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Paweł Dziewulski
University of Sussex
p.k.dziewulski@sussex.ac.uk

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A comprehensive revealed preference approach to approximate utility maximisation*

Paweł Dziewulski[†]

August, 2021

Abstract

We develop a comprehensive revealed preference method for studying *approximate utility maximisation*, where an alternative is selected from a menu only if its utility is not significantly lower than that of any other available option. We show that this model characterises choices that violate transitivity of indifferences, but preserve transitivity of the revealed strict preferences. More importantly, although the individual may fail to maximise their utility exactly, it is possible to recover their true preferences from the observable data, make out-of-sample predictions and welfare comparisons. Our results require minimal assumptions on the empirical framework and are applicable, amongst others, to the study of choices over consumption bundles, state-contingent consumption, and lotteries.

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1 Introduction

Stemming from Samuelson (1938), Richter (1966), and Afriat (1967), numerous developments in the revealed preference literature provided tools for a non-parametric analysis of utility maximisation with limited choice data. An important feature of this approach is the ability to dispense of any ancillