on the boundary of $\Gamma(t)$ and $\Gamma(t')$, respectively, as depicted in Figures 1(a), with the entire line connecting x' and $x \vee x'$ outside of $\Gamma(t')$.

3 Normal demand and monotone marginal cost

It is commonplace to hear that a firm has found it necessary to raise the price of its output because of increases in the price of raw materials. In this section we explain the precise conditions under which such a claim is correct.[10] In formal terms, we discuss the relationships linking the parallelogram order, factor demand, and marginal cost. We also examine the closely related issue of characterizing normal demand for consumers.

3.1 Conditional factor demand and marginal costs

The firm has a continuous and increasing production function $F : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$. We denote the range of F by $Q := \{F(x) : x \in \mathbb{R}^{\ell}_{+}\}$.[11] For any $q \in Q$, let

$$U(q) \ := \ \Big\{x \in \mathbb{R}^{\ell} : F(x) \geq q\Big\}.$$

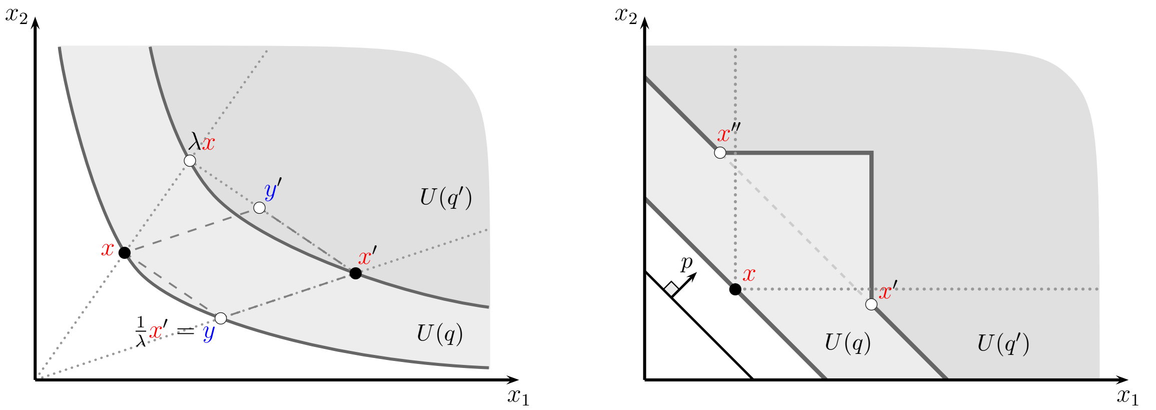
We refer to U as the *upper contour correspondence of* F. The set $U(q)$ is upward comprehensive, bounded from below by 0, and closed (since F is continuous).

Definition. The function $F : \mathbb{R}^{\ell} \to \mathbb{R}$ is $\mathcal{P}$*-increasing* in K if its upper contour correspondence $U : Q \to \mathbb{R}$ is $\mathcal{P}$-increasing in K (where Q is the range of F). The function is *quasi-*$\mathcal{P}$*-increasing* whenever the correspondence $q \to \mathrm{co}\, U(q)$ is $\mathcal{P}$-increasing in K.

We assume that the firm is a price-taker in the market for factors/inputs and faces *strictly* positive input prices $p = (p_1, p_2, \ldots, p_\ell)$. The *conditional factor/input demand* at

[10] We would like to thank the anonymous referee who suggested this motivation.

[11] Our analysis could be performed for functions F defined over a general domain $X \subseteq \mathbb{R}^{\ell}_{+}$. However, to keep the exposition simple, we restrict our attention to the special case of $X = \mathbb{R}^{\ell}_{+}$.



(a) A homothetic function is $\mathcal{P}$-increasing.

(b) The set of cost-minimisers is not $\mathcal{P}$-increasing in $\{1, 2\}$, but its convex hull is.

Figure 2: Illustrations for Examples 3 (left) and 4 (right).

p and output $q \in Q$ refers to those bundles that achieve output of at least q with the least cost. Formally, input demand is the correspondence $H : \mathbb{R}^\ell_{++} \times Q \to \mathbb{R}^\ell_+$, where $H(p, q) = \operatorname{argmin}\{p \cdot x : x \in U(q)\}$. Our assumptions on F guarantee that $H(p, q)$ is nonempty and compact, for all (p, q) in $\mathbb{R}^\ell_{++} \times Q$. The associated cost function is $C(p, q) = \min\{p \cdot x : x \in U(q)\}$.

Theorem 1 tells us that if U is $\mathcal{P}$-increasing in K then the conditional factor demand $H(p, \cdot)$ is also $\mathcal{P}$-increasing in K. In particular, the factor demand is increasing in K (as target output q increases); the conventional usage in this context would say that demand is *normal* (more precisely, *jointly normal*) *in* K. We may also conclude that, for each p, $H(p, \cdot)$ admits a selection $h(p, q) \in H(p, q)$, for all $q \in Q$, such that $h_K(p, q') \geq h_K(p, q)$ whenever $q' \geq q$ (see Proposition 2). If F is quasiconcave (so that $U(q)$ is convex) we obtain the converse result that U is $\mathcal{P}$-increasing in K if, at every factor price $p \in \mathbb{R}^\ell_{++}$, the factor demand for each $i \in K$ is normal. This follows from Theorem 3 (Remark 2.6). The following is a basic example of a $\mathcal{P}$-increasing production function; more examples are presented at the end of this subsection.

Example 3. A production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is homothetic if $F(x') \geq F(x)$ implies $F(\lambda x') \geq F(\lambda x)$, for any scalar $\lambda > 0$. It is well-known that if F is homothetic then its factor demand is jointly normal; in fact, the following (much stronger) property holds: if $x \in H(p, q)$, then $tx \in H(p, q')$, where $t > 1$ if $q' < q$ and $t > 1$ if $q' < q$. Thus a production function that is homothetic and quasiconcave must be $\mathcal{P}$-increasing.

This can be directly confirmed. Suppose $x \in U(q)$ and $x' \in U(q')$, with $q' > q$. We focus on the case where $F(x) < q' \leq F(x')$ and x' and x are not ordered (the other cases being straightforward). Since F is continuous and increasing, there is a scalar $a > 1$ such that $F(x) = F(x'/a)$ and, by the homotheticity of F, we obtain $F(ax) = F(x')$ (see Figure 2a). Set $y = x'/a$ and $y' = x' + (x - (x'/a))$. The bundle y' is on the line segment joining ax and x' and, since F is quasiconcave, $y' \in U(q')$. We also have $y \leq x'$, $x \leq y'$ and $x + x' = y + y'$, as required by the parallelogram property.

The cost function C has increasing differences in (p_K, q) if, for any input prices $p'_K \geq p_K$, p_{-K}, and output levels $q' \geq q$, we have

$$C\big((p'_K, p_{-K}), q'\big) - C\big((p'_K, p_{-K}), q\big) \;\geq\; C\big((p_K, p_{-K}), q'\big) - C\big((p_K, p_{-K}), q\big). \tag{3}$$

Thus, the increase in cost when output is raised from q to q' is greater at the input prices (p'_K, p_{-K}) compared to the prices (p_K, p_{-K}), when $p'_K \geq p_K$. When C is differentiable in output, this is equivalent to $\partial C/\partial q\big((p_K, p_{-K}), q\big)$ being increasing in p_K, i.e., an increase in the price of a factor in the set K leads to a higher marginal cost.

By Theorem 3, the cost function C has increasing differences in (p_K, q) if, and only if, $\operatorname{co} U$ is $\mathcal{P}$-increasing in K (equivalently, the function F is quasi-$\mathcal{P}$-increasing). This is enough to guarantee that demand is normal in $\{i\}$ for each $i \in K$, but not enough to guarantee that demand is *jointly* normal in K (which requires U to be $\mathcal{P}$-increasing in K). The next example illustrates this distinction.

Example 4. The production function $F : \mathbb{R}^2_+ \to \mathbb{R}$ has isoquants depicted in Figure 2b. It is quasi-$\mathcal{P}$-increasing, since $\operatorname{co} U(q) = \big\{(x_1, x_2) : x_1 + x_2 \geq q\big\}$, for all $q \in Q$. In line with statement (iii) in Theorem 3, factor demand is normal in factor 1 and in factor 2 separately. For example, as shown in the figure, when $p_1 = p_2$ and given $x \in H(p, q)$, there is $x' \in H(p, q')$ such that $x'_1 \geq x_1$, and $x'' \in H(p, q')$ such that $x''_2 \geq x_2$. However, U is not $\mathcal{P}$-increasing for $K = \{1, 2\}$ and joint normality in both goods does not hold: there is no bundle in $H(p, q')$ that is higher than x in both goods. Nonetheless, due to the quasi-$\mathcal{P}$-increasing property, the cost function has increasing differences in $\big((p_1, p_2), q\big)$.

Our ability to sign the impact of higher factor prices on marginal cost allows us to predict how the firm's profit-maximizing output will change. Let $R : Q \to \mathbb{R}_+$ be the revenue that the firm earns when it produces $q \in Q$. The firm chooses $q \in Q$ to maximize profit $\Pi(p, q) = R(q) - C(p, q)$. When C has increasing differences in (p_K, q), a rise in the price of any factor in K raises marginal cost and reduces the profit-maximizing output. Conversely, if the optimal output decreases with p_K for any increasing revenue function R, then C must have increasing differences in (p_K, q). The following result (which we prove in the Appendix) summarizes our claims.

Proposition 3. *For any continuous and increasing production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$, the following statements are equivalent.*

(i) *F is quasi-$\mathcal{P}$-increasing in K.*

(ii) *For all $p \in \mathbb{R}^\ell_{++}$, $H(p, \cdot)$ is normal in $\{i\}$, for each $i \in K$.*

(iii) *C has increasing differences in (p_K, q).*

(iv) *The set* $\operatorname{argmax}\left\{R(q) - C(p, q) : q \in Q\right\}$ *is decreasing in $p_K \gg 0$ in the strong set order, for any function $R : Q \to \mathbb{R}$.*

Going back to the issue we raised at the beginning of this section, Proposition 3 states that the quasi-$\mathcal{P}$-increasing property in K of the technology F is sufficient and (in a sense) necessary for the profit-maximizing output to fall when p_K increases. Whenever the price of output is determined by a downward sloping demand curve, this also implies that the firm charges a higher price for its output when p_K increases.

We end this subsection with three more examples of $\mathcal{P}$-increasing production functions (besides the homothetic case already considered in Example 3).

Example 5. A function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is *increasing in $K \subseteq \{1, 2, \ldots, \ell\}$ by the $\mathcal{C}$-flexible set order* if the correspondence $q \to U(q)$ is increasing in K by the $\mathcal{C}$-flexible set order; this property implies that F is $\mathcal{P}$-increasing in K. (Recall the discussion on the $\mathcal{C}$-flexible set order at the end of Section 2.) It is known that F is increasing in K by the $\mathcal{C}$-flexible set order if it is continuous, increasing, supermodular, and concave in x_{-i}, for $i \in K$; for a proof see Quah (2007) or Section S.3 of the Online Supplement. For example, $F(x_1, x_2, x_3) := \sqrt{x_1 x_2} + x_3$ for $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ is increasing, supermodular, and concave and thus is increasing (in $K = \{1, 2, 3\}$) by the $\mathcal{C}$-flexible set order.

The next two examples give economically interpretable ways of constructing new $\mathcal{P}$-increasing production functions from other production functions with that property.

Example 6. Suppose that one unit of output can be produced from one unit each of n intermediate goods. Each intermediate good is produced from ℓ factors, with $f^j : \mathbb{R}^\ell_+ \to \mathbb{R}$ being the production function for the jth intermediate good. Assuming that any bundle x of factors is efficiently assigned, the firm's production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is

$$F(x) := \max\left\{ \min_{j=1,\dots,n} \{f^j(y^j)\} : x \geq \sum_{j=1}^n y^j \right\}.$$

It is straightforward to check that its upper contour correspondence U satisfies $U(q) = \sum_{j=1}^n U^j(q)$, where U^j is the upper contour correspondence associated with f^j. By Proposition 1, U is $\mathcal{P}$-increasing in K if each U^j is $\mathcal{P}$-increasing in K. For example, $F(x_1, x_2, x_3) := \min\{x_1^2, x_2 + x_3\}$ is $\mathcal{P}$-increasing since $f^1(x) = x_1^2$ and $f^2(x) = x_2 + x_3$ are both homothetic and thus $\mathcal{P}$-increasing. (In fact, both functions are also increasing in the $\mathcal{C}$-flexible set order.) While F is $\mathcal{P}$-increasing, it is clearly not homothetic and neither is it increasing in the $\mathcal{C}$-flexible set order.[12]

Example 7. Another aggregation procedure preserving the parallelogram order is as follows. Given production functions $f^j : \mathbb{R}^\ell_+ \to \mathbb{R}$ ($j = 1, 2 \dots, n$), define $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ by

$$F(x) \;:=\; \max\left\{ G\Big(f^1(y^1), f^2(y^2), \dots, f^n(y^n)\Big) : x \geq \sum_{j=1}^n y^j \right\}, \tag{4}$$

where $G : \times_{j=1}^n Q^j \to \mathbb{R}$ is an increasing function that aggregates the values of f^j, and Q^j contains the range of f^j. We show in the Appendix that F is $\mathcal{P}$-increasing in K provided that f^j is continuous, concave, and is $\mathcal{P}$-increasing in K (for each j), and the aggregating function G is increasing in the $\mathcal{C}$-flexible set order.[13]

For example, if $G(q^1, q^2, \dots, q^n) = \sum_{k=1}^n q^k$, then G is increasing in the $\mathcal{C}$-flexible set order. In this case F can be interpreted as the production function of a firm that allocates its output efficiently across n plants, with plant j having the production function f^j. Thus, the following functions are $\mathcal{P}$-increasing.

(i) $F(x_1, x_2, x_3, x_4) := \sqrt{x_1\, x_2} + \sqrt{x_3 + x_4}$;

(ii) $F(x_1, x_2, x_3) := \max\left\{ \sqrt{x_1\, y} + \sqrt{x_3 + z} : x_2 \geq y + z \right\}$.

[12] If $x = (\sqrt{5}, 0, 5)$ and $x' = (\sqrt{6}, 6, 0)$, then $F(x) = 5$, $F(x') = 6$, $F(x \wedge x') = F\big((\sqrt{5}, 0, 0)\big) = 0$, and $F(x \vee x') = F\big((\sqrt{6}, 6, 5)\big) = 6$. For F to be increasing in the $\mathcal{C}$-flexible set order, we must find $\lambda \in [0, 1]$ such that $F\big(\lambda x' + (1 - \lambda)(x \wedge x')\big) \geq F(x) = 5$ and $F\big(\lambda x + (1 - \lambda)(x \vee x')\big) \geq F(x') = 6$. But this is impossible since $F(x \wedge x') < 5$ and if $\lambda > 0$, $F\big(\lambda x + (1 - \lambda)(x \vee x')\big) < 6$.

[13] This example does not generalize Example 6: while the 'min' function is increasing in the $\mathcal{C}$-flexible set order, Example 6 does not require f^j to be concave, which is required in this example.

In (i) the two plants produce output with completely distinct factors (the first plant with factors 1 and 2 and the second with factors 3 and 4) while in case (ii), one factor (factor 2) is used by both plants. Notice that both plants have homothetic production functions, but since one plant has constant returns to scale while the other has diminishing returns, F is not homothetic in both (i) and (ii).

3.2 Sequential optimization and efficient bundles

Our basic results tell us that, if two constraint sets are ordered by the parallelogram order, then so are the minimizers of a linear objective over those sets. This feature makes our results applicable to decision procedures where linear objectives are sequentially applied. For example, in the context of factor demand, the firm may choose, among those bundles that minimize cost, the ones that minimize the usage of one or a combination of factors belonging to $C \subseteq \{1, 2, \ldots, \ell\}$ (perhaps for environmental reasons, or to minimize the firm's vulnerability to supply disruptions). In that case, the firm's factor demand at factor prices $p \in \mathbb{R}^\ell_{++}$ and output q is $H^*(p, q) := \text{argmin}\left\{\sum_{i\in C} x_i : x \in H(p, q)\right\}$. By Theorem 1, if F is $\mathcal{P}$-increasing, then $H(p, \cdot)$ is $\mathcal{P}$-increasing, and so is $H^*(p, \cdot)$.

Another application is to guarantee normality for efficient bundles. For a given production function F, a bundle $x \in \mathbb{R}^\ell_+$ is *efficient at producing* q if $x \in U(q)$ and $x' < x$ implies $x' \notin U(q)$, for any alternative x'. Let $E(q)$ denote the set of bundles that produce q efficiently. Given $x \in E(q)$ and $q' > q$, we ask whether there is $x' \in E(q')$ such that $x' \geq x$. For example, suppose x represents the effort levels of ℓ team members in a joint project, and gives an efficient way of allocating the effort among the team members to produce q. If the output target is now higher at q', is there a way of producing this efficiently, while ensuring that no team member contributes strictly less? This is not always possible. For example, suppose $U(q) = \left\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 x_2 = 1\right\}$ and $U(q') = \left\{(x_1, x_2) \in \mathbb{R}^2_+ : \min\{x_1, x_2\} \geq 2\right\}$. Then $\{(2, 2)\} = E(q')$ and $(3, 1/3) \in E(q)$, but clearly $(2, 2) \ngeq (3, 1/3)$.

The situation in this example cannot occur when the upper contour sets are convex sets ranked by the parallelogram order. Indeed, according to Che et al. (2020), if $x \in E(q)$ and $U(q)$ is convex, then there is a sequence of non-zero weights $p^1, p^2, \ldots, p^{m-1} \in \mathbb{R}^\ell_+$ and $p^m \in \mathbb{R}^\ell_{++}$ such that $x \in \Phi^m(q)$, where $\Phi^k(q) := \text{argmin}\left\{p^k \cdot x : x \in \Phi^{k-1}(q)\right\}$ and

$\Phi^0(q) = U(q)$. If $U(q')$ dominates $U(q)$ in the parallelogram order then, by consecutive application of Theorem 1, we know that $\Phi^m(q')$ dominates $\Phi^m(q)$ in the parallelogram order. Thus (subject to standard conditions guaranteeing that $\Phi^m(q')$ is nonempty), there is $x' \in \Phi^m(q')$ such that $x' \geq x$. Since $p^m \gg 0$, the bundle $x' \in E(q')$.[14]

3.3 Consumer demand

Our results on conditional factor demand can be straightforwardly re-formulated to guarantee that Marshallian demand is normal. Apart from being an intrinsically appealing property for most product categories, normality plays an important role in many model settings. For example, normality is used in Bergstrom et al. (1986) to guarantee the uniqueness of Nash equilibria in a public goods game; the results on general equilibrium comparative statics in Nachbar (2002) and Quah (2003) hinge on their assumption that demand is normal; in Blundell et al. (2005), normality helps to determine how provision of a public good varies with intra-household bargaining power; and normality simplifies the non-parametric estimation of demand functions in Blundell et al. (2003). Thus it is important to have a thorough understanding of the foundations of this property.

Suppose a consumer has a utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$, defined over bundles of ℓ commodities. At prices $p \in \mathbb{R}^\ell_{++}$ and income $m \geq 0$, the consumer chooses a consumption bundle $x \in \mathbb{R}^\ell_+$ that is affordable and maximizes her utility; the solution to this problem is captured by the *Marshallian demand correspondence* $D : \mathbb{R}^\ell_{++} \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$, where $D(p, m) := \operatorname{argmax} \{u(x) : p \cdot x \leq m\}$. The indirect utility function $v : \mathbb{R}^\ell_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is given by $v(p, m) := \max \{u(x) : p \cdot x \leq m\}$.

Given the range V of u, the *Hicksian demand* $H : \mathbb{R}^\ell_{++} \times V \to \mathbb{R}^\ell_+$ maps prices p and utility levels v to those bundles that minimize the expenditure $p \cdot x$ over all alternatives satisfying $u(x) \geq v$. Obviously, the Hicksian demand is formally identical to the input demand in the production context, while the analog to the cost function is the expenditure function $e : \mathbb{R}^\ell_{++} \times V \to \mathbb{R}_+$, where $e(p, v) := \min \{p \cdot x : u(x) \geq v\}$.

[14] There are other manifestations of this result. For example, suppose V' and V are two convex sets representing the utility-possibilities of ℓ agents. If V' dominates V in the parallelogram order, then (using essentially the same argument) we may conclude that for every Pareto optimal utility allocation $v' \in V'$ there is a Pareto optimal allocation $v \in V$ such that $v' \geq v$. In other words, suppose the initial allocation is v' and the economy shrinks from V' to V; then there is a new allocation v which is Pareto optimal and which involves all agents sharing in the loss – no one is strictly better off.

Suppose that utility u is continuous and locally non-satiated.[15] In such a case, correspondences D and H are well-defined. Moreover, we have $p \cdot x = m$, for all $x \in D(p,m)$, while the two demands are related by the identity $D(p,m) = H\big(p, v(p,m)\big)$, for any prices p and income m (see Proposition 3.E.1 in Mas-Colell et al., 1995).

Let $K \subseteq \{1,2,\ldots,\ell\}$. We say that D is *normal* in K if, for any prices p, income levels m, m', and $x \in D(p,m)$, there is $x' \in D(p,m')$ such that $m' \geq m$ implies $x'_K \geq x_K$ and $m' \leq m$ implies $x'_K \leq x_K$. The demand D is normal in input i if it is normal in $K = \{i\}$. Finally, if $K = \{1,2,\ldots,\ell\}$, we simply say that D is normal.

The equivalence of demands D and H allows us to translate normality results on Hicksian demand into results on Marshallian demand. In particular, the following result on utility functions that are $\mathcal{P}$-increasing (in the sense defined in Section 3.1) follows immediately from Theorem 3 and Remark 2.6.

Proposition 4. *Let $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ be a continuous and increasing utility function. If u is $\mathcal{P}$-increasing in $K \subseteq \{1,2,\ldots,\ell\}$ then, for any prices $p \in \mathbb{R}^\ell_{++}$, the correspondence $m \to D(p,m)$ is $\mathcal{P}$-increasing in K. In particular, D is normal in K. Furthermore, if u is quasiconcave and D is normal in K, then u is $\mathcal{P}$-increasing in K.*

It is well-known that if a Marshallian demand function d is normal for good i, then the demand for i obeys the *law of demand*, i.e., function $d_i\big((p_i,p_{-i}),m\big)$ is decreasing in p_i, for all p_{-i} and m. We know that if, for all $p \in \mathbb{R}^\ell_{++}$, the Marshallian demand $D(p,\cdot)$ is $\mathcal{P}$-increasing in K, then D admits a selection d that satisfies normality (see Proposition 2). Therefore, an immediate consequence of Proposition 4 is the following: *for any continuous and locally non-satiated utility $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ that is $\mathcal{P}$-increasing in K, there is $d(p,m) \in D(p,m)$ that obeys the law of demand for every good $i \in K$.*

Example 8. Consider an agent who lives for ℓ periods and has a preference over consumption streams $x = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell_+$. In this context, it is natural to assume that this agent's utility has a recursive form, where

$$u(x) \ := \ h_1\Bigg(x_1, h_2\Bigg(x_2, h_3\Big(x_3, \ldots h_{\ell-1}\big(x_{\ell-1}, x_\ell\big)\Big)\Bigg)\Bigg)$$

[15] Utility function u is *locally non-satiated* if, for any bundle $x \in \mathbb{R}^\ell_+$, there is another bundle y arbitrarily close to x such that $u(y) > u(x)$.

Let $h_i : \mathbb{R}^2_+ \to \mathbb{R}_+$ be continuous, increasing, concave, and supermodular. Then h_i is increasing in the $\mathcal{C}$-flexible set order (see Example 5); consequently, the map from $(x_{\ell-2}, x_{\ell-1}, x_\ell)$ to $h_{\ell-2}\big(x_{\ell-2}, h_{\ell-1}(x_{\ell-1}, x_\ell)\big)$ is $\mathcal{P}$-increasing[16] and concave, because both $h_{\ell-2}$ and $h_{\ell-1}$ are concave functions. Repeating this argument, we eventually conclude that u is concave and $\mathcal{P}$-increasing, and hence it has a normal Marshallian demand.

3.4 Related results on normal demand

Some version of the equivalence between normality of demand and monotonicity of marginal costs is known at least since Fisher (1990). Fisher's original argument assumes that F is differentiable and generates a unique demand; we dispense with these assumptions. Our argument that normality implies increasing marginal costs (which is the proof that statement (iii) implies (iv) in Theorem 2) does not significantly break new ground, but our converse result is stronger, because it does not assume that demand is unique or that F is quasiconcave. Our proof goes through statement (i) and, thus, hinges on our characterization of normality using the parallelogram order.[17]

Alarie et al. (1990) and Bilancini and Boncinelli (2010) also characterize normal demand, under the condition that the objective function $F : \mathbb{R}^\ell_{++} \to \mathbb{R}$ is strictly increasing, strictly quasiconcave, and twice-differentiable. Note that these conditions on F are sufficient for demand to be strictly positive for all goods and a smooth *function* of prices. Let $\mathbf{G}(x)$ and $\mathbf{J}(x)$ denote the gradient and the Hessian of F, respectively, at some bundle x. The corresponding bordered Hessian $\tilde{\mathbf{J}}(x)$ is given by

$$\tilde{\mathbf{J}}(x) \;:=\; \begin{bmatrix} \mathbf{J}(x) & \mathbf{G}(x) \\ \mathbf{G}'(x) & 0 \end{bmatrix},$$

where $\mathbf{G}'(x)$ is the transposition of the column vector $\mathbf{G}(x)$. Let $\tilde{\mathbf{J}}_{i,j}(x)$ denote the (i,j)-th minor of $\tilde{\mathbf{J}}(x)$ and $\big|\tilde{\mathbf{J}}_{i,j}(x)\big|$ be its determinant. Bilancini and Boncinelli (2010) show

[16] We obtain this by applying the result in Example 7 with $G = h_{\ell-2}$, $f^1(x_{\ell-2}, x_{\ell-1}, x_\ell) = x_{\ell-2}$, and $f^2(x_{\ell-2}, x_{\ell-1}, x_\ell) = h_{\ell-1}(x_{\ell-1}, x_\ell)$.

[17] One could obtain a weaker version of (ii) from (iii) through a direct argument. Given any $\hat{p}_{-i}$, $p''_i > p'_i$, and $q'' > q'$, $H_i(p_i, \hat{p}_{-i}, q)$ is unique at $q = q', q''$, for almost every $p_i \in [p'_i, p''_i]$. We claim that, if (iv) holds, then there cannot be a generic violation of normality, in the sense of having $H_i(p_i, \hat{p}_{-i}, q') > H_i(p_i, \hat{p}_{-i}, q'')$ for almost every $p_i \in [p'_i, p''_i]$. Indeed, since $C(p''_i, \hat{p}_{-i}, q) - C(p'_i, \hat{p}_{-i}, q) = \int_{p'_i}^{p''_i} H_i(p_i, \hat{p}_{-i}, q)\, dp_i$, we would obtain $C(p''_i, \hat{p}_{-i}, q') - C(p'_i, \hat{p}_{-i}, q') > C(p''_i, \hat{p}_{-i}, q'') - C(p'_i, \hat{p}_{-i}, q'')$, which violates (iii). Obviously, this conclusion is weaker than (ii) (as stated in Proposition 3), where normality holds at every price for each good $i \in K$, whether or not demand is unique.

that the demand function is normal in $K \subseteq \{1, 2, \ldots, \ell\}$ if, and only if,

$$(-1)^{i-1} \big| \tilde{\mathbf{J}}_{i,(\ell+1)}(x) \big| \;\geq\; 0, \tag{5}$$

for all $i \in K$ and each $x \in \mathbb{R}^{\ell}_{++}$. By combining Proposition 4 with this result, we conclude that the condition (5) is equivalent to F being $\mathcal{P}$-increasing in K, when F satisfies the ancillary smoothness assumptions in their setup.

4 Factor complementarity

In this section, we show how our results can be applied to investigate complementary demand in the standard quasilinear optimizing model. This model appears in many contexts in economic modeling, but to keep the exposition focused, we shall refer to the problem as one of a firm choosing ℓ factors from a set X, a nonempty, closed subset of $\mathbb{R}^{\ell}_{+}$ to maximize profit. We denote the firm's production function by $F : X \to \mathbb{R}$ and the asymptotic cone generated by its production possibility set by $\mathbf{A}^F$.[18] We assume that F is continuous and $\mathbf{A}^F = \mathbb{R}^{\ell+1}_{-}$. The last condition requires that $F(x)$ cannot grow too quickly as x increases; it holds if F is uniformly bounded above but this is not a necessary condition.[19] In some models of quasilinear demand (see, for example, Ausubel and Milgrom (2002)) the domain X is finite or discrete and we allow for that here. Note also that F need not be concave or quasiconcave.

At factor prices $p \in \mathbb{R}^{\ell}_{++}$, the firm's (unconditional) *input/factor demand* are those bundles that maximize profit, i.e.,

$$\mathcal{H}(p) \;:=\; \operatorname{argmax} \Big\{ F(x) - p \cdot x : x \in X \Big\}.$$

Our assumptions on X and F guarantee that $\mathcal{H}(p)$ is nonempty for all $p \in \mathbb{R}^{\ell}_{++}$.[20] The firm's profit is $\pi(p) = F(x) - p \cdot x$, for $x \in \mathcal{H}(p)$.

Definition (Complements)**.** For any j and $K \subseteq \{1, 2, \ldots, \ell\}$, the set K is a *complement* of j if, for any $p, p' \in \mathbb{R}^{\ell}_{++}$ satisfying $p_{-j} = p'_{-j}$, and any $x \in \mathcal{H}(p)$, there is $x' \in \mathcal{H}(p')$

[18] The production possibility set associated with F is $\big\{z \in \mathbb{R}^{\ell+1} : z \leq \big(-x, F(x)\big),\ \text{for } x \in X\big\}$.

[19] Since the production possibility set is downward comprehensive, we have $\mathbb{R}^{\ell+1}_{-} = \mathbf{A}^F$. Loosely speaking, the two sets are equal when F grows slowly. For example, $\mathbf{A}^F = \mathbb{R}^2_{-}$ if $x \in \mathbb{R}_+$ and $F(x) = \sqrt{x}$.

[20] In formal terms, this claim is similar to the claim that Φ is nonempty for any $p \in \mathbb{R}^{\ell}_{++}$ under the assumptions of Theorem 3 (see footnote 8). We prove this claim in Section S.2 of the Online Supplement. For general results linking the domain of $\mathcal{H}$ and $\mathbf{A}^F$, see, for example, Neuefeind (1980).

such that $x'_K \geq x_K$ if $p'_j \leq p_j$ and $x'_K \leq x_K$ if $p'_j \geq p_j$. The factors in K are *joint complements* if K is a complement of j, for any $j \in K$.

Consider a change in factor prices from (p_K, p_{-K}) to (p'_K, p_{-K}) (with factor prices outside the set K being fixed). Clearly, this price change can be broken into $|K|$ steps, with the price of one good in K changing at each step. With this observation, we conclude that the factors in K are joint complements if, and only if, for any $x \in \mathcal{H}(p_K, p_{-K})$, there is $x' \in \mathcal{H}(p'_K, p_{-K})$ such that $x'_K \geq x_K$ if $p'_K \leq p_K$, and $x'_K \leq x_K$ if $p'_K \geq p_K$.

We first apply our method to answer a fundamental question: *if factor i is a complement of factor j, then is j a complement of i?* A well-known and widely-applied result in monotone comparative statics states that the set of all factors are joint complements if F is a supermodular function (see Topkis, 1978). Obviously, in this case the question we pose does not arise, but when complementarity patterns are more complicated, is it still true that complementarity is a symmetric property?

To answer this question, define the correspondence Γ^j with the domain $T = \mathbb{R}_-$, by

$$\Gamma^j(t) \;:=\; \Big\{(y,v) \in \mathbb{R}^{\ell+1} : (y,v) \geq \big(x, -F(x) - tx_j\big) \text{ for some } x \in X\Big\}. \tag{6}$$

It is straightforward to check that, for any $p \in \mathbb{R}^\ell_{++}$,

$$x \in \mathcal{H}(p_j - t, p_{-j}) \quad\Longleftrightarrow\quad \big(x, -F(x) - tx_i\big) \in \operatorname{argmin}\Big\{(p,1)\cdot y : y \in \Gamma^j(t)\Big\}$$

and thus $\pi(p_j - t, p_{-j}) = -\min\big\{(p,1)\cdot y : y \in \Gamma^j(t)\big\}$. Theorem 3 (with $K = \{i\}$) guarantees that the following are equivalent: (i) $\operatorname{co}\Gamma^j$ is $\mathcal{P}$-increasing in $\{i\}$; (ii) i is a complement of j; and is (iii) $-\pi(p_j - t, p_{-j})$ has increasing differences in (t, p_i). Notice that condition (iii) is equivalent to π being supermodular in (p_i, p_j), other prices being fixed. Since supermodularity is a symmetric property, we conclude that i is a complement of j if, and only if, j is a complement of i, with both equivalent to the supermodularity of π in (p_i, p_j). The following result summarizes our discussion.

Proposition 5. *Let X be a closed set in $\mathbb{R}^\ell_+$ and $F : X \to \mathbb{R}_+$ a continuous function with $\mathbf{A}^F = \mathbb{R}^{\ell+1}_-$.*[21] *Then, for any distinct $i, j \in \{1, 2, \ldots, \ell\}$, factor i is a complement of j if, and only if, j is a complement of i.*

[21] It is straightforward to check that these properties on F guarantee that Γ^j (as defined by (6)) satisfies the ancillary conditions required for the application of Theorem 3.

Let $K \subseteq \{1, 2, \ldots, \ell\}$. Then, any two factors in K are complements if, and only if, $\pi(p_K, p_{-K})$ is a supermodular function of p_K, for any fixed p_{-K}.[22]

We now turn to the conditions on F which guarantee that complementarity holds. Let $K \subseteq \{1, 2, \ldots, \ell\}$. By Theorem 1, if Γ^j (as defined by (6)) is $\mathcal{P}$-increasing in K then the map from $t \in \mathbb{R}_-$ to $\mathcal{H}(p_j - t, p_{-j})$ is $\mathcal{P}$-increasing in K. In other words, the set of factors K is complementary to j. The following property on F is sufficient to guarantee that Γ^j is $\mathcal{P}$-increasing in K, for all $j \in K$.

Definition. Let X be a subset of $\mathbb{R}^\ell$. The function $F : X \to \mathbb{R}$ is *super*modular in* $K \subseteq \{1, \ldots, \ell\}$ if, for any $x, x' \in X$ there is $y, y' \in X$ such that $(x \wedge x')_K \geq y_K$, $y'_K \geq (x \vee x')_K$, $x + x' = y + y'$, and $F(x) + F(x') \leq F(y) + F(y')$.

When F is super*modular in $K = \{1, 2, \ldots, \ell\}$, we simply refer to it as super*modular. A supermodular function is clearly super*modular, since the condition required by the latter holds simply by choosing $y = x \wedge x'$ and $y' = x \vee x'$. In general, super*modularity is a strictly weaker condition.[23] Implicit in our definition of super*modularity is that the domain X is *lattice-like in* K in the sense that, for any $x, x' \in X$ there is $y, y' \in X$ such that $(x \wedge x')_K \geq y_K$ and $x + x' = y + y'$. This property is strictly weaker than requiring X to be a sublattice of $\mathbb{R}^\ell$ (see Example 10).

The next result states that super*modularity is a sufficient and (under certain ancillary conditions) a necessary condition for complementarity.

Proposition 6. *Let $X \subseteq \mathbb{R}^\ell_+$ and $F : X \to \mathbb{R}$ be a super*modular function in $K \subseteq \{1, \ldots, \ell\}$. Then, the map from p_K to $\mathcal{H}(p_K, p_{-K})$ is $\mathcal{P}$-decreasing in K.*[24] *Moreover, if $X = \mathbb{R}^\ell_+$ and F is continuous, increasing, and concave, then F is super*modular in K if the factors in K are joint complements.*

[22] Theorem 10 in Ausubel and Milgrom (2002) states that all inputs are substitutes (rather than complements) if, and only if, the profit function π is submodular with respect to their prices. However, their definition of substitutes applies only to those prices at which the demand is a singleton, whereas our definition of complementarity (and its obvious modification for substitutability) applies at *all* prices. Modifying our proof of Proposition 5 in the obvious way will allow us to conclude that (a) for $i \neq j$, i is a substitute of j if and only if j is a substitute of i, and (b) π is submodular if and only if i and j are substitutes for any $i \neq j$. See Section S.4 of the Online Supplement for a fuller discussion.

[23] For example of a super*modular function that is not supermodular, suppose $X = \{0, 1, 2, 3\} \times \{0, 1\}$, and define $F : X \to \mathbb{R}$ by $F(x_1, 0) = x_1$ and $F(0, 1) = 1$, $F(1, 1) = F(2, 1) = 2$, $F(3, 1) = 4$. Since $3 = F(1, 0) + F(2, 1) < F(1, 1) + F(2, 0) = 4$, the function is *not* supermodular. However, one could check that F is super*modular; in particular, $F(3, 1) + F(0, 0) = F(1, 1) + F(2, 0)$.

[24] This means that $\mathcal{H}(p_K, p_{-K})$ dominates $\mathcal{H}(p'_K, p_{-K})$ in K by the parallelogram order if $p'_K \geq p_K$.

We know that if F is supermodular, then the map from p to $\mathcal{H}(p)$ is decreasing in the strong set order (see Topkis, 1978). In Proposition 6, this map is $\mathcal{P}$-decreasing, which is a weaker property that follows from the weaker assumption that F is super*modular (in $K = \{1, 2, \dots, \ell\}$). Nonetheless, this property on $\mathcal{H}$ suffices to guarantee that the factors in K are joint complements.

We end this section with three examples of super*modular functions.

Example 9 (Representative Agent)**.** It is well-known that the aggregate demand of multiple agents with quasilinear objectives is in turn representable by a single agent with a quasilinear objective. Formally, suppose $\mathcal{H}(p)$ is the factor demand at prices $p \in \mathbb{R}^{\ell}_{++}$ generated by $F^n : X^n \to \mathbb{R}$ (where $X^n \subseteq \mathbb{R}^{\ell}_{+}$) for $n = 1, \dots, N$. Then the aggregate demand $\mathcal{H}(p) = \sum_{i=1}^{N} \mathcal{H}^n(p)$ admits a *representative agent* in the sense that $\mathcal{H}(p) = \mathcal{H}^*(p)$, where $\mathcal{H}^*$ is the factor demand generated by $F^* : X^* \to \mathbb{R}$, where $X^* = \sum_{n=1}^{N} X^n$ and $F^*(x) := \max\left\{ \sum_{n=1}^{N} F^n(y^n) : \sum_{n=1}^{N} y^n = x \right\}$.

If each F^n is a supermodular function, then $\mathcal{H}^n$ has the property that all factors are joint complements, which in turn guarantees that all factors are joint complements for aggregate demand $\mathcal{H}$. This raises the question of whether the representative agent's utility function F^* is also supermodular. This is not generally true.[25] However, super*modularity *is* preserved by aggregation, so F^* will be super*modular if F^n is super*modular (and, in particular, supermodular) for all n. If we interpret F^* as the production function of a firm and F^n as the production of the nth team (or plant or division) of the firm, then our observation means that one has to be careful in modeling the firm's (overall) production function as supermodular even if its sub-units have that property, because only super*modularity is preserved by aggregation.[26]

We now show that F^* is super*modular. For any $x, x' \in X$, there is $u^n, u^{n\prime} \in X^n$, for $n = 1, \dots N$, such that $x = \sum_n u^n$, $x' = \sum_n u^{n\prime}$, $F^*(x) = \sum_n F^n(u^n)$, and $F^*(x') = \sum_n F^n(u^{n\prime})$. If F^n is super*modular, then there is $v^n, v^{n\prime} \in X^n$ satisfying [a] $u^n_K, u^{n\prime}_K \geq v^n_K$, [b] $v^n + v^{n\prime} = u^n + u^{n\prime}$, and [c] $F^n(v^n) + F^n(v^{n\prime}) \geq F^n(u^{n\prime}) + F^n(u^{n\prime})$. Denote $y = \sum_n v^n$, $y' = \sum_n v^{n\prime}$. Summing across n, [a] guarantees that $x_K, x'_K \geq y_K$,

[25] Suppose $F^1(x_1, x_2, x_3) = x_1^{1/2} x_2^{1/4}$ and $F^2(x_1, x_2, x_3) = x_1^{1/2} x_3^{1/4}$ for $(x_1, x_2, x_3) \in \mathbb{R}^3_+$. Then one could check that $F^*(x_1, x_2, x_3) = \sqrt{x_1}\sqrt{\sqrt{x_2} + \sqrt{x_3}}$. While F^1 and F^2 are supermodular functions, F^* is not, since the cross derivative of x_2 and x_3 is negative.

[26] We may contrast this with a property such as concavity which is more robust in the sense that F^* is concave if each function F^n is concave.

[b] implies that $x + x' = y + y'$, and [c] gives

$$F^*(x)+F^*(x') \;=\; \sum_n F^n(u^n)+\sum_n F^n(u^{n\prime}) \;\le\; \sum_n F^n(v^{n\prime})+\sum_n F^n(v^{n\prime}) \;\le\; F^*(y)+F^*(y').$$

Thus, the function F^* is super*modular in K.

Example 10. Let $X = \{B \cdot z : z \in \mathbb{R}^k_+\}$, where B be an $\ell \times k$ matrix B with positive entries. Given a function $f : \mathbb{R}^k_+ \to \mathbb{R}$, define $F : X \to \mathbb{R}$ by $F(x) := \max\{f(z) : B \cdot z = x\}$. We can interpret F as the production function of a two-stage production process: the final output is produced from k intermediate goods and governed by the production function f; in turn, each intermediate good is produced from ℓ factors, with the jth column of B being the vector of factors needed to produce one unit of the jth intermediate good.

We claim that F is super*modular if f is super*modular. Indeed, take any $x, x' \in X$ and $z, z' \in Z$ such that $x = B \cdot z$, $x' = B \cdot z'$, $f(z) = F(x)$ and $f(z') = F(x')$. Since all entries in B are positive, $B \cdot (z \wedge z') \le B \cdot z$ and $B \cdot (z \wedge z') \le B \cdot z'$, implying $B \cdot (z \wedge z') \le (B \cdot z) \wedge (B \cdot z') = x \wedge x'$. Since f is super*modular, there is $\tilde{z}, \tilde{z}' \in Z$ such that $(z \wedge z') \ge \tilde{z}'$, $z + z' = \tilde{z} + \tilde{z}'$, and $f(\tilde{z}) + f(\tilde{z}') \ge f(z) + f(z')$. Note that $B \cdot \tilde{z}' \;\le\; B \cdot (z \wedge z') \;\le\; x \wedge x'$. We also have $(B \cdot \tilde{z}) + (B \cdot \tilde{z}') = x + x'$. Let $y = B \cdot \tilde{z}$ and $y' = B \cdot \tilde{z}'$. Then $x \wedge x' \ge y'$, $x + x' = y + y'$, and

$$F(x) + F(x') \;=\; f(z) + f(z') \;\le\; f(\tilde{z}) + f(\tilde{z}') \;\le\; F(y) + F(y').$$

Note that while F is a super*modular function, it need not be supermodular, even when f is supermodular. In fact, the set X need not even be a sublattice of $\mathbb{R}^\ell$.[27]

Unlike supermodularity, which allows us to analyze only complementarities between *all* inputs jointly, super*modularity can be employed to the study of complementarities between subsets of inputs. In the next example, we discuss one such case.

Example 11. Let $I_1, I_2, \ldots, I_n$ be a partition of $\{1, \ldots, \ell\}$ and $X_j \subseteq \mathbb{R}^{|I_j|}$, for all $j = 1, \ldots, n$. Suppose that $g_j : X_j \to Y_j$ (where $Y_j \subseteq \mathbb{R}$) is $\mathcal{P}$-increasing in $\{i_j\} \in I_j$. Then for any increasing and supermodular function $A : \times_j Y_j \to \mathbb{R}$, the function $F : \times_j X_j \to \mathbb{R}$,

$$F(x) \;:=\; A\Big(g_1(x_{I_1}), g_2(x_{I_2}), \ldots, g_n(x_{I_n})\Big),$$

[27] For example, suppose three factors are used to produce two intermediate goods and B is the matrix with two columns $(1,1,0)$ and $(0,1,1)$. Let $x = B \cdot (1,0)^T = (1,1,0)^T$ and $x' = B \cdot (0,1)^T = (0,1,1)^T$. Note that there is no $z \in Z$ for which $B \cdot z = (0,1,0)^T = x \wedge x'$. Hence, X is not a sublattice of $\mathbb{R}^2$.

is super*modular in $K = \{i_1, \ldots, i_n\}$. We prove this claim in the Appendix. Notice that the factors within each group I_j are not necessarily complements, even if g_j is $\mathcal{P}$-increasing in I_j (rather than just $\{i_j\}$). For example, suppose $F(x) = (x_1 + x_2)(x_3 + x_4)$. Then $A = y_1 y_2$ is supermodular, and $h_1 = x_1 + x_2$ and $h_2 = x_3 + x_4$ are both supermodular and concave functions (and hence $\mathcal{P}$-increasing in $\{1, 2\}$ and $\{3, 4\}$ respectively). Thus factors *across* groups – such as 1 and 3 or 1 and 4 – are joint complements. However, factors within a group are clearly not complements. Indeed suppose there is a drop in the price of factor 1, so it goes from being above to below the price of 2. The firm would use only factor 2 initially since it is cheaper than 1, and it switches completely to factor 1 at the new price. Thus the demand for 1 increases but that of 2 drops to zero.

5 First order stochastic dominance under ambiguity

We consider an agent making decisions in an uncertain environment. Suppose that the possible states of the world are represented by a set $S \subseteq \mathbb{R}$; to keep our exposition focused on the essentials we assume that the set $S = \{s_1, s_2, \ldots, s_\ell, s_{\ell+1}\}$ is finite, where $s_1 < s_2 < \ldots < s_\ell < s_{\ell+1}$. We denote the set of distributions on S by $\triangle_S$. We represent these distributions by their cumulative distribution functions. Let λ, $\mu : S \to \mathbb{R}$ be two cumulative distribution functions. The distribution λ *first order stochastically dominates* μ if $\lambda(s) \leq \mu(s)$ for all $s \in S$; we denote this by $\lambda \succeq \mu$. An important feature of $(\triangle_S, \succeq)$ is that it is a lattice. For distributions λ and λ' their meet and join are defined by $(\lambda \wedge \lambda')(s) = \max\left\{\lambda(s), \lambda'(s)\right\}$ and $(\lambda \vee \lambda')(s) = \min\left\{\lambda(s), \lambda'(s)\right\}$, respectively.

The concept of first order stochastic dominance (FSD) allows us to compare distributions by expected utility; indeed, $\lambda \succeq \mu$ if, and only if, $\int_S u(s)d\lambda(s) \geq \int_S u(s)d\mu(s)$ for all increasing functions $u : S \to \mathbb{R}$. This basic result also has a simple and widely-used corollary that allows us to compare the actions of an agent maximizing expected utility. Suppose this agent chooses an action from a set $X \subseteq \mathbb{R}$ and her utility from action x is $g(x, s)$ when state s is realized. Let $\lambda(\cdot, t)$ be a distribution over S (parameterized by t in a poset T) which captures the agent's belief about the likelihood of different states. Then the expected utility of taking action x is $f(x, t) = \int_S g(x, s)d\lambda(s, t)$. Suppose that $g(x, s)$ has increasing differences in (x, s) (equivalently, is supermodular in (x, s)) and λ is ordered by first order stochastic dominance in the sense that $\lambda(\cdot, t') \succeq \lambda(\cdot, t)$ whenever

$t' \geq t$. In such a case, $x' \geq x$ implies that

$$f(x', t) - f(x, t) \;=\; \int_S \big[g(x', \tilde{s}) - g(x, \tilde{s})\big] d\lambda(\tilde{s}, t)$$

is increasing in t, since $s \to \big[g(x', s) - g(x, s)\big]$ is increasing in s. In other words, f has increasing differences in (x, t), which guarantees that argmax $\big\{f(x, t) : x \in X\big\}$ increases with t in the strong set order (see Topkis (1978) or Milgrom and Shannon (1994)).

Our objective in this section is to extend this simple result on comparative statics to some widely-used multi-prior models of decision-making under uncertainty.

5.1 First order stochastic dominance (FSD) in the maxmin model

In the *maxmin* model of Gilboa and Schmeidler (1989), the agent evaluates an uncertain environment with a convex *set* of distributions over $S \subseteq \mathbb{R}$. If $g(x, s)$ is the utility from action $x \in X \subseteq \mathbb{R}$ when $s \in S$ is realized, then the agent's utility (ex ante) is

$$f(x, t) \;:=\; \min \left\{ \int_S g(x, s) d\lambda(s) : \lambda \in \Lambda(t) \right\}, \tag{7}$$

where $\Lambda(t)$ is a convex set of distributions parameterized by $t \in T$. This leads naturally to the following question: when g is a supermodular function, what shift in the set $\Lambda(t)$ would guarantee that the agent chooses a higher action? The next definition gives the set generalization of first order stochastic dominance that is appropriate for this purpose.

Definition. Let T be a poset. The correspondence $\Lambda : T \to \Delta_S$ is *FSD-increasing by the parallelogram order* (or $\mathcal{P}_{FSD}$*-increasing*, for short) if for any $t' \geq_T t$ and distributions $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, there is some $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that

$$\lambda' \succeq \mu, \quad \mu' \succeq \lambda, \quad \text{and} \quad \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' = \tfrac{1}{2}\mu + \tfrac{1}{2}\mu'.$$

Our set-generalization of first order stochastic dominance is clearly just a version of the parallelogram order. To be precise, the correspondence $\Gamma : T \to \mathbb{R}^\ell$ given by

$$\Gamma(t) \;:=\; \Big\{ y \in \mathbb{R}^\ell : y_i = -\lambda(s_i), \text{ for all } i = 1, 2, \ldots, \ell \text{ and } \lambda \in \Lambda(t) \Big\} \tag{8}$$

is $\mathcal{P}$-increasing if, and only if, Λ is $\mathcal{P}_{\text{FSD}}$-increasing. In particular, this tells us that if Λ is $\mathcal{P}_{\text{FSD}}$-increasing, then so is $\operatorname{co}\Lambda$.

Proposition 7. *Suppose that $\Lambda : T \to \triangle_S$ has compact and convex values. Then $f : X \times T \to \mathbb{R}$, as defined by (7), has increasing differences in (x, t) for any supermodular function $g : X \times S \to \mathbb{R}$ if, and only if, Λ is $\mathcal{P}_{\mathrm{FSD}}$-increasing.*[28]

Remark 5.1. We show in the Appendix that Proposition 7 remains true if S is a compact interval of $\mathbb{R}$ and function $g(x, \cdot)$ is Riemann-Stieltjes integrable with respect to each $\lambda \in \Lambda(t)$, for all $x \in X$ and $t \in T$. This holds if any of the following conditions are satisfied: (a) function $g(x, s)$ is continuous in $s \in S$; (b) $g(x, s)$ is bounded on S and has only finitely many discontinuities in s, and all distributions in $\Lambda(t)$ are atomless; or (c) $g(x, s)$ is bounded and monotone on S, and all distributions in $\Lambda(t)$ are atomless.

Remark 5.2. Proposition 7 can be equivalently formulated as saying that the map from (x, t) to $\max\left\{\int_S g(x, s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ has increasing differences in (x, t), for any supermodular g if, and only if, Λ is $\mathcal{P}_{\mathrm{FSD}}$-increasing.[29] The α-maxmin model by Ghirardato et al. (2004) allows for both ambiguity averse and ambiguity loving behavior, with the agent's utility function having the form

$$\alpha \ \min\left\{\int_S g(x, s)d\lambda(s) : \lambda \in \Lambda(t)\right\} \ + \ (1-\alpha) \ \max\left\{\int_S g(x, s)d\lambda(s) : \lambda \in \Lambda(t)\right\},$$

for some $\alpha \in [0, 1]$. This function has increasing differences in (x, t) if Λ is $\mathcal{P}_{\mathrm{FSD}}$-increasing, since both elements of the sum have increasing differences; this in turn guarantees that the set of optimal actions increases with t in the strong set order.

Remark 5.3. The $\mathcal{P}_{\mathrm{FSD}}$-increasing property is one possible set-generalization of first order stochastic dominance. As pointed out at the beginning of this section, if the distributions $\lambda(\cdot, t)$ are FSD-increasing with t, then the map $t \to \int u(s)d\lambda(s, t)$ is increasing, for any increasing function u. This leads naturally to the analogous question for multiple priors: what conditions on Λ will guarantee that the function $t \to \min\left\{\int u(s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ is increasing, for any increasing utility function u? We show in that this leads to a set-

[28] Milgrom and Shannon (1994) show that a weaker condition on f than supermodularity, called *single crossing differences* in (x, t), is sufficient for $\operatorname{argmax}\{f(x, t) : x \in X\}$ to be increasing in the strong set order. We show in Section S.5 of the Online Supplement that $\mathcal{P}_{\mathrm{FSD}}$ monotonicity of Λ is also necessary for f to have single crossing differences in (x, t), for any supermodular function g.

[29] Indeed, Proposition 7 guarantees that $\min\left\{\int_S -g(x, s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ has decreasing differences in (x, t) since $-g(x, s)$ is submodular; therefore $\max\left\{\int_S g(x, s)d\lambda(s) : \lambda \in \Lambda(t)\right\} = -\min\left\{\int_S -g(x, s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ has increasing differences in (x, t).

generalization of first order stochastic dominance that is weaker than the $\mathcal{P}_{\text{FSD}}$-increasing property.[30]

Remark 5.4. Whenever $g(x, s)$ is increasing in s, we can assume, without loss of generality, that Λ is upper comprehensive, i.e., if $\lambda \in \Lambda(t)$ and $\lambda' \succeq \lambda$, then $\lambda' \in \Lambda(t)$.[31] In Section S.7 of the Online Supplement, we show that when Λ is upper comprehensive, the $\mathcal{P}_{\text{FSD}}$-increasing property on Λ remains *necessary* for f to have increasing differences for all $g(x, s)$ that are supermodular in (x, s) and increasing in s.

Proof of Proposition 7. Define $\Gamma : T \to \mathbb{R}^\ell$ by (8). If Λ is $\mathcal{P}_{\text{FSD}}$-increasing, then Γ is $\mathcal{P}$-increasing. For any function $g : X \times S \to \mathbb{R}$ and distribution λ,

$$\begin{aligned}\int_S g(x,s)d\lambda(s) &= g(x,s_1)\lambda(s_1) + \sum_{i=1}^{\ell} g(x,s_{i+1})\big[\lambda(s_{i+1}) - \lambda(s_i)\big] \\ &= g(x,s_{\ell+1}) + \sum_{i=1}^{\ell} \big[g(x,s_{i+1}) - g(x,s_i)\big]\big[-\lambda(s_i)\big]. \quad (9)\end{aligned}$$

Given $x' \geq x$, we define $p, p' \in \mathbb{R}^\ell$ by $p_i = g(x, s_{i+1}) - g(x, s_i)$ and $p'_i = g(x, s_{i+1}) - g(x, s_i)$, for $i = 1, 2, \ldots, \ell$. Then inequality (9) gives

$$f(x,t') - f(x,t) = \min\Big\{p \cdot y : y \in \Gamma(t')\Big\} - \min\Big\{p \cdot y : y \in \Gamma(t)\Big\}, \quad (10)$$

with a similar formula for $f(x', t') - f(x', t)$. If g is supermodular, then $p' \geq p$ and Theorem 2 guarantees that $\min\{p \cdot y : y \in \Gamma(t')\} - \min\{p \cdot y : y \in \Gamma(t)\}$ is less than $\min\{p' \cdot y : y \in \Gamma(t')\} - \min\{p' \cdot y : y \in \Gamma(t)\}$. Thus, $f(x, t') - f(x, t) \leq f(x', t') - f(x', t)$, and so f has increasing differences.

To show the converse suppose Λ is not $\mathcal{P}_{\text{FSD}}$-increasing and thus Γ is not $\mathcal{P}$-increasing. By Theorem 2, there are vectors $p' \geq p$ in $\mathbb{R}^\ell$ and $t' \geq_T t$ such that

$$\begin{aligned}&\min\Big\{p \cdot y : y \in \Gamma(t')\Big\} - \min\Big\{p \cdot y : y \in \Gamma(t)\Big\} \\ &\qquad > \min\Big\{p' \cdot y : y \in \Gamma(t')\Big\} - \min\Big\{p' \cdot y : y \in \Gamma(t)\Big\}. \quad (11)\end{aligned}$$

[30] This weaker notion of stochastic dominance is not sufficient to guarantee comparative statics except in special cases. One such special case is considered in Ui (2015) which studies global games with ambiguity. Section S.6 of the Online Supplement explains Ui's formulation in greater detail.

[31] Given a correspondence Λ, let $\bar{\Lambda}(t) = \big\{\lambda \in \triangle_S : \lambda \succeq \lambda', \text{ for } \lambda' \in \Lambda(t)\big\}$. It is clear that $\bar{\Lambda}$ is upper comprehensive and that $\min\big\{\int_S g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\big\} = \min\big\{\int_S g(x,s)d\lambda(s) : \lambda \in \bar{\Lambda}(t)\big\}$.

Take any $x, x' \in X$ satisfying $x' > x$.[32] Define a supermodular function $g : X \times S \to \mathbb{R}$ as follows: $g(y, s_1) = 0$ for all $y \in X$ and for $i > 2$, $g(y, s_i) := \sum_{j=1}^{i-1} p_i$ if $y \leq x$ and $g(y, s_i) := \sum_{j=1}^{i-1} p'_i$ otherwise. The formula (10), together with (11), gives $f(x, t') - f(x, t) > f(x', t') - f(x', t)$, so f violates increasing differences. $\square$

The following are examples of $\mathcal{P}_{\text{FSD}}$-increasing correspondences.

Example 12 (Strong set order)**.** Suppose Λ is increasing in the strong set order, i.e., for any $t' \geq t$, $\lambda \in \Lambda(t)$, and $\lambda' \in \Lambda(t')$, we have $\lambda \wedge \lambda' \in \Lambda(t)$ and $\lambda \vee \lambda' \in \Lambda(t')$. By setting $\mu = \lambda \wedge \lambda'$ and $\mu' = \lambda \vee \lambda'$, we conclude that Λ is $\mathcal{P}_{\text{FSD}}$-increasing. For example, let $\bar{\nu}^t$ and $\underline{\nu}^t$ be distributions on $S \subset \mathbb{R}$ that increase with respect to first order stochastic dominance in t and satisfy $\bar{\nu}^t \succeq \underline{\nu}^t$. for all t. Then $\Lambda(t) = [\bar{\nu}^t, \underline{\nu}^t]$, which consists of all distributions ordered between $\bar{\nu}^t$ and $\underline{\nu}^t$, is convex-valued, and the correspondence Λ increases with t in the strong set order.

For another simple example, let $\Lambda(t)$ be a set of normal distributions with a fixed variance and means drawn from a set $M^\Lambda(t) \subset \mathbb{R}$. In this case, the family of normal distributions is totally ordered by the mean, i.e, one distribution first order stochastically dominates another distribution if and only if the former has a higher mean than the latter. Then it is clear that $\Lambda(t)$ increases with t in the strong set order if and only if $M^\Lambda(t)$ (as sets in $\mathbb{R}$) increases with t in the strong set order.[33]

Example 13 (Increasing mean)**.** Take an increasing function $h : S \to \mathbb{R}$ and suppose that $\Lambda(t)$ consists of all distributions over S for which the expected value of h is equal to t, i.e., $\Lambda(t) = \left\{\lambda \in \triangle_S : \int_S h(s)d\lambda(s) = t\right\}$. It is clear that Λ is not increasing in the strong set order since the supremum or infimum of two distributions μ and μ' will not generally have the same mean as μ or μ'. However, we show in the Appendix that Λ is $\mathcal{P}_{\text{FSD}}$-increasing because Γ (as defined by (8)) is increasing in the $\mathcal{C}$-flexible set order.

Example 14 (Convex combinations)**.** Since the $\mathcal{P}_{\text{FSD}}$-increasing property is just a version of the parallelogram order, it is also preserved by convex combinations, i.e., if Λ_1

[32] Clearly, the proof requires that X is nonempty and contains at least two elements.

[33] It is common in applications to model ambiguity with a parametric family of distributions having different means, while keeping other parameters unchanged (see Bianchi et al. (2018) and Ilut and Schneider (2022)). In these cases, the distributions are often totally ordered by first order stochastic dominance, so our example applies beyond the family of normal distributions.

and Λ_2 are $\mathcal{P}_{\text{FSD}}$-increasing, then so is $\Lambda := \alpha\Lambda_1 + (1-\alpha)\Lambda_2$, for $\alpha \in (0,1)$ (see Proposition 1). In particular, instances of the $\mathcal{P}_{\text{FSD}}$-increasing property given in the two previous examples could be combined to generate more examples. We give two instances where such combinations occur naturally.

Firstly, in ϵ-contamination models of ambiguity aversion (see, for example, Epstein and Wang (1994) and Nishimura and Ozaki (2004)), the agent has a set of priors which is the convex combination of the set of all distributions on S and a single distribution interpreted as the agent's belief, held with incomplete confidence. The weight on the former is ϵ so, in our notation, $\Lambda_1(t)$ is the set of all priors, $\Lambda_2(t)$ is a singleton set, and $\alpha = \epsilon$. Clearly, if the distribution $\Lambda_2(t)$ is FSD-increasing in t, then $\Lambda(t)$ has the $\mathcal{P}_{\text{FSD}}$-increasing property.

Secondly, prior sets which are convex combinations of other sets of distributions could arise because of set predictions. For example, suppose a firm uses a model to forecast future demand for its product. This model gives a set prediction of demand levels conditional on the prevailing state of the economy $\omega \in \Omega$ and some other parameter $t \in T$ (such as the firm's advertising expenditure in the current period). We denote by $A(\omega, t)$ the finite set of demand forecasts at (ω, t). Suppose that, for any ω, $A(\omega, t)$ increases with t in the strong set order and let $\Lambda^\omega(t)$ be the set of degenerate probability distributions corresponding to $A(\omega, t)$.[34] Assuming that the firm knows that ω occurs with probability $\pi(\omega)$, the set of possible demand distributions (for a given t and before the realization of ω) is $\Lambda(t) = \sum_{\omega\in\Omega} \pi(\omega)\Lambda^\omega(t)$. In other words, a typical element of $\Lambda(t)$ is a distribution where some $s^\omega \in A(\omega, t)$ occurs with probability $\pi(\omega)$. Since Λ^ω is increasing in the strong set order, Λ, and thus also $\operatorname{co}\Lambda$, is $\mathcal{P}_{\text{FSD}}$-increasing.[35] However, Λ need not increase in the strong set order, nor in the $\mathcal{C}$-flexible sense.[36]

We conclude this subsection with three economic applications. Further applications are found in Section S.9 of the Online Supplement, where we apply our results to formulate

[34] For example, suppose the firm models the possible demand outcomes (given (ω, t)) as the optimal choices of a representative agent with the quasilinear utility function $Q(s, \omega, t) = \phi(s, \omega, t) - s$, where ϕ is supermodular in (s, t). Then $A(\omega, t) = \operatorname{argmax}_{s\in S} Q(s, \omega, t)$ increases with t in the strong set order.

[35] Elements of $\operatorname{co}\Lambda(t)$ have a natural interpretation: each element is a distribution over demand that arises from choosing a distribution over $A(\omega, t)$ (for each ω), with ω occurring with probability $\pi(\omega)$. The maxmin model requires the set of priors to be convex, but it makes no difference here whether the set of priors is $\Lambda(t)$ or $\operatorname{co}\Lambda(t)$, since the value of $f(x, t)$ (as defined by (7)) is the same in either case.

[36] Section S.8 of the Online Supplement provides a specific example.

conditions for monotone decision rules for ambiguity averse agents choosing in a dynamic context; this generalizes known results on monotone decision rules (see Hopenhayn and Prescott, 1992) for agents maximizing discounted expected utility.

Example 15 (Optimal savings)**.** An agent lives for two periods, with income m in period 1 and uncertain income s in period 2.[37] With saving $x \in [0, m]$ in period 1, the agent's utility conditional on s is $g(x, s) = u(m - x) + \beta u\big(x(1 + r) + s\big)$, where u is the per-period utility, β is the discount rate, and r is the interest. The function g is increasing in s if u is increasing, and it is submodular in (x, s) (equivalently, $g_{xs} \leq 0$) if u is concave. Suppose the agent has maxmin preferences of the form (7). Since g is submodular, f has decreasing differences in (x, t) if Λ is $\mathcal{P}_{\text{FSD}}$-increasing. It follows that the agent saves less with higher t; formally, $\text{argmax}_{x\in[0,m]} f(x, t)$ falls with t in the strong set order.

In particular, suppose that through the news and other channels, this agent is confident that the *mean* income in period 2 is t, but is not confident of the precise distribution that s takes. In this case, he may behave as though $\Lambda(t)$ consists of all distributions with mean t (as in Example 13); if so, any news that raises the agent's belief about the mean income in period 2 will cause him to save less in period 1.

Example 16 (Portfolio problem)**.** An investor divides her wealth $m > 0$ between a *safe asset*, that pays out $r > 0$ for sure, and a *risky asset* with an uncertain return of s in $S \subseteq \mathbb{R}_+$. The investor's beliefs over the risky return is captured by the correspondence Λ. The investor chooses to invest $x \in X \subseteq \mathbb{R}$ in the risky asset, with the rest of his wealth invested in the safe security. We allow the investor to go short on either asset but require her to be solvent, i.e., it must be that $xs + (m - x)r \geq 0$, for all $s \in S$ and $x \in X$. Assuming that her Bernoulli index is $u : \mathbb{R}_+ \to \mathbb{R}$ and the investor is ambiguity averse, the investor's utility at $x \in X$ is

$$f(x, t) \;:=\; \min \left\{ \int_S u\big(xs + (m - x)r\big) d\lambda(s) : \lambda \in \Lambda(t) \right\}. \tag{12}$$

We know that f has increasing differences in (x, t) if Λ is $\mathcal{P}_{\text{FSD}}$-increasing and $g(x, s) := u\big(xs + (m - x)r\big)$ is supermodular. Assuming that u is strictly increasing, concave, and twice continuously differentiable, it is straightforward to check that g is supermodular if

[37] For other discussions of the two-period savings problem with ambiguity aversion, see **?** and Ilut and Schneider (2022). The latter also contains a review of infinite horizon consumption-saving problems with ambiguity aversion and of the evidence of ambiguity aversion in household survey data.

the coefficient of relative risk aversion of u is less than 1.[38] With this condition on u, f has increasing differences in (x,t) and (consequently) the investor's holding in the risky asset increases with t. This is valid even if her preference has the α-maxmin form.[39]

To be more specific, suppose that the investor uses different models of the return on the risky asset and these models returning an interval of distributions $[\underline{\nu},\bar{\nu}]$, with $\bar{\nu}$ being the most optimistic and $\underline{\nu}$ the least. This return attracts tax and we denote by t the proportion of the return that is retained after tax. Then $\Lambda(t) = [\underline{\nu}^t,\bar{\nu}^t]$ is the set of distributions after tax, where $\underline{\nu}^t$ and $\bar{\nu}^t$ are the after-tax return distributions corresponding to $\underline{\nu}$ and $\bar{\nu}$ respectively. Clearly, $\underline{\nu}^t$ and $\bar{\nu}^t$ are both FSD-increasing with t and thus Λ is $\mathcal{P}_{\text{FSD}}$-increasing (see Example 12).[40]

The next example has a different flavor from Examples 15 and 16: it has both x and t as choice variables and exploits the fact that supermodularity is preserved by the sum.

Example 17. A firm operating in uncertain market conditions must decide how much to produce and how much to spend on advertising. In period 1, the marginal cost of production is $c > 0$ and the marginal cost of advertising is $a > 0$. If the firm chooses t units of advertising, its belief on the demand for its output s is given by a set of distributions $\Lambda(t)$. We assume that higher advertising leads to greater demand in the sense that Λ is $\mathcal{P}_{\text{FSD}}$-increasing. For an example of how this could arise, see Example 14.

In period 2, s is realized and the firm has to meet this demand even if it exceeds its period 1 output; the profit in period 2 is $\pi(x,s) := s - \kappa(\max\{s-x,0\})$. Function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ should be interpreted as the cost of producing the additional units to meet demand in period 2. At the same time, goods for which there is no demand can be freely disposed. Also, notice that $\pi(x,s)$ need not be increasing in s.

The firm chooses $x \geq 0$ and $t \geq 0$ in period 1 to maximize

$$\Pi(x,t,c,a) \ := \ \min\left\{\int_S \pi(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\} - cx - at.$$

[38] Note that, since x can take negative values, function g does not increase in s.

[39] There are other discussions of the portfolio choice model under ambiguity. For example, Gollier (2011) examines how the demand for the risky asset changes with the level of ambiguity aversion, in the context of the smooth ambiguity model. Cherbonnier and Gollier (2015) study both the smooth ambiguity model and the α-maxmin model; the authors provide conditions under which the demand for the risky asset increases with respect to initial wealth. See also the survey of Ilut and Schneider (2022).

[40] Assuming that only positive returns are taxed proportionately, we obtain $\underline{\nu}^t(s) = \underline{\nu}(s)$, for $s \leq 0$, and $\underline{\nu}^t(s) = \underline{\nu}(s/t)$ otherwise.

It is straightforward to check that π is supermodular if κ is increasing, convex, and $\kappa(0) = 0$.[41] Proposition 7 guarantees that $f(x,t) = \min\left\{\int_S \pi(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ is a supermodular function of (x,t) and therefore Π is supermodular in (x,t). Furthermore, Π has increasing differences in $\big((x,t),(-c,-a)\big)$. Thus $\operatorname{argmax}_{(x,t)\in\mathbb{R}^2_+} \Pi(x,t,c,a)$ decreases with (c,a) in the strong set order, i.e., a fall in advertising cost or a fall in the period 1 cost of production leads to more advertising and greater output.

5.2 Variational and multiplier preferences

Proposition 7 can be extended to cover a broader class of preferences. Maccheroni et al. (2006) introduce and axiomatize a generalization of the maxmin model, called *variational preferences*. In this model, the utility of action $x \in X \subset \mathbb{R}$ is

$$f(x,t) \;=\; \min\left\{\int_S g(x,s)d\lambda(s) + c(\lambda,t) : \lambda \in \triangle_S\right\}, \tag{13}$$

where $c(\cdot,t)$ is a convex function parameterized by $t \in T$. Loosely speaking, the agent's utility from action x is obtained by minimizing her expected utility over the set of all probability distributions; unlike the maxmin model where the agent is restricted to a subset of $\triangle_S$, any distribution in $\triangle_S$ could be 'picked' in the variational preferences model, though each distribution λ is associated with a different cost $c(\lambda,t)$. The next result identifies those shifts in the cost function c which guarantee that the agent's utility has increasing differences in (x,t).

Proposition 8. *Let $c : \triangle_S \times T \to \mathbb{R}_+$ be a continuous and convex function on $\triangle_S$, for all $t \in T$. The following statements are equivalent.*

(i) *The function c satisfies the following property:*

(C) *for any $t' \geq t$ in T and λ, λ' in $\triangle_S$ there is μ, μ' in $\triangle_S$ such that*

$$\lambda' \succeq \mu,\ \mu' \succeq \lambda,\ \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' = \tfrac{1}{2}\mu + \tfrac{1}{2}\mu',\ \text{and } c(\lambda,t) + c(\lambda',t') \geq c(\mu,t) + c(\mu',t').$$

(ii) *The function $f : X \times T \to \mathbb{R}$ defined in (13) is supermodular, for any supermodular function $g : X \times S \to \mathbb{R}$.*[42]

[41] Take any $x' \geq x$ and consider three cases. If (i) $s \leq x$, then $\delta(s) := \big[\pi(x',s) - \pi(x,s)\big] = 0$; whenever (ii) $x < s \leq x'$, then $\delta(s) = \kappa(s-x)$; and finally (iii) $s > x'$ implies $\delta(s) = \kappa(s-x) - \kappa(s-x')$. In either case, under the assumptions imposed on κ, the function δ is increasing in s.

[42] As in the case of Proposition 7, statement (ii) in Proposition 8 is equivalent to $f(x,t)$ having single crossing differences in (x,t), for all supermodular functions g that are increasing in s.

To better understand condition (C), which may seem opaque initially, notice that it captures the change in the function c that leads to an upward revision in the agent's belief about the state. To be specific, suppose that λ_* and λ'_* are distributions that minimize $\int_S g(x,s)d\lambda(s) + c(\lambda,t)$ and $\int_S g(x,s)d\lambda(s) + c(\lambda,t')$, respectively, with $t' \geq_T t$. (C) guarantees that there are distributions μ_* and μ'_* such that $\lambda'_* \succeq \mu_*$, $\mu'_* \succeq \lambda_*$, and

$$\int_S g(x,s)d\lambda_*(s) + \int_S g(x,s)d\lambda'_*(s) + c(\lambda_*,t) + c(\lambda'_*,t') \;\geq$$
$$\int_S g(x,s)d\mu_*(s) + \int_S g(x,s)d\mu'_*(s) + c(\mu_*,t) + c(\mu'_*,t').$$

Thus, μ_* also minimizes $\int_S g(x,s)d\lambda(s)+c(\lambda,t)$ and μ'_* minimizes $\int_S g(x,s)d\lambda(s)+c(\lambda,t')$. In other words, as t increases the distribution the agent uses to evaluate the utility of an action x shifts up in the sense of first order stochastic dominance (from λ_* to μ'_*).

The proof of Proposition 8 is in the Appendix. Note that (C) can be thought of as a generalization of the $\mathcal{P}_{\text{FSD}}$-increasing property imposed on $\Lambda : T \to \triangle_S$. Indeed, given Λ, let $c(\lambda,t) = 0$ if $\lambda \in \Lambda(t)$, and ∞ otherwise. Then c obeys (C) if, and only if, Λ is $\mathcal{P}_{\text{FSD}}$-increasing (while (13) reduces to the maxmin form (7) in this case). Below are two more examples of cost functions that satisfy property (C).[43]

Example 18 (Submodular cost and decreasing differences)**.** Let $c : \triangle_S \times T \to \mathbb{R}_+$ be a submodular function of λ that has decreasing differences in (λ,t). Then, for all λ, $\lambda' \in \triangle_S$ and t, $t' \in T$ with $t' \geq_T t$, we have

$$c(\lambda',t) - c(\lambda' \wedge \lambda, t) \;\geq\; c(\lambda' \vee \lambda, t) - c(\lambda,t) \;\geq\; c(\lambda' \vee \lambda, t') - c(\lambda,t')$$

and condition (C) holds, if we choose $\mu = \lambda \wedge \lambda'$ and $\mu' = \lambda \vee \lambda'$.

An important sub-class of variational preferences are *multiplier preferences*, which were used in Sargent and Hansen (2001) and axiomatized by Strzalecki (2011a). In this case, the cost function is $c(\lambda,t) = \theta R\big(\lambda \| \lambda^*(\cdot,t)\big)$, for $\theta \geq 0$ and $\lambda^*(\cdot,t) \in \triangle_S$, where

$$R\big(\lambda \| \lambda^*(\cdot,t)\big) \;:=\; \int_S \ln\left(\frac{d\lambda(s)}{d\lambda^*(s,t)}\right) d\lambda(s)$$

is the *relative entropy*.[44] Note that $d\lambda(s)$, $d\lambda^*(s,t)$ denote the probability of state s in the distribution λ, $\lambda^*(\cdot,t)$, respectively. This representation can be interpreted in the

[43] Note that (C) restricts how $c(\lambda,t)$ varies jointly with λ and t; for a *fixed* t, it has no content.

[44] See Strzalecki (2011b) for a detailed discussion on the relation between variational preferences, multiplier preference, and subjective expected utility.

following manner. The decision maker has a belief over the states of the world given by a *reference* or *benchmark* distribution $\lambda^*(\cdot, t)$, but she is not completely confident that she is exactly correct. To accommodate this concern, the decision maker takes all distributions in $\triangle_S$ into account when evaluating her utility from a given action, though distributions further away from $\lambda^*(\cdot, t)$ cost more and are thus less likely to be the distribution that solves the minimization problem in (13).

We show in the Appendix that *for multiplier preferences, c is a submodular function of λ. Furthermore, c has decreasing differences in (λ, t) if $\lambda^*(\cdot, t)$ is increasing in t with respect to the monotone likelihood ratio* (MLR).[45] For many commonly used distributions (such as the normal, lognormal, or exponential distributions) the MLR condition is satisfied if t is the mean of the distribution. In other words, for reference distributions drawn from one of these classes, an increase in its mean is sufficient to guarantee an increase in the optimal choice of the action x.

Example 19. Suppose that $\tilde{c} : \mathbb{R} \times T \to \mathbb{R}$ has decreasing differences in (m, t) and the cost function $c : \triangle_S \times T \to \mathbb{R}$ is evaluated by $c(\lambda, t) := \tilde{c}\left(\int_S h(s) d\lambda(s), t\right)$ for some increasing function $h : S \to \mathbb{R}$. In other words, the cost function depends only on the mean of the random variable h with respect to the distribution λ, and the parameter t. We claim that c satisfies (C). Let $t' \geq_T t$; take any λ, λ' in $\triangle_S$ and denote the mean of function h corresponding to each distribution by m, m', respectively. Suppose that $m' \geq m$; then there are distributions μ, μ' with means m, m', respectively, such that $\lambda' \succeq \mu$, $\mu' \succeq \lambda$, and $(1/2)\lambda + (1/2)\lambda' = (1/2)\mu + (1/2)\mu'$.[46] Since $c(\lambda, t) = c(\mu, t)$ and $c(\lambda', t') = c(\mu', t')$, we obtain (as required) $c(\lambda, t) + c(\lambda', t') = c(\mu, t) + c(\mu', t')$. If $m' < m$, then choose $\mu = \lambda'$ and $\mu' = \lambda$; since $\tilde{c}$ has decreasing differences in (m, t) we obtain

$$c(\lambda, t) + c(\lambda', t') \;=\; \tilde{c}(m, t) + \tilde{c}(m', t') \;\geq\; \tilde{c}(m', t) + \tilde{c}(m, t') \;=\; c(\mu, t) + c(\mu', t').$$

In Examples 15, 16, and 17, we gave economic applications of Proposition 7, which assumes that the agent has maxmin utility. It is clear that, by appealing to Proposition 8, the conclusions in those examples will continue to hold, mutatis mutandi, if the agent has variational or multiplier preferences.

[45] This requires that, for any $t' \geq t$, the ratio $d\lambda^*(s, t')/d\lambda^*(s, t)$ be increasing with s. This property implies $\lambda^*(\cdot, t') \succeq \lambda^*(\cdot, t)$, hence, it is stronger than the first order stochastic dominance.

[46] For a proof of this claim, see the proof of Example 13 in the Appendix.

Appendix

Proofs for Example 1 Let $d\bar{x}_2/dx_1$ be decreasing in $\tilde{t}$. Take $t' \geq_T t$ and $x \in \Gamma(t), x' \in \Gamma(t')$. Let $\bar{x}_2(x_1, t) = x_2$ and $\bar{x}_2(x'_1, t') = x'_2$. Γ is $\mathcal{P}$-increasing if we can find $y \in \Gamma(t)$, $y' \in \Gamma(t')$ such that $y' \geq x$, $x' \geq y$, and $x + x' = y + y'$. If $x'_1 \geq x_1$, then $x'_2 = \bar{x}_2(x'_1, t') \geq \bar{x}_2(x_1, t) = x_2$, and we can choose $y' = x'$ and $y = x$. If $x'_1 < x_1$, let y be given by $y_1 = x'_1$ and $y_2 = \bar{x}_2(x'_1, t) \leq \bar{x}_2(x'_1, t') = x'_2$. Therefore, $x' \geq y$ and $y \in \Gamma(t)$. Set $y' = x + x' - y$. Since $d\bar{x}_2/dx_1$ decreases in $\tilde{t}$, we obtain $\bar{x}_2(x_1, t) - \bar{x}_2(x'_1, t) \geq \bar{x}_2(x_1, t') - \bar{x}_2(x'_1, t')$, which implies that $y'_2 \geq \bar{x}_2(x_1, t') = \bar{x}_2(y'_1, t')$, and so $y' \in \Gamma(t')$.[47]

If the function $x_1 \to \bar{x}_2(x_1, \tilde{t})$ is C^1 and convex (in x_1), then the converse is also true. Otherwise, there is $t' \geq_T t$ and z_1 such that $d\bar{x}_2/dx_1(z_1, t) < d\bar{x}_2/dx_1(z_1, t')$; then, since $\bar{x}_2$ is C^1, there is $z'_1 < z_1$ such that $d\bar{x}_2/dx_1(z_1, t) < d\bar{x}_2/dx_1(z'_1, t')$. By convexity of $\bar{x}_2$, $d\bar{x}_2/dx_1(v_1, t) < d\bar{z}_2/dx_1(v'_1, t')$ for any $v'_1 \geq x'_1$ and $v_1 \leq z_1$. Thus $\bar{x}_2(z_1, t) - \bar{x}_2(y_1, t) < \bar{x}_2(y'_1, t') - \bar{x}_2(z'_1, t')$, for any y_1, y'_1 such that $y'_1 \geq z_1$, $z'_1 \geq y_1$, and $z_1 + z'_1 = y_1 + y'_1$. This guarantees that there is no $y \in \Gamma(t), y' \in \Gamma(t')$ such that $y'_1 \geq z_1$, $z'_1 \geq y_1$, and $z + z' = y + y'$ and thus Γ is not $\mathcal{P}$-increasing in $K = \{1\}$. $\square$

Proof of Proposition 2 Without loss of generality, suppose that $K = \{1, 2, \ldots, n\}$, for some $n \leq \ell$. Let $\phi(t) := \{x \in \Phi(t) : x >_{lex} y, \text{ for all } y \in \Phi(t)\}$, where $>_{lex}$ denotes the lexicographic order.[48] Take any $p \in \mathbb{R}^\ell$. Since Φ is compact-valued, ϕ is well-defined and $\phi(t) \in \Phi(t)$, for all $t \in T$. We claim that $\phi_K(t') \geq \phi_K(t)$, for any $t' \geq_T t$. Since Φ is $\mathcal{P}$-increasing in K, there is $y \in \Phi(t)$, $y' \in \Phi(t')$ such that $\phi(t) - y = y' - \phi(t')$ and $y'_K \geq \phi_K(t)$, $\phi_K(t') \geq y_K$. If $\phi_K(t') \ngeq \phi_K(t)$, then $\phi_K(t) \neq y_K$, and so $\phi_K(t) >_{lex} y$. Thus, there is $j \leq n$ such that $\phi_i(t) = y_i$, for all $i \leq j$, and $\phi_j(t) > y_j$. However, $\phi_K(t) - y = y' - \phi_K(t')$, and so $y'_i = \phi_i(t')$, for all $i \leq j$, and $y'_j > \phi_j(t')$. Hence, $y' >_{lex} \phi(t')$, which contradicts the definition of $\phi(t')$. $\square$

Continuation of the proof of Theorem 2 We show that statement (iii) implies (i) by contradiction. Suppose $\operatorname{co}\Gamma$ is not $\mathcal{P}$-increasing. There is $t' \geq_T t$ and $x \in \operatorname{co}\Gamma(t)$, $x' \in \operatorname{co}\Gamma(t')$ for which there is no $y \in \operatorname{co}\Gamma(t)$, $y' \in \operatorname{co}\Gamma(t')$ satisfying $x + x' = y + y'$ and

[47] It is clear from this proof that so long as $d\bar{x}_2/dx_1$ is decreasing in $\tilde{t}$, then Γ is $\mathcal{P}$-increasing in $\{1\}$. The further assumption that $\bar{x}_2$ is increasing in (x_1, t) guarantees that Γ is $\mathcal{P}$-increasing in $\{1, 2\}$.

[48] By definition, $x >_{lex} y$ if $x_i = y_i$, for all $i \leq j$, and $x_j > x_j$, for some $j \leq \ell$.

$x'_K \geq y_K$, $y'_K \geq x_K$. Take any such x, x' and define

$$\begin{aligned} C &:= \Big\{(x-y', x'-y) \in \mathbb{R}^\ell \times \mathbb{R}^\ell : y \in \operatorname{co}\Gamma(t) \text{ and } y' \in \operatorname{co}\Gamma(t')\Big\} \text{ and} \\ D &:= \Big\{(d, d') \in \mathbb{R}^\ell \times \mathbb{R}^\ell : d + d' = 0 \text{ and } d'_K \geq 0\Big\}. \end{aligned}$$

Clearly, both sets are closed, convex, and $C \cap D = \emptyset$. Moreover, since C is compact, one can show that the difference $D - C$ is closed.[49] By the strong separating hyperplane theorem, there are non-zero vectors $p, p' \in \mathbb{R}^\ell$ and a number b that satisfy

$$\sup\Big\{p \cdot c + p' \cdot c' : (c, c') \in C\Big\} \;<\; b \;<\; \inf\Big\{p \cdot d + p' \cdot d' : (d, d') \in D\Big\}. \tag{A1}$$

Since $(0,0) \in D$, we have $b < 0$. Let $\epsilon_i \in \mathbb{R}^\ell_+$ be the vector with the i'th entry equal to 1 and zeros elsewhere, for $i = 1, 2, \ldots, \ell$. Given that $\alpha(-\epsilon_i, \epsilon_i) \in D$, for all numbers $\alpha \geq 0$ and $i \in K$, we have $p'_K \geq p_K$. Since $\alpha(-\epsilon_i, \epsilon_i)$ belongs to D, for all α and $i \notin K$, it must be that $p'_{-K} = p_{-K}$. The first inequality in (A1) gives $p \cdot x + p' \cdot x' < b + p \cdot y' + p' \cdot y$, for all $y' \in \Gamma(t')$ and $y \in \Gamma(t)$. Therefore, we obtain

$$\begin{aligned} f(p,t) + f(p',t') \;&\leq\; p \cdot x + p' \cdot x' \\ &<\; \min\Big\{p \cdot z : z \in \Gamma(t')\Big\} + \min\Big\{p' \cdot z : z \in \Gamma(t)\Big\} \;=\; f(p,t') + f(p',t), \end{aligned}$$

which contradicts our assumption that $f(p,t)$ has increasing differences in (p_K, t). □

Proof of Theorem 3 Implication (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) follows directly from Theorem 2. It suffices to show that (iv) ⇒ (i). Suppose that this property is violated for some $x \in \operatorname{co}\Gamma(t)$, $x' \in \operatorname{co}\Gamma(t')$. Define the nonempty closed sets C and D as in the the proof of Theorem 2. Since $\mathbf{A}\Gamma(t) = \mathbb{R}^\ell_+$ and $\Gamma(t)$ is upward comprehensive, we have $\mathbf{A}\big(\operatorname{co}\Gamma(t)\big) = \mathbb{R}^\ell_+$.[50] Thus, we have $\mathbf{A}C = \mathbb{R}^{2\ell}_-$. Since $\mathbf{A}D = D$ and any nonzero element of D must have entries with strictly different signs, it must be that $\mathbf{A}C \cap \mathbf{A}D = \{0\}$; this suffices for $D - C$ to be closed (see Border, 1985, Proposition 2.38). By the strong separating hyperplane theorem, there are non-zero vectors $p, p' \in \mathbb{R}^\ell$ and a number $b > 0$ that satisfy (A1), and $p'_K \geq p_K$, $p_K = p'_K$, as in the proof of Theorem 2. Finally, since C is downward comprehensive, $p, p' \geq 0$.

[49] We denote $(D - C) := \big\{d - c : d \in D \text{ and } c \in C\big\}$.

[50] For the proof of this claim, see Proposition S.1 in Section S.2 of the Online Supplement.

Let $\tilde{p} := (p+\delta\mathbf{1})$ and $\tilde{p}' := (p'+\delta\mathbf{1})$, where $\delta > 0$ and $\mathbf{1} \in \mathbb{R}^\ell$ is the unit vector. Clearly, $\tilde{p}, \tilde{p}' \in \mathbb{R}^\ell_{++}$, $\tilde{p}'_K \geq \tilde{p}_K$, and $\tilde{p}'_{-K} = \tilde{p}_{-K}$. Since $d+d' = 0$, we obtain $\tilde{p}\cdot d+\tilde{p}'\cdot d' = p\cdot d+p'\cdot d'$ for any $(d, d') \in D$ and $\delta > 0$. Thus, $\inf\{p \cdot d + p' \cdot d' : (d, d') \in D\} = \inf\{\tilde{p} \cdot d + \tilde{p}' \cdot d' : (d, d') \in D\}$. Note that $\mathbf{1} \cdot y$ is uniformly bounded below over $\Gamma(t)$ and $\Gamma(t')$, since both $\Phi(t)$ and $\Phi(t')$ are nonempty for $p = \mathbf{1}$. Thus, $\sup\{\tilde{p}\cdot c+\tilde{p}'\cdot c' : (c, c') \in C\}$ is arbitrarily close to $\sup\{p\cdot c+p'\cdot c' : (c, c') \in C\}$ for an arbitrarily small $\delta > 0$, and we can guarantee that the former term (like the latter) is strictly lower than b. We conclude that (A1) still holds, even with $\tilde{p}, \tilde{p}' \in \mathbb{R}^\ell_{++}$ taking the place of p and p'. Re-tracing the proof that (iv) $\Rightarrow$ (i) in Theorem 2, we obtain $f(\tilde{p}, t) + f(\tilde{p}', t') < f(\tilde{p}, t') + f(\tilde{p}', t)$, contradicting the assumption that f has increasing differences in (p_K, t). $\square$

Proof of Proposition 3 The equivalence of statements (i), (ii), and (iii) follows from Theorem 3. That (iii) implies (iv) follows from Topkis (1978). It remains to show that (iv) implies (iii). Suppose (iii) fails and there is $p''_i \geq p'_i$ and $q'' \geq q'$ such that $C\big((p''_i, p_{-i}), q''\big) - C\big((p'_i, p_{-i}), q''\big) < C\big((p''_i, p_{-i}), q'\big) - C\big((p'_i, p_{-i}), q'\big)$, for some p_{-i}. Let $R(q) := C\big((p''_i, p_{-i}), q\big)$, for all $q < q''$, and $R(q) := C\big((p''_i, p_{-i}), q''\big)$, for all $q \geq q''$. Since $C\big((p''_i, p_{-i}), q\big)$ is increasing in q, at price (p''_i, p_{-i}) the firm is maximizing profit (which equals zero) at $q = q''$. However, the profit is *not* maximized at any $q \geq q''$ when (p'_i, p_{-i}), since $R(q') - C\big((p'_i, p_{-i}), q'\big) > R(q'') - C\big((p'_i, p_{-i}), q''\big) \geq R(q) - C(p, q)$, for any $q \geq q'$, since R is constant for $q \geq q''$ and C is increasing in q. $\square$

Proof of Example 7 Take any $q' \geq q$ and $x, x' \in X$ satisfying $F(x) \geq q$, $F(x') \geq q'$. If $F(x) \geq F(x')$, set $y := x'$, $y' := x$, which trivially satisfy the required condition. Consider the case where $F(x') > F(x)$. Take any vectors $a^j, a^{j\prime} \in X$, $j = 1, \ldots, n$, such that $x \geq \sum_{j=1}^n a^j$, $x' \geq \sum_{j=1}^n a^{j\prime}$ and $F(x) = G\big(f^1(a^1), \ldots, f^n(a^n)\big)$, $F(x') = G\big(f^1(a^{1\prime}), \ldots, f^n(a^{n\prime})\big)$. Suppose $f^j(a^{j\prime}) \geq f^j(a^j)$, for all $j = 1, \ldots, n$. Since f^j is $\mathcal{P}$-increasing, there is $b^j, b^{j\prime} \in X$ such that $f^j(b^j) \geq f^j(a^j)$, $f^j(b^{j\prime}) \geq f^j(a^{j\prime})$, as well as $a^j + a^{j\prime} = b^j + b^{j\prime}$ and $a^{j\prime}_K \geq b^j_K$, for all $j = 1, \ldots, n$. Define $y := \sum_{j=1}^n b^j$ that satisfies $x'_K \geq \sum_{j=1}^n a^{j\prime}_K \geq \sum_{j=1}^n b^j_K = y_K$. Since G is an increasing function,

$$G\Big(f^1(b^1), f^2(b^2), \ldots, f^n(b^n)\Big) \;\geq\; G\Big(f^1(a^1), f^2(a^2), \ldots, f^n(a^n)\Big) \;=\; F(x) \;\geq\; q.$$

which implies $y \in U(q)$. Let $y' := x + x' - y$. Since $x'_K \geq y_K$, we obtain $y'_K \geq x_K$. Furthermore, $y' \geq \sum_{j=1}^n b^{j\prime}$, which guarantees that $y' \in U(q')$.

We turn to the second case where $f^j(a^{j\prime}) < f^j(a^j)$, for some j. Denote the set of all such indices by L, and let M be its complement. Given that the function G is increasing and $F(x') > F(x)$, the set M is non-empty. Let $v := \big(f^j(a^j)\big)_{j=1}^n$ and $v' := \big(f^j(a^{j\prime})\big)_{j=1}^n$; v and v' are unordered since M and L are both nonempty. Since G is increasing in the $\mathcal{C}$-flexible set order, there is $\lambda \in [0,1]$ such that $G\big(\lambda v' + (1-\lambda)(v \wedge v')\big) \geq q$ and $G\big(\lambda v + (1-\lambda)(v \vee v')\big) \geq q'$. Let $\tilde{v} := \lambda v' + (1-\lambda)(v \wedge v')$ and $\tilde{v}' := \lambda v + (1-\lambda)(v \vee v')$. Note that $f^j(a^j) = v_j = \tilde{v}'_j$ and $f^j(a^{j\prime}) = v'_j = \tilde{v}_j$, for all $j \in L$.

For each $j \in L$, let $\tilde{b}^j := a^{j\prime}$ and $\tilde{b}^{j\prime} := a^j$, so $f^j(\tilde{b}^j) = \tilde{v}_j$ and $f^j(\tilde{b}^j) = \tilde{v}'_j$. For each $j \in M$, set $\tilde{b}^j := \big[\lambda a^{j\prime} + (1-\lambda) b^j\big]$ and $\tilde{b}^{j\prime} := \big[\lambda a^j + (1-\lambda) b^{j\prime}\big]$, where b^j, $b^{j\prime}$ are chosen as in the first case. Since f^j is concave, $f^j\big(\tilde{b}^j\big) \geq \tilde{v}_j$ and $f^j\big(\tilde{b}^{j\prime}\big) \geq \tilde{v}'_j$. Furthermore, $a^j + a^{j\prime} = \tilde{b}^j + \tilde{b}^{j\prime}$ and $a^{j\prime}_K \geq \tilde{b}^j_K$. Define $y := \sum_{j=1}^n \tilde{b}^j$, and note that $x'_K \geq \sum_{j=1}^n a^{j\prime}_K \geq \sum_{j=1}^n \tilde{b}^j_K = y_K$. Since G is monotone, $G\big(f^1(\tilde{b}^1), \ldots, f^n(\tilde{b}^n)\big) \geq G(\tilde{v}) \geq q$, and so $y \in U(q)$. Let $y' := x + x' - y$. Since $x'_K \geq y_K$, we obtain $y'_K \geq x_K$. Furthermore, $y' \geq \sum_{j=1}^n b^{j\prime}$, $G\big(f^1(\tilde{b}^{1\prime}), \ldots, f^n(\tilde{b}^{n\prime})\big) \geq G(\tilde{v}') \geq q'$, and so $y' \in U(q')$. $\square$

Proof of Proposition 6 Define the correspondence $\Gamma^K : \mathbb{R}^K_- \to \mathbb{R}^{\ell+1}$ by

$$\Gamma^K(t) \;:=\; \Big\{(y,v) \in \mathbb{R}^\ell \times \mathbb{R} : (y,v) \geq \big(x, -F(x) - t \cdot x_K\big),\; \text{for some } x \in X\Big\}.$$

Mimicking the argument we made in the main part of the paper concerning Γ^j, we may conclude that the factors in K are joint complements if, and only if, Γ^K is $\mathcal{P}$-increasing in K. We first show that Γ^K has this property if F is super*modular in K.

Let $t' > t$, $\big(x, -F(x) - t \cdot x_K\big) \in \Gamma(t)$, and $\big(x', -F(x') - t' \cdot x'_K\big) \in \Gamma(t')$, for $x, x' \in X$. By super*modularity, there is $y, y' \in X$ such that $(x \wedge x')_K \geq y_K$, $y'_K \geq (x \vee x')_K$, $x + x' = y + y'$, and $F(x) + F(x') \leq F(y) + F(y')$. Then

$$F(x) - F(y) + t \cdot (x_K - y_K) \;\leq\; F(x) - F(y) + t' \cdot (x_K - y_K) \;\leq\; F(y') - F(x') + t' \cdot (y'_K - x'_K)$$

and so $-F(y) - t \cdot y_K - F(y') - t' \cdot y'_K \leq -F(x) - t \cdot x_K - F(x') - t' \cdot x'_K$. Thus we can choose $w \geq -F(y) - t \cdot y_K$, $w' \geq -F(y') - t' \cdot y'_K$ such that $w + w' = -F(x) - t \cdot x_K - F(x') - t' \cdot x'_K$. Since $(y, w) \in \Gamma(t)$ and $(y', w') \in \Gamma(t')$, this proves that Γ is $\mathcal{P}$-increasing in K.

To prove the converse, let $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ be a continuous, increasing and concave function and suppose that F fails super*modularity at $x, x' \in \mathbb{R}^\ell_+$. Let

$$Z \;:=\; \Big\{ z \in \mathbb{R}^{|K|} \times \mathbb{R} : z_K \leq (x_K - y_K), z_{|K|+1} \leq F(y) + F(y') - F(x) - F(x'), \\ \text{for some } y, y' \in X \text{ such that } x'_K \geq y_K \text{ and } y + y' \leq x + x' \Big\}.$$

Z is nonempty (since $(x_K - x'_K, 0) \in Z$), convex (because F is concave), closed (because F is continuous and X is closed and bounded from below), and downward comprehensive (by construction). Furthermore, we have $0 \notin Z$. Otherwise, there is y and y' such that $y_K \leq (x \wedge x')_K$, $y + y' \leq x + x'$ and $F(x) + F(x') \leq F(y) + F(y')$. Since F is increasing we can always find $y'' \geq y'$ such that $x + x' = y + y''$ and $F(x) + F(x') \leq F(y) + F(y'')$, contradicting our assumption about x and x'. By the strong separating hyperplane theorem, there is a vector $p > 0$ and a number b such that $p \cdot z < b < 0$, for all $z \in Z$, where $p_{|K|+1} > 0$.[51] Without loss of generality, let $p_{|K|+1} = 1$ and $t = -p_K$, $t' = 0$. Thus, there is $t' \geq t$ such that, for any $y, y' \in X$ satisfying $x'_K \geq y_K$ and $y + y' \leq x + x'$,

$$\big[-F(x) - t \cdot x_K\big] + \big[-F(x') - t' \cdot x'_K\big] \;<\; \big[-F(y) - t \cdot y_K\big] + \big[-F(y') - t' \cdot y'_K\big],$$

which is incompatible with Γ^K being $\mathcal{P}$-increasing in K. □

Continuation of Example 11 Take any $x, x' \in X$ and denote $t_j = g_j(x_{I_j})$, $t'_j = g_j(x'_{I_j})$, for all $j = 1, \ldots, n$. Since g_j is $\mathcal{P}$-increasing in $\{i_j\}$, there is some $y_{I_j}, y'_{I_j} \in X_j$, such that $g(y'_{I_j}) \geq t_j \vee t'_j$, $g(y_{I_j}) \geq t_j \wedge t'_j$, $(x \wedge x')_{i_j} \geq y_{i_j}$ and $x_{I_j} + x'_{I_j} = y_{I_j} + y'_{I_j}$. Let $y = (y_{I_j})_{j=1}^n$ and $y' = (y'_{I_j})_{j=1}^n$. Clearly, $(x \wedge x')_{i_j} \geq y_{i_j}$, for all j, and $x + x' = y + y'$. Since A is supermodular and increasing,

$$\begin{aligned} F(x) + F(x') \;&=\; A(t) + A(t') \;\leq\; A(t \wedge t') + A(t \vee t') \\ &\leq\; A\big(g_1(y_{I_1}), \ldots, g_n(y_{I_n})\big) + A\big(g_1(y'_{I_1}), \ldots, g_n(y'_{I_n})\big) \;=\; F(y) + F(y'). \end{aligned}$$

Thus F is super*modular in $K = \{i_1, \ldots, i_n\}$. □

Proof of Remark 5.1 Suppose $S = [a, b]$. Let $\{s^n_i\}_{i=0}^n$ be a sequence with $n+1$ terms such that $a = s^n_0 < s^n_1 < \ldots < s^n_{n-1} < s^n_n = b$. Since at each (x, t), function $g(x, \cdot)$ is the

[51] Suppose that $p_{|K|+1} = 0$. Since X is lattice-like in K, there is y, y' such that $(x \wedge x')_K \geq y_K$ and $x + x' = y + y'$. Thus, there is $\tilde{z} \in Z$ such that $\tilde{z}_K \geq 0$, which leads to $p \cdot \tilde{z} \geq 0$, yielding a contradiction.

Riemann-Stieltjes integrable with respect to $\lambda \in \Lambda(t)$, we can choose $\{s_i^n\}_{i=0}^n$ so that

$$\int_S g(x,s)d\lambda(s) \;=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} g(x,s_{i+1})\big[\lambda(s_{i+1}) - \lambda(s_i)\big]$$

for all $\lambda \in \Lambda(t)$. This guarantees that $\lim_{n\to\infty} f_n(x,t) = f(x,t)$ for all (x,t), where

$$f_n(x,t) \;:=\; \min\left\{\sum_{i=0}^{n-1} g(x,s_{i+1}^n)\big[\lambda(s_{i+1}^n) - \lambda(s_i^n)\big] : \lambda \in \Lambda(t)\right\}.$$

We know, from the case where S is finite, that $f_n : X \times T \to \mathbb{R}$ is a supermodular function. Since supermodularity is preserved by pointwise convergence, f is supermodular. $\square$

Continuation of Example 13 Take any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$. Since $\int_S h(s)d(\lambda \wedge \lambda')(s) \leq \int_S h(s)d\lambda'(s) = t$ and $\int_S h(s)d\lambda'(s) = t'$, there is $\alpha \in [0,1]$ such that

$$\alpha \int_S h(s)d\lambda'(s) + (1-\alpha)\int_S h(s)d(\lambda \wedge \lambda')(s) \;=\; t.$$

Let $\mu = \alpha\lambda' + (1-\alpha)(\lambda \wedge \lambda')$ and $\mu' = \alpha\lambda + (1-\alpha)(\lambda \vee \lambda')$. Clearly, $\mu \in \Lambda(t)$, $\lambda' \succeq \mu$, and $\lambda \succeq \mu'$. Since $\lambda + \lambda' = (\lambda \vee \lambda') + (\lambda \wedge \lambda')$, we also obtain $\lambda + \lambda' = \mu + \mu'$. Hence,

$$\int_S h(s)d\mu'(s) \;=\; \int_S h(s)d\lambda(s) + \int_S h(s)d\lambda'(s) - \int_S h(s)d\mu(s) \;=\; t + t' - t \;=\; t'.$$

Thus $\mu' \in \Lambda(t')$. We conclude that $\Lambda(t')$ dominates $\Lambda(t)$ in the $\mathcal{C}$-flexible set order. $\square$

Proof of Proposition 8 Let $t'' >_T t'$ and let M satisfy $M > \max\big\{c(\lambda,t'), c(\lambda,t'')\big\}$, for all $\lambda \in \triangle_S$. The correspondence $\Gamma : T' \to \mathbb{R}^{\ell+1}$, where $T' = \{t', t''\}$, is defined by

$$\Gamma(t) \;:=\; \Big\{y \in \mathbb{R}^{\ell+1} : y_i = -\lambda(s_i),\ \text{for } i = 1,2,\ldots,\ell,\ \text{and } y_{\ell+1} \in \big[c(\lambda,t), M\big] \text{ for } \lambda \in \triangle_S\Big\}.$$

Since c is convex, Γ is convex-valued and it is straightforward to check that Γ is $\mathcal{P}$-increasing if, and only if, c obeys (C). Indeed, if c obeys (C), define $p', p \in \mathbb{R}^\ell$ by $p'_i = g(x', s_{i+1}) - g(x', s_i)$ and $p_i = g(x, s_{i+1}) - g(x, s_i)$, for $i = 1, \ldots, \ell$, for some supermodular function g. By supermodularity of g, we have $p' \geq p$ when $x' \geq x$. By Theorem 2 and the integration formula (9), for any $x' \geq x$ and $t' \geq_T t$, we obtain

$$\begin{aligned} f(x,t') - f(x,t) \;&=\; \min\Big\{(p,1)\cdot y : y \in \Gamma(t')\Big\} - \min\Big\{(p,1)\cdot y : y \in \Gamma(t)\Big\} \\ &\leq\; \min\Big\{(p',1)\cdot y : y \in \Gamma(t')\Big\} - \min\Big\{(p',1)\cdot y : y \in \Gamma(t)\Big\} \;=\; f(x',t') - f(x',t). \end{aligned}$$

We prove the converse by contradiction. Suppose c violates (C) and so Γ is not $\mathcal{P}$-increasing. By Theorem 2, the function $\tilde{f} : \mathbb{R}^{\ell+1} \times T' \to \mathbb{R}$, where $\tilde{f}\big((\tilde{p}, q), t\big) := \min\big\{(\tilde{p}, q) \cdot y \,:\, y \in \Gamma(t)\big\}$, must violate increasing differences in $(\tilde{p}, t)$, i.e., there is $p, p' \in \mathbb{R}^\ell$, $t, t' \in T$, and $q \in \mathbb{R}$ such that $p' \geq p$, $t' >_T t$ and

$$\tilde{f}\big((p', q), t\big) - \tilde{f}\big((p, q), t\big) \;>\; \tilde{f}\big((p', q), t'\big) - \tilde{f}\big((p', q), t'\big). \tag{A2}$$

If $q \leq 0$, then $\tilde{f}\big((p, q), t\big) = \tilde{f}\big((p, q), t'\big)$ and $\tilde{f}((p', q), t) = \tilde{f}((p', q), t')$, so we only need to consider $q > 0$. Given this, we can assume with no loss of generality that $q = 1$, so that $\tilde{f}\big((\tilde{p}, 1), t\big) = \min\Big\{\sum_{i=1}^{\ell} \tilde{p}_i\big[-\lambda(s_i)\big] + c(\lambda, t) : \lambda \in \Delta_S\Big\}$. Define the function $g : X \times S \to \mathbb{R}$ as in the proof of Proposition 7. Using (9), we obtain $f(x', t) - f(x, t) = \tilde{f}\big((p', 1), t\big) - \tilde{f}\big((p, 1), t\big)$ and $\tilde{f}\big((p', 1), t'\big) - \tilde{f}\big((p, 1), t'\big) = f(x', t') - f(x, t')$, in which case (A2) implies that f violates increasing differences. $\square$

Continuation of Example 18 We first show that $R\big(\lambda \| \lambda^*(\cdot, t)\big)$ is submodular in λ. Let λ, $\lambda' \in \triangle_S$ and denote $\mu' = \lambda \vee \lambda'$ and $\mu = \lambda \wedge \lambda'$. To abbreviate the notation, let $p_i^*(t)$, p_i, p_i', q_i, q_i' be the probability of state s_i, for all $i = 1, 2, \ldots, (\ell+1)$, corresponding to the cumulative distribution of $\lambda^*(t)$, λ, λ', μ, and μ', respectively. $R\big(\lambda \| \lambda^*(\cdot, t)\big)$ is submodular in λ if, for all i, $c(p_i) + c(p_i') \geq c(q_i) + c(q_i')$, where $c(x) = x \ln x - x \ln p_i^*(t)$. Clearly, this inequality holds for $i = 1$. Consider $i > 1$. With no loss of generality, let $\mu(s_{i-1}) = \lambda(s_{i-1})$ and $\mu'(s_{i-1}) = \lambda'(s_{i-1})$. Consider two cases. Assume that (i) $p_i' + \lambda'(s_{i-1}) \leq p_i + \lambda(s_{i-1})$, so that $\mu(s_i) = \lambda(s_i)$ and $\mu'(s_i) = \lambda'(s_i)$. Then $q_i = p_i$ and $q_i' = p_i'$ and $c(p_i) + c(p_i') = c(q_i) + c(q_i')$ holds. Suppose, instead, that (ii) $p_i' + \lambda'(s_{i-1}) > p_i + \lambda(s_{i-1})$, which implies $\mu(s_i) = \lambda'(s_i)$ and $\mu'(s_i) = \lambda(s_i)$. Let $\delta = \lambda(s_{i-1}) - \lambda'(s_{i-1})$ and notice that $0 \leq \delta < p_i' - p_i$. Since $q_i = p_i' - \delta$ and $q_i' = p_i + \delta$, and c is convex, $c(q_i) + c(q_i') \leq c(p_i) + c(p_i')$.

To show that $R\big(\lambda \| \lambda^*(\cdot, t)\big)$ has decreasing differences in (λ, t), take any distribution $\lambda' \succeq \lambda$, $t' \geq t$, and notice that

$$\Big[R\big(\lambda' \| \lambda^*(\cdot, t')\big) - R\big(\lambda \| \lambda^*(\cdot, t')\big)\Big] - \Big[R\big(\lambda' \| \lambda^*(\cdot, t)\big) - R\big(\lambda \| \lambda^*(\cdot, t)\big)\Big]$$
$$= \sum_{i=1}^{\ell} \big[\ln p_i^*(t') - \ln p_i^*(t)\big]\big[p_i - p_i'\big].$$

This is nonpositive since $\ln p_i^*(t') - \ln p_i^*(t)$ is increasing in i (because $\lambda^*(t)$ is increasing in t with respect to the monotone likelihood ratio order) and $\lambda' \succeq \lambda$. $\square$

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Supplement to: "Comparative statics with linear objectives: normality, complementarities, and ranking multi-prior beliefs"

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Abstract

This supplement contains additional results related to Dziewulski and Quah (2022). These notes should be read in conjunction with the main paper.

In this supplement, we present proofs of the claims made in the article. In addition, in Section S.9 we apply the results from the main paper to dynamic programming under ambiguity. Throughout, we employ the notation introduced in the main body.

S.1 Anti-symmetry of the parallelogram order

As argued in Remark 2.3 of the main paper, the parallelogram order is transitive and reflexive. In this section we show that it is also anti-symmetric within the class of compact and convex subsets of $\mathbb{R}^\ell$. That is, for any compact and convex sets A, $A' \subseteq \mathbb{R}^\ell$, if A' dominates A, and A dominates A' by the parallelogram order for $K = \{1, \ldots, \ell\}$, then the two sets are equal. First, we state one auxiliary result. We say that $x \in A$ is an *extreme point* of A if it is not a convex combination of any other points in A.

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Lemma S.1. *Take any convex sets $A, A' \subseteq \mathbb{R}^\ell$ such that $A \not\subseteq A'$. Then, there is an extreme point of* $\text{co}(A \cup A')$ *that belongs to $x \in A \setminus A'$.*

Proof. Recall that a convex hull of a set consists of all convex combinations of its extreme points. Specifically, it must be that $\text{co}(A \cup A')$ consists of convex combinations of extreme points in A and A'. Towards contradiction, suppose that all such extreme points are in A'. Since A' is convex, we have $A \subseteq \text{co}(A \cup A') = \text{co}\, A' = A'$, yielding a contradiction. Therefore, there must be at least one extreme point of $\text{co}(A \cup A')$ in $A \setminus A'$. □

We continue with our main argument. Towards contradiction, suppose that A' dominates A, and A dominates A' by parallelogram order, but $A \not\subseteq A'$. By the lemma above, there is an extreme point $x \in \text{co}(A \cup A')$ such that $x \in A \setminus A'$. By Theorem 12.7 in Soltan (2015), there are vectors $p^1, \ldots, p^N$, such that $\Phi_A^n = \text{argmax}\left\{p^n \cdot y : y \in \Phi_A^{n-1}\right\}$, for all $n = 1, \ldots, N$, and $\{x\} = \Phi_A^N$, where $\Phi_A^0 = A$. Let $\Phi_{A'}^N$ be the set induced as above for A', for the same vectors $p^1, \ldots, p^N$. By successive application of Theorem 2 of the main paper, it must be that $\Phi_{A'}^N$ dominates Φ_A^N by the parallelogram order. In particular, there must be some $x' \in \Phi_{A'}^N$ such that $x' \geq x$. Similarly, the set $\Phi_A^N = \{x\}$ dominates $\Phi_{A'}^N$ by the parallelogram order. Thus, it must be that $x \geq x'$. However, the two inequalities imply that $x = x'$, which contradicts that $x \notin A$.

S.2 Asymptotic cones, convex hulls, and closed sets

In this section we discuss the relationship between asymptotic cones, convex hulls and closed sets. We prove Proposition S.1 which we use to prove that statement (iv) in Theorem 3 implies statement (i) (see the Appendix to the main paper). We also prove Proposition S.2 which provides sufficient conditions under which the optimisation problems discussed in Sections 2–4 to have a solution.

We prove Proposition S.1 through three lemmas that follow. Recall that an *asymptotic cone* $\mathbf{A}Y$ of the set $Y \subseteq \mathbb{R}^\ell$ is the set of limits of all sequences of the form $\{\lambda_n x_n\}$, for positive numbers $\lambda_n \to 0$ and $x_n \in Y$, for all n.

Lemma S.2. *The set $Y \subset \mathbb{R}^\ell$ satisfies $\mathbf{A}Y \subseteq \mathbb{R}^\ell_+$. The sequence $\{x_n\}$ is given by $x_n = \sum_{i=1}^k \alpha_n^i y_n^i$, where $\alpha_n^i \geq 0$ and $y_n^i \in Y$, for all i and n. Suppose that $\{x_n\}$ and (for every i) $\{\alpha_n^i\}$ are bounded sequences. Then, for each i, the sequence $\{\alpha_n^i y_n^i\}$ is bounded.*

Proof. Towards contradiction, suppose there is a set $I = \{1, \dots, m\}$ such that the sequence $\{\alpha_n^i y_n^i\}$ is unbounded, for all $i \in I$. Note that, the set must have at least two elements; otherwise $\{x_n\}$ would be unbounded. Similarly, the sum $\sum_{i \in I} \alpha_n^i y_n^i$ must be bounded. After taking subsequences if necessary, suppose that the sequence $\alpha_n^1 y_n^1 / L_n^1$ converges to $y^1 \neq 0$, where L_n^1 denotes the norm of $\alpha_n^1 y_n^1$. The limit y^1 must belong to $\mathbf{A}Y \subseteq \mathbb{R}_+^\ell$ since $\alpha_n^1 / L_n^1 \to 0$. Thus, the sequence

$$\frac{\sum_{i=2}^m \alpha_n^i y_n^i}{L_n^1}$$

converges to $-y^1 < 0$, since $\sum_{i \in I} \alpha_n^i y_n^i / L_n^1 \to 0$. If each term $\alpha_n^i y_n^i / L_n^1$ for $i \neq 1$ is bounded, then one of them will have a limit outside of $\mathbb{R}_+^\ell \supseteq \mathbf{A}Y$, yielding a contradiction. Alternatively, suppose that $\alpha_n^2 y_n^2 / L_n^1$ is unbounded, without loss of generality. As previously, the sequence $\alpha_n^2 x_n^2 / (L_n^1 \, L_n^2)$, where L_n^2 denotes the norm of $\alpha_n^2 y_n^2 / L_n^1$, has a limit in $\mathbf{A}Y = \mathbb{R}_+^\ell$, which implies that the sequence

$$\frac{\sum_{i=3}^m \alpha_n^i y_n^i}{L_n^1 \, L_n^2}$$

has a limit in $\mathbb{R}_-^\ell \setminus \{0\}$. If each sequence $\alpha_n^i x_n^i / (L_n^1 \, L_n^2)$ is convergent, then one of them has a limit that is not in $\mathbb{R}_+^\ell \supseteq \mathbf{A}Y$, yielding a contradiction. Otherwise, we can continue the argument which will eventually lead to a contradiction. □

The next lemma introduces a general class of sets that admit a *closed* convex hull.

Lemma S.3. *Whenever the set $Y \subseteq \mathbb{R}^\ell$ is closed, upward comprehensive, and $\mathbf{A}Y = \mathbb{R}_+^\ell$, then its convex hull $\operatorname{co} Y$ is closed.*[1]

Proof. Let $\{x_n\}$ be a sequence in $\operatorname{co} Y$ converging to x. By Carathéodory's theorem, we may assume that $x_n = \sum_{i=1}^{\ell+1} \alpha_n^i y_n^i$, for $y_n^i \in Y$, $\alpha_n^i \geq 0$, and $\sum_{i=1}^{\ell+1} \alpha_n^i = 1$, for all $i = 1, \dots, \ell + 1$ and n, without loss of generality. Moreover α_n^i converges to $\alpha^i \geq 0$, for all $i = 1, \dots, \ell + 1$. By shifting Y by a constant if necessary, we can also assume that $x = 0$. It suffices to show that $x \in \operatorname{co} Y$.

We partition the sequences of indices $i = 1, \dots, \ell + 1$ into two groups: (i) those i for which the sequence $\{y_n^i\}$ is bounded, and (ii) those i for which the sequence $\{\alpha_n^i y_n^i\}$ is

[1] Whenever $Y \subseteq \mathbb{R}^\ell$ is upward comprehensive and $\mathbf{A}Y \subseteq \mathbb{R}_+^\ell$, then $\mathbf{A}Y = \mathbb{R}_+^\ell$. Take any $y \in Y$ and $x \in \mathbb{R}_+^\ell$. Since Y is upward comprehensive, we have $(y + 1/\lambda x) \in Y$, for any $\lambda > 0$. Moreover, $\lambda(y + 1/\lambda x) \to x$ as $\lambda \to 0$. Since x was arbitrary, this proves that $\mathbb{R}_+^\ell \subseteq \mathbf{A}Y$.

bounded, but $\{y_n^i\}$ is not. Denote the two sets by I, I', respectively. By Lemma S.2, these are the only two cases that we need to consider.

For each $i \in I$, we may assume that the sequence $\{y_n^i\}$ has the limit y^i which belongs to Y (since Y closed). For each $i \in I'$, denote the limit of $\{\alpha_n^i x_n^i\}$ by z^i, which exists by assumption. In particular, it must be that $\alpha_n^i \to 0$, and so $z^i \in \mathbf{A}Y \subseteq \mathbb{R}_+^\ell$. As a result, we have $z = \sum_{i \in I'} z^i \geq 0$ and $z + \sum_{i \in I} \alpha^i y^i = x = 0$. Thus, we have $\sum_{i \in I} \alpha^i y^i = -z \leq 0$. Since we can always re-normalise the weights so that $\sum_{i \in I} \alpha^i y^i \in \operatorname{co} Y$, and since $\operatorname{co} Y$ is upward comprehensive, this suffices to show that $x = 0 \in \operatorname{co} Y$. □

Next, we establish a relationship between asymptotic cones and convex hulls.

Lemma S.4. *If $\mathbf{A}Y \subseteq \mathbb{R}_+^\ell$ then $\mathbf{A}(\operatorname{co} Y) \subseteq \mathbb{R}_+^\ell$, for any $Y \subseteq \mathbb{R}^\ell$.*

Proof. Suppose that $\lambda_n x_n \to z$, where $x_n \in \operatorname{co} Y$, for all n, and $\lambda_n \to 0$. We claim that $z \geq 0$. By Carathéodory's theorem, we may assume (without loss of generality) that $x_n = \sum_{i=1}^{\ell+1} \alpha_n^i y_n^i$, where $\alpha_n^i \geq 0$, $y_n^i \in Y$, and $\sum_{i=1}^{\ell+1} \alpha_n^i = 1$, for all $i = 1, \ldots, \ell+1$ and n. Thus, $\lambda_n \alpha_n^i \to 0$, for all $i = 1, \ldots, \ell+1$. Moreover, by Lemma S.2 and our assumption, each sequence $\{(\lambda_n \alpha_n^i) y_n^i\}$ is convergent to some $z^i \in \mathbf{A}Y \subseteq \mathbb{R}_+^\ell$, and so $\lambda_n x_n = \lambda_n \sum_{i=1}^{\ell+1} \alpha_n^i y_n^i$ converges to $z = \sum_{i=1}^{\ell+1} z^i \geq 0$. □

The next proposition follows from the previous two lemmas.

Proposition S.1. *If $Y \subseteq \mathbb{R}^\ell$ is closed, upward comprehensive, and $\mathbf{A}Y = \mathbb{R}_+^\ell$, then $\mathbf{A}(\operatorname{co} Y) = \mathbb{R}_+^\ell$.*

Proof. By Lemmas S.3, S.4, the set $\operatorname{co} Y$ is closed and $\mathbf{A}(\operatorname{co} Y) \subseteq \mathbb{R}_+^\ell$. Since $\operatorname{co} Y$ is upward comprehensive, we have $\mathbb{R}_+^\ell \subseteq \mathbf{A}(\operatorname{co} Y)$, proving the claim. □

The following proposition establishes sufficient conditions under which the minimum of any strictly positive linear functional over a set Y is well-defined. This is used extensively in Sections 2–4, where we focus on minimisation problems with linear objectives.

Proposition S.2. *Let $Y \subseteq \mathbb{R}_+^\ell$ be closed and $\mathbf{A}Y \subseteq \mathbb{R}_+^\ell$. Then, for all $p \in \mathbb{R}_{++}^\ell$, the set $\operatorname{argmin} \{p \cdot y : y \in Y\}$ is nonempty and closed, and (thus) the function $f : \mathbb{R}_{++}^\ell \to \mathbb{R}$, where $f(p) := \min \{p \cdot y : y \in Y\}$ is well-defined.*

Proof. Suppose there is some $\bar{p} \gg 0$ for which the minimization problem has no solution. Choose any $\bar{y} \in Y$ and consider the set $Y' = \{y \in Y : \bar{p} \cdot y \leq \bar{p} \cdot \bar{y}\}$. If the minimization problem has no solution then Y' is unbounded; indeed, if it is bounded then it is both closed and bounded and there will be y^* that minimizes $\bar{p} \cdot y$ in Y', and also in Y.

If Y' is unbounded, then it contains an unbounded sequence $\{y_n\}$. Let $\hat{y}_n = y_n/\|y_n\|$ have a limit given by $\hat{y} \neq 0$, which is in $\mathbf{A}Y$. Since $\bar{p} \cdot \hat{y}_n \leq \bar{p} \cdot \bar{y}/\|y_n\|$, by taking limits we obtain $\bar{p} \cdot \hat{y} \leq 0$, which is impossible since $\bar{p} \gg 0$ and $\hat{y} > 0$. □

At the beginning of Section 4 of the main paper, we claim that the profit function $\pi(p) := \max\{F(x) - p \cdot x : x \in X\}$ of the firm is well-defined for any strictly positive price p, whenever the asymptotic cone $\mathbf{A}^F$ of the production possibility set $P = \{(z, y) \in \mathbb{R}^\ell \times \mathbb{R} : (z, y) \leq (-x, F(x)),\text{ for } x \in X\}$ is $\mathbb{R}^{\ell+1}_-$. Indeed, since

$$\pi(p) \;:=\; \max\Big\{F(x) - p \cdot x : x \in X\Big\} \;=\; \max\Big\{(p, 1) \cdot (z, y) : (z, y) \in P\Big\},$$

Proposition S.2 guarantees that the function is well-defined for any $p \in \mathbb{R}^\ell_{++}$.

S.3 Continuation of Example 5

Let $X \subseteq \mathbb{R}^\ell$ be a convex lattice. In the main paper we claim that a function $F : X \to \mathbb{R}$ is increasing in the $\mathcal{C}$-flexible order for $K \subseteq \{1, \ldots, \ell\}$ if it is continuous, increasing, supermodular, and concave in x_{-i}, for all $i \in K$. This result can be found in Quah (2007); we prove it here for easy reference.

Take any $q' \geq q$ and $x, x' \in X$ such that $x'_K \not\geq x_K$ and $F(x) \geq q$, $F(x') \geq q'$. We show that there is a $\lambda \in [0, 1]$ satisfying $F\big(\lambda x' + (1-\lambda)(x \wedge x')\big) \geq q$, $F\big(\lambda x + (1-\lambda)(x \vee x')\big) \geq q'$. This suffices for F to be increasing in the $\mathcal{C}$-flexible order for K.

Consider two cases. (i) If $F(x \wedge x') \geq q$, set $\lambda = 0$. By monotonicity of F, we have $F(x \wedge x') \geq F(x') \geq q'$. Alternatively, let (ii) $F(x \wedge x') < q$. Since $q \leq q' \leq F(x')$, by continuity of F there is some $\lambda \in [0, 1]$ such that $F\big(\lambda x' + (1 - \lambda)(x \wedge x')\big) = q$. Denote $v = x' - (x \wedge x') = (x \vee x') - x$, which is a positive vector. Since $x'_K \not\geq x_K$, there is some $i \in K$ such that $v_i = 0$. In particular, we obtain

$$\begin{aligned} q' - q \;\leq\; F(x') - F\big(\lambda x' + (1 - \lambda)(x \wedge x')\big) \;&=\; F(x') - F\big(x \wedge x' + \lambda v\big) \\ &\leq\; F(x \vee x') - F\big(x + \lambda v\big) \;\leq\; F\big((x \vee x') - \lambda v\big) - F(x), \end{aligned}$$

where the second inequality follows from supermodularity of F and the third is implied by the fact that F is concave in x_{-i} and $v_i = 0$.[2] Therefore, since $F(x) \geq q$, it must be that $q' \leq F\big((x \vee x') - \lambda v\big) = F\big(\lambda x + (1-\lambda)(x \vee x')\big)$. This suffices to show that $\mathcal{C}$-flexible order is stronger that parallelogram order. $\square$

S.4 Substitutes in production

In Section 4 of the main paper (see footnote 22) we argued that our results can be applied to a study of technologies that exhibit substitutability of inputs. Here, we explore this claim. As in Section 4, let X be a non-empty and closed subset of $\mathbb{R}^\ell_+$, and $F : X \to \mathbb{R}$ be a production function such that the asymptotic cone generated by the corresponding production possibility set is $\mathbf{A}^F$. At factor prices $p \in \mathbb{R}^\ell_{++}$, the firm's (unconditional) *input/factor demand* is given by $\mathcal{H}(p) := \operatorname{argmax}\big\{F(x) - p \cdot x : x \in X\big\}$.

For any two distinct inputs $i, j = 1, \ldots, \ell$, we say that i is a *substitute* of j, if, for any $p, p' \in \mathbb{R}^\ell_{++}$ satisfying $p_{-j} = p'_{-j}$, and any $x \in \mathcal{H}(p)$, there is $x' \in \mathcal{H}(p')$ such that $x'_i \geq x_i$ if $p'_j \geq p_j$, and $x'_i \leq x_i$ if $p'_j \leq p_j$. Consider the following claim.

Proposition S.3. *Let $X \subseteq \mathbb{R}^\ell_+$ be a closed set and $F : X \to \mathbb{R}$ be a continuous function with $\mathbf{A}^F = \mathbb{R}^\ell_-$. For any distinct $i, j = 1, \ldots, \ell$, the following statements are equivalent:* (i) *factor i is a substitute of j;* (ii) *factor j is a substitute of i;* (iii) *the profit function $\pi : \mathbb{R}^\ell_{++} \to \mathbb{R}$, where $\pi(p) := \max\big\{F(x) - p \cdot x : x \in X\big\}$, is submodular in (p_i, p_j) (keeping other prices fixed).*

Proof. Analogously to our argument in Section 4 of the main paper, we answer this question by defining the correspondence Γ^j with the domain $T = \mathbb{R}_+$, by

$$\Gamma^j(t) \ := \ \Big\{(y, v) \in \mathbb{R}^{\ell+1} : (y, v) \geq \big(x, -F(x) + t x_j\big) \text{ for some } x \in X\Big\}. \tag{S.1}$$

It is straightforward to check that, for any $p \in \mathbb{R}^\ell_{++}$,

$$x \in \mathcal{H}(p_j + t, p_{-j}) \iff \big(x, -F(x) + t x_i\big) \in \operatorname{argmin}\Big\{(p, 1) \cdot y : y \in \Gamma^j(t)\Big\}$$

and thus $\pi(p_j + t, p_{-j}) = -\min\big\{(p, 1) \cdot y : y \in \Gamma^j(t)\big\}$. Theorem 3 (with $K = \{i\}$) guarantees that the following are equivalent: (i) $\operatorname{co}\Gamma^j$ is $\mathcal{P}$-increasing in $\{i\}$; (ii) i is a

[2] We are making use of the fact that when f is concave in direction v, we have $f(x - v) - f(x) \geq f(x - v - tv) - f(x - tv)$, for any $x \in X$ and scalar $t > 0$.

substitute of j; and (iii) $-\pi(p_j + t, p_{-j})$ has increasing differences in (t, p_i). Notice that condition (iii) is equivalent to π being submodular in (p_i, p_j), other prices being fixed. Since submodularity is a symmetric property, we conclude that i is a substitute of j if, and only if, j is a substitute of i, with both equivalent to submodularity of π in (p_i, p_j). $\square$

S.5 Comment on Proposition 7

In Proposition 7 we concluded that the belief correspondence Λ is $\mathcal{P}_{\text{FSD}}$-increasing if, and only if, the resulting value function $f(x,t) := \min\left\{ \int g(x,s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$ is supermodular in (x,t), for any supermodular function g. As we pointed out in footnote 28 of the main paper, the $\mathcal{P}_{\text{FSD}}$-increasing property is also necessary for the function f to satisfy a weaker conditions — *single crossing differences* — for any supermodular function g.[3] As shown in Milgrom and Shannon (1994), single crossing differences alone are sufficient for the set of maximisers of f with respect to x to be increasing in t. Below, we state the formal proof of our claim stated in footnote 28.

Suppose that $\tilde{f}(\tilde{x},t) := \min\left\{ \int_S \tilde{g}(\tilde{x},s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$ violates increasing differences, for some function $\tilde{g}$. In particular, for some $x' \geq x$ and $t' \geq t$,

$$v \;:=\; \tilde{f}(x',t) - \tilde{f}(x,t) \;>\; \tilde{f}(x',t') - \tilde{f}(x,t').$$

Define the function g by $g(y,s) = \tilde{g}(y,s)$, for $y \leq x$, and $g(y,s) = \tilde{g}(y,s) - v$ otherwise. Clearly, g is supermodular, but f given by $f(\tilde{x},t) := \min\left\{ \int_S g(\tilde{x},s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$ violates single crossing differences since $0 = f(x',t) - f(x,t) > f(x',t') - f(x,t')$. Therefore, the maxmin value function violates increasing differences for some supermodular function $\tilde{g}$ if, and only if, it violates single-crossing differences for another function. Clearly, this suffices to show that the beliefs Λ are $\mathcal{P}_{\text{FSD}}$-increasing if, and only if, the value function obeys single-crossing differences, for any supermodular function g.

S.6 Continuation of Remark 5.3

In Remark 5.3 of the main paper, we discussed an alternative generalization of first order stochastic dominance to multi-prior beliefs. Formally, we are interested in conditions on

[3] The function $g : X \times S \to \mathbb{R}$ has single crossing differences if $g(x',s') \geq (>)\, g(x,s')$ implies $g(x',s) \geq (>) g(x,s)$, for any $x' \geq x$ and $s' \geq s$, where we assume that $X, S \subseteq \mathbb{R}$.

the belief correspondence $\Lambda : T \to \triangle_S$ such that, for any increasing function $u : S \to \mathbb{R}$, the value function $v : T \to \mathbb{R}$, given by

$$v(t) \ := \ \min\left\{\int u(s)d\lambda(s) : \lambda \in \Lambda(t)\right\}, \tag{S.2}$$

is increasing in t. Below, we characterize this property.

Proposition S.4. *Suppose the correspondence $\Lambda : T \to \triangle_S$ has compact and convex values. Then, the following statements are equivalent.*

(i) *Correspondence Λ satisfies the following property:*

(F) *if $t' \geq_T t$, then for any $\lambda' \in \Lambda(t')$ there is some $\lambda \in \Lambda(t)$ such that $\lambda' \succeq \lambda$.*

(ii) *For any increasing function $u : S \to \mathbb{R}$, the function v in (S.2) increases in t.*

Proof. To show that (i) $\Rightarrow$ (ii), take any $t' \geq_T t$ and $\lambda' \in \Lambda(t')$. By (F), there is some $\lambda \in \Lambda(t)$ such that $\lambda' \succeq \lambda$. Thus, for any increasing u,

$$\int_S u(s)d\lambda'(s) \ \geq \ \int_S u(s)d\lambda(s) \ \geq \ \min\left\{\int_S u(s)d\nu(s) : \nu \in \Lambda(t)\right\}.$$

Taking the minimum over the left term gives us the result.

To show (ii) $\Rightarrow$ (i), suppose (F) fails. Then there is $t' \geq t$ and $\lambda' \in \Lambda(t')$ such that $\lambda' \not\succeq \lambda$, for all $\lambda \in \Lambda(t)$. Let $V = \left\{y \in \mathbb{R}^\ell : y_i \geq \lambda'(s_i),\ \text{for } i = 1, \ldots, \ell\right\}$. Since $V \cap \Lambda(t') = \emptyset$ and $\left(V - \Lambda(t')\right)$ is closed and convex, by the strong separating hyperplane theorem, $\min\left\{\sum_{i=1}^{\ell} \hat{p}_i y_i : y \in V\right\} > \max\left\{\sum_{i=1}^{\ell} \hat{p}_i \lambda(s_i) : \lambda \in \Lambda(t')\right\}$, for some $\hat{p} \in \mathbb{R}^\ell$. Given that V is upward comprehensive, $\hat{p} > 0$ and $\sum_{i=1}^{\ell} \hat{p}_i \lambda'(s_i) = \min\left\{\hat{p} \cdot y : y \in V\right\}$. Define $u : S \to \mathbb{R}$ by $u(s_1) = \hat{p}_1$ and $u(s_{i+1}) = u(s_i) + \hat{p}_{i+1}$, for $i = 1, \ldots, \ell$, which is an increasing function. Since $\int_S u(s)d\mu(s) = u(s_{\ell+1}) - \sum_{i=1}^{\ell} \hat{p}_i \mu(s_i)$, for any $\mu \in \triangle_S$,

$$\begin{aligned}
\min\left\{\int_S u(s)d\lambda(s) : \lambda \in \Lambda(t)\right\} \ &= \ u(s_{\ell+1}) - \max\left\{\sum_{i=1}^{\ell} \hat{p}_i \lambda(s_i) : \lambda \in \Lambda(t)\right\} \\
&> \ u(s_{\ell+1}) - \sum_{i=1}^{\ell} \hat{p}_i \lambda'(s_i) \ \geq \ u(s_{\ell+1}) - \max\left\{\sum_{i=1}^{\ell} \hat{p}_i \lambda(s_i) : \lambda \in \Lambda(t')\right\} \\
&= \ \min\left\{\int_S u(s)d\lambda(s) : \lambda \in \Lambda(t')\right\}.
\end{aligned}$$

Thus (F) is indeed necessary for monotone maxmin utility. $\square$

Notice that, property (F) is strictly weaker than $\mathcal{P}_{\text{FSD}}$-increasing property. Clearly, any correspondence that increases in the latter sense satisfies (F), but the converse does not hold. In fact, as we show below, (F) is not even sufficient for monotone comparative statics. That is, this property alone does not guarantee that the set of maximisers of the function $f(x,t) := \min\left\{\int g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ with respect to x is increasing in the parameter t, for all supermodular functions g. Therefore, it is not sufficient for $f(x,t)$ to be supermodular, for all supermodular functions g.

Example S.1. Suppose that $X = \{0,1\}$ and $S = \{s_1, s_2, s_3\}$. The distribution λ is given by $\lambda(s_1) = 1/2$ and $\lambda(s_2) = 3/4$, while λ' satisfies $\lambda'(s_1) = \lambda'(s_2) = 1/2$ and μ is given by $\mu(s_1) = 1/4$, $\mu(s_2) = 7/8$. Suppose that $T = \{t, t'\}$, where $t' >_T t$, and $\Lambda(t') = \{\lambda'\}$ and $\Lambda(t) = \text{co}\{\lambda, \mu\}$. Since $\lambda' \succeq \lambda$, correspondence Λ obeys stochastic dominance in the sense given by (F). Let $g : X \times S \to \mathbb{R}$ be such that $g(0,s_1) = g(0,s_2) = 5$, $g(0,s_3) = 21$, $g(1,s_1) = 0$, $g(1,s_2) = 8$, and $g(1,s_3) = 24$; note that $g(x,s)$ is increasing in s and supermodular in (x,s). Since $\int_S g(0,s)d\lambda'(s) > \int_S g(1,s)d\lambda'(s)$, we have $\{0\} = \text{argmax}\left\{f(x,t') : x \in X\right\}$. However, given that

$$\int_S g(0,s)d\lambda(s) \;>\; \int_S g(1,s)d\mu(s) \;=\; \int_S g(1,s)d\lambda(s) \;>\; \int_S g(0,s)d\mu(s),$$

it must be that $\{1\} = \text{argmax}\left\{f(x,t) : x \in X\right\}$.

Even though property (F) is not sufficient for monotone comparative statics within a general class of supermodular functions g, it may suffice in certain special cases of g. For example, suppose X consists of only two actions – 0 and 1 – with $g(1,s)$ increasing in s and $g(0,s)$ decreasing in s, then obviously $f(1,t) - f(0,t)$ is increasing in t if Λ satisfies (F), since $f(1,t)$ and $f(0,t)$ are separately increasing and decreasing in t. In the study of global games with ambiguity by Ui (2015), this is precisely the assumption imposed on (what we call) g, which then allows the author to conclude that the higher action is chosen by players in the game when they receive a higher signal.

S.7 Continuation of Remark 5.4

Next, we turn to the claim in Remark 5.4. Recall that, whenever the function $g(x,s)$ is increasing in s, one can assume that the belief correspondence Λ has upward comprehensive values, without affecting the maxmin representation of preferences. In such a case, the $\mathcal{P}_{\text{FSD}}$ monotonicity remains necessary for f in (7) to have increasing differences, for all $g(x,s)$ that are supermodular in (x,s) and increasing in s.

Proposition S.5. *Suppose that correspondence $\Lambda : T \to \triangle_S$ has compact, convex, and upward comprehensive values. Then the following statements are equivalent.*

(i) Λ *is $\mathcal{P}_{\text{FSD}}$-increasing.*

(ii) *The function f in (7) is supermodular in (x,t), for all supermodular functions g that are increasing in s.*

Proof. Implication (i) $\Rightarrow$ (ii) follows from Proposition 7. We prove the converse in two steps. First, using Theorem 3 and an argument analogous to the one in the proof of Proposition 7, we can show that the function f satisfies increasing differences only if the correspondence $\Gamma : T \to \mathbb{R}^\ell$, defined as

$$\Gamma(t) \ := \ \Big\{ y \in \mathbb{R}^\ell : y_i \geq -\lambda(s_i), \text{ for all } i = 1, \ldots, \ell \text{ and some } \lambda \in \Lambda(t) \Big\},$$

increases in the parallelogram order. This means that for any $t' \geq_T t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, there is $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$, θ and $\theta' \in \mathbb{R}^\ell$ such that $\theta_i \leq \mu(s_i)$, $\theta'_i \leq \mu'(s_i)$, $\lambda(s_i) + \lambda'(s_i) = \theta_i + \theta'_i$, and $\theta_i \geq \lambda'(s_i)$ for all i. Therefore, Λ has the following property, which we shall refer to as $(\star)$: for any $t' \geq_T t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, there is $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that $(1/2)\lambda + (1/2)\lambda' \succeq (1/2)\mu + (1/2)\mu'$ and $\lambda' \succeq \mu$.

To complete the proof we show that $(\star)$ implies $\mathcal{P}_{\text{FSD}}$ monotonicity when Λ is upper comprehensive. $(\star)$ states that for any $t' \geq t$, $\lambda \in \Lambda(t)$, and $\lambda' \in \Lambda(t')$, there is $\mu \in \Lambda(t)$ and $\mu' \in \Lambda(t')$ such that $\mu(s_i) \geq \lambda'(s_i)$ and $\mu(s_i) + \mu'(s_i) \geq \lambda(s_i) + \lambda'(s_i)$ for all i. We modify μ and μ' state-by-state such that the condition holds with equality. Suppose $\mu(s_1) + \mu'(s_1) > \lambda(s_1) + \lambda'(s_1)$. If it is possible, choose $\nu^1(s_1)$ in the interval $\big[\lambda'(s_1), \mu(s_1)\big]$ such that $\nu^1(s_1) + \mu'(s_1) = \lambda(s_1) + \lambda'(s_1)$ and then set $\nu'^1(s_1) = \mu'(s_1)$. If, after setting $\nu^1(s_1) = \lambda'(s_1)$, we have $\nu^1(s_1) + \mu'(s_1) > \lambda(s_1) + \lambda'(s_1)$, then set $\nu'^1(s_1) = \lambda(s_1)$. Let $\nu^1(s_i) = \mu(s_i)$ and $\nu'^1(s_i) = \mu(s_i)$ for $i \geq 2$. Note that ν^1 and ν'^1 are bona fide distributions (i.e., both functions are increasing with the state) and, since Λ is upper comprehensive, $\nu^1 \in \Lambda(t)$, $\nu'^1 \in \Lambda(t')$. Furthermore, ν^1 and ν'^1 satisfy the conditions required by $(\star)$ and $\nu^1(s_1) + \nu'^1(s_1) = \lambda(s_1) + \lambda'(s_1)$. Now define ν^2 and ν'^2 by $\nu^2(s_i) = \nu^1(s_i)$ and $\nu'^2(s_i) = \nu'^1(s_i)$, for all $i \neq 2$. If possible, set $\nu^2(s_2) \in$

$\big[\max\{\lambda'(s_2), \nu^1(s_1)\}, \mu(s_2)\big]$ so that $\nu^2(s_1)+\nu'^1(s_2) = \lambda(s_2)+\lambda'(s_2)$ and then set $\nu'^2(s_2) = \nu'^1(s_2)$. Otherwise, set $\nu^2(s_2) = \max\{\lambda'(s_2), \nu^1(s_1)\}$ and set $\nu'^2(s_2)$ so that $\nu^2(s_2) + \nu'^2(s_2) = \lambda(s_2) + \lambda'(s_2)$. Note that both ν^2 and ν'^2 are distributions, with $\nu^2 \in \Lambda(t)$, $\nu'^2 \in \Lambda(t')$, and $\nu(s_i) \geq \lambda'(s_i)$ for all i; furthermore, $\nu^2(s_i) + \nu'^2(s_i) \geq \lambda(s_i) + \lambda'(s_i)$ for all i, with equality in the case of $i = 1, 2$. By repeating this process we eventually obtain $\nu \in \Lambda(t)$ and $\nu' \in \Lambda(t')$ with the required property. Thus, $\mathcal{P}_{\text{FSD}}$ monotonicity holds. $\square$

S.8 Continuation of Example 14

In this section we revisit the class of multi-prior beliefs presented in Example 14 of the main paper. As we have shown, such correspondences are $\mathcal{P}_{\text{FSD}}$-increasing, however, in general, they do not increase in the strong set order or in the $\mathcal{C}$-flexible sense.

For example, let $\Omega = \{\omega_1, \omega_2\}$, $\pi(\omega_1) = \pi(\omega_2) = 1/2$, and $T = \{t, t'\}$, with $t' >_T t$. Let the correspondence A be given by $A(\omega_1, t) = \{0\}$, $A(\omega_1, t') = \{0, 3\}$, and $A(\omega_2, t) = A(\omega_2, t') = \{1, 4\}$. Therefore, the set $A(\omega, t')$ dominates $A(\omega, t)$ in the strong sense, for all $\omega \in \Omega$. Let $\Lambda^\omega(\tilde{t})$ denote the set of degenerate probability distributions over $A(\omega, \tilde{t})$, and $\Lambda(\tilde{t}) = \sum_{i=1,2} \pi(\omega_i)\Lambda^\omega(\tilde{t})$, for all $t \in T$. We claim that the correspondence Λ does not increase in the strong set order. Take distributions

$$\lambda(z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{1}{2} & \text{if } 0 \leq z < 4 \\ 1 & \text{otherwise;} \end{cases} \quad \text{and} \quad \lambda'(z) = \begin{cases} 0 & \text{if } z < 1 \\ \frac{1}{2} & \text{if } 1 \leq z < 3 \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, we have $\lambda \in \Lambda(t)$ since the measure is obtained by mixing degenerate measures concentrated at 0 and 4 with weights equal to $\pi(\omega_1)$ and $\pi(\omega_2)$, respectively. Similarly, we have $\lambda' \in \Lambda(t')$. However, $\lambda \wedge \lambda'$ and $\lambda \vee \lambda'$ are given by

$$(\lambda \wedge \lambda')(z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{1}{2} & \text{if } 0 \leq z < 3 \\ 1 & \text{otherwise;} \end{cases} \quad \text{and} \quad (\lambda \vee \lambda')(z) = \begin{cases} 0 & \text{if } z < 1 \\ \frac{1}{2} & \text{if } 1 \leq z < 4 \\ 1 & \text{otherwise.} \end{cases}$$

Since the support of $\lambda \wedge \lambda'$ is $\{0, 3\}$, it could not belong to $\Lambda(t)$ consisting of distributions with the support in $\{0, 1, 4\}$. For the same reason, there is no convex combination of $\lambda \wedge \lambda'$ and λ' that belongs to $\Lambda(t)$, since the supports of $\lambda \wedge \lambda'$ and λ' contain 3. Hence, the correspondence increases neither in the strong set order, nor in the $\mathcal{C}$-flexible sense.

S.9 Dynamic programming under ambiguity

In an influential paper, Hopenhayn and Prescott (1992) used the tools of monotone comparative statics to analyze stationary dynamic optimization problems. In this section, we show how those results could be extended to the case where the agent has a multi-prior belief, by applying the results from the main part of paper.

Consider an agent who faces a stochastic control problem where X and S are the sets of endogenous and exogenous state variables, respectively. To keep the exposition simple, we shall assume that X is a sublattice of a Euclidean space and S is a subset of another Euclidean space. The evolution of s over time follows a Markov process with the transition function λ. The agent's problem can be formulated in the following way (see Stokey et al., 1989). At each period τ, given the current state $(x_\tau, s_\tau) \in X \times S$, the agent chooses the endogenous variable $x_{\tau+1}$ for the following period; $x_{\tau+1}$ is chosen from a non-empty feasible set $B(x_\tau, s_\tau) \subseteq X$ which may depend on the current state. The single-period return is given by the function $F : X \times S \times X \to \mathbb{R}$; $F(x, s, y)$ is the payoff when (x, s) is the state variable in period τ and y is the endogenous state variable in period $\tau + 1$ chosen in period τ. Finally, we assume that the payoffs are discounted by a constant factor $\beta \in (0, 1)$.

The agent's objective is to maximize her expected discounted payoffs over an infinite horizon, given the initial condition (x, s). We denote the value of this optimization problem by $v^*(x, s)$. Under standard assumptions — in particular, the continuity and boundedness of F and the continuity of B — this problem admits a recursive representation, where $v = v^*$ is the unique solution to the Bellman equation

$$v(x, s) \;=\; \max \left\{ F(x, s, y) + \beta \int_S v(y, \tilde{s}) d\lambda(\tilde{s}, s) : y \in B(x, s) \right\},$$

where $\lambda(\cdot, s)$ is a cumulative probability distribution over states of the world in the following period, conditional on the current state s.[4] The function v^* is bounded and continuous. Moreover, whenever we define operator $\mathscr{T} : \mathcal{B} \to \mathcal{B}$ by

$$\big(\mathscr{T} v\big)(x, s) \;=\; \max \left\{ u(x, s, y) + \beta \int_S v(y, \tilde{s}) d\lambda(\tilde{s}, s) : y \in B(x, s) \right\},$$

that maps the space $\mathcal{B}$ of bounded and continuous real-valued functions over $X \times S$ into itself, then beginning at *any* bounded and continuous function $v \in \mathcal{B}$, function $(\mathscr{T}^n v)$

[4] See Theorem 9.6 in Stokey et al. (1989) for details.

converges uniformly to v^* as n tends to infinity.[5] Furthermore, the set

$$\Phi(x,s) \;:=\; \arg\max\left\{F(x,s,y)+\beta\int_S v^*(y,\tilde{s})d\lambda(\tilde{s},s) : y\in B(x,s)\right\}$$

is non-empty and compact, for all $(x,s)\in X\times S$, and the correspondence $\Phi : X\times S\to X$ is upper hemi-continuous. We refer to any optimal control problem in which v^* and Φ have the properties listed in this paragraph as a *well-behaved* problem.

Given a well-behaved problem, Hopenhayn and Prescott (1992) (henceforth HP) apply Theorem 4.3 in Topkis (1978) to show that the value $v^*(x,s)$ *is supermodular in* x *and has increasing differences in* (x,s) under the following assumptions: (i) $F(x,s,y)$ is supermodular in (x,y) and has increasing differences in $\big((x,y),s\big)$; (ii) the graph of B is a sublattice of $X\times S\times X$; (iii) $\lambda(\cdot,s)$ is increasing in s with respect to the first order stochastic dominance. The properties of v^* in turn guarantee that the function

$$f(x,s,y) \;:=\; F(x,s,y)+\beta\int_S v^*(y,\tilde{s})d\lambda(\tilde{s},s)$$

is supermodular in y and has increasing differences in $\big(y,(x,s)\big)$. By Theorem 6.1 in Topkis (1978), $\Phi(x,s)$ is a compact sublattice of X and is increasing in (x,s).[6] This in turn guarantees the existence of the greatest optimal selection

$$\phi(x,s) \;:=\; \Big\{y\in\Phi(x,s) : y\geq_X z,\ \text{for all } z\in\Phi(x,s)\Big\},^7$$

that is increasing and Borel measurable. Lastly, the policy function ϕ induces a Markov process on $X\times S$, where, for measurable sets $Y\subseteq X$ and $T\subseteq S$, the probability of $Y\times T$ conditional on (x,s) is the probability of T conditional on s if $\phi(x,s)\in Y$, and it is zero otherwise. HP make use of the monotonicity of ϕ to guarantee that this Markov process has a stationary distribution.[8] We now consider a stochastic control problem identical to the one we just described, except that we allow the agent to be ambiguity averse. Since at each period τ the exogenous variable is drawn from the set S, the set of

[5] By $\mathscr{T}^n$ we denote the n'th orbit of the operator $\mathscr{T}$, i.e., we have $(\mathscr{T}^{n+1}v)=\mathscr{T}\big(\mathscr{T}^n v\big)$.

[6] Condition (ii) on B guarantees that $B(x,s)$ is sublattice of X and that it increases with (x,s) in the strong set order. Given with the properties on f, we know that $\Phi(x,s)$ is a sublattice and that it increases with (x,s); this follows from Theorem 6.1 in Topkis (1978).

[7] Function is well-defined because Φ is compact-valued and a sublattice.

[8] The focus in this section is on primitive conditions guaranteeing the monotonicity of the policy function. Readers who are interested in how the distribution over (x,s) evolves over time (under monotonicity or weaker assumptions) should consult Huggett (2003). HP and Stachurski and Kamihigashi (2014) also discuss uniqueness and other issues relating to the stationary distribution.

all possible realizations of the exogenous variable over time is given by S^∞. An expected utility maximizer behaves as though she is guided by a distribution over S^∞; to obtain the utility of a given plan of action, the agent evaluates the discounted utility on every possible path, i.e., over every element in S^∞ and takes the average across paths, weighing each path with its probability. When the agent has a maxmin preference, her behavior can be modeled by a *set* of distributions $\mathcal{M}$ over S^∞. The utility of a plan is then given by the minimum of the expected discounted utility for every distribution in $\mathcal{M}$.

In contrast to expected discounted utility, it is known that the agent's utility in the maxmin model will not generally have a recursive representation. However, there is a condition on $\mathcal{M}$ called *rectangularity* which is sufficient (and effectively necessary) for this to hold (see Epstein and Schneider, 2003). Furthermore, it is known that a time-invariant version of rectangularity is also sufficient to guarantee that the agent's problem can be solved through the Bellman equation, in a way analogous to that for expected discounted utility (see Iyengar, 2005). This condition says that the agent's belief over the possible value of the exogenous variable in the following period, after observing s in the current period, is given by a *set* of distribution functions $\Lambda(s)$; this set depends on the current realization of the exogenous variable and is time-invariant. The set $\mathcal{M}$, given an initial value s_0, is obtained by concatenating the transition probabilities. Therefore, the probability of a path $(s_1, s_2, s_3, \ldots)$ is $\prod_{i=1}^{\infty} p_i$, where p_1 is the probability of s_1 for some distribution in $\Lambda(s_0)$, p_2 is the probability of s_2 for some distribution in $\Lambda(s_2)$, etc.

With this assumption on $\mathcal{M}$ in place, and some other standard conditions, one could guarantee that the value $v^*(x, s)$ of the control problem with the initial state (x, s), is the unique solution to the Bellman equation

$$v(x,s) \;=\; \max\Big\{F(x,s,y) + \beta(Av)(y,s) : y \in B(x,s)\Big\},$$

where $(Av)(y,s) = \min\big\{\int_S v(y,s)d\lambda(s) : \lambda \in \Lambda(s)\big\}$ (see Iyengar, 2005). Furthermore, the problem is *well-behaved* in the sense defined at the beginning of this section.

With this basic set-up, we are almost in a position to recover a monotone result of the HP type; all that is needed is a condition guaranteeing that $(Av)(y,s)$ is a supermodular function of (y,s), whenever v is supermodular. When X and S are one-dimensional, Proposition 7 tells us that this holds if belief $\Lambda(t)$ is $\mathcal{P}_{\text{FSD}}$-increasing.

Proposition S.6. *Consider a well-behaved optimal control problem where X, $S \subseteq \mathbb{R}$, with X compact and S finite. Let $F(x,s,y)$ be supermodular in (x,s,y), $\Lambda : S \to \triangle_S$ be $\mathcal{P}_{\mathrm{FSD}}$-increasing, and the graph of $B : X \times S \to X$ be a sublattice; then the value function $v^*(x,s)$ is supermodular, and the correspondence $\Phi : X \times S \to \mathbb{R}$, where*

$$\Phi(x,s) \;:=\; \arg\max\Big\{F(x,s,y) + \beta(Av^*)(y,s) : y \in B(x,s)\Big\}$$

is sublattice-valued and increasing in the strong set order. Finally, the greatest selection $\phi : X \times S \to \mathbb{R}$ of Φ is well-defined, increasing, and Borel measurable.

Proof. Let $v : X \times S \to \mathbb{R}$ be a continuous and bounded function. Since the problem is well-behaved we know that the function $(\mathscr{T}v)$, given by

$$\big(\mathscr{T}v\big)(x,s) \;=\; \max\Big\{F(x,s,y) + \beta(Av)(y,s) : y \in B(x,s)\Big\},$$

is a continuous function on $X \times S$ and $\mathscr{T}^n v$ converges uniformly to v^* as $n \to \infty$. By Proposition 7 in the main paper, whenever function v is supermodular, then so is Av. This implies that $F(x,s,y) + \beta(Av)(y,s)$ is supermodular over $X \times S \times X$. Given that the graph of correspondence B is a sublattice, by Theorem 4.3 in Topkis (1978), the function $\mathscr{T}v$ is supermodular in (x,s). Since supermodularity is preserved under uniform convergence, we conclude that $v^* = \mathscr{T}v^*$ is a supermodular function of (x,s). The set $\Phi(x,s)$ consists of elements y that maximize $F(x,s,y) + \beta(Av^*)(x,s)$ over $B(x,s)$. Since the objective function is supermodular, while values of correspondence B are complete sub-lattices of X, by Theorem 6.1 in Topkis (1978), set $\Phi(x,s)$ is a complete sub-lattice of X. Furthermore, since B increases over $X \times S$ in the strong set order, so does Φ. As the problem is well-behaved, $\Phi(x,s)$ admits the greatest selection $\phi(x,s)$ and this selection is increasing. That ϕ is Borel measurable follows from standard arguments (see HP). $\square$

Below we discuss an application of this result.

Example S.2. Consider the following dynamic optimization problem of a firm. In each period, the firm collects revenue $\pi(x,s)$, where $s \in S$ denotes the realized exogenous state of the world and $x \in \mathbb{R}_+$ is the level of capital stock currently available to the firm. Once s is revealed to the firm and the revenue collected, the firm may invest $a \in [0,K]$ at a cost $c(a)$, K being a finite positive number. With this investment, capital stock in the next period is $y = \delta x + a$, where $\delta \in [0,1]$ denotes the fraction of non-depreciated capital. Therefore, the dividend in each period is

$$F(x, s, y) \; := \; \pi(x, s) - c(y - \delta x),$$

where the firm chooses y from the interval $B(x, s) = [\delta x, \delta x + K]$. We know from HP that if the firm is an expected utility maximizer and the optimal control problem is well-behaved, the firm has a policy function that is increasing in (x, s) under these additional conditions: the transition *function* $\Lambda : S \to \triangle_S$ is increasing with respect to first order stochastic dominance and F is supermodular; the latter is guaranteed if π is supermodular and c is concave. Proposition S.6 goes further by saying that this remains true if the firm has a maxmin preference, so long as the belief Λ is $\mathcal{P}_{\text{FSD}}$-increasing.

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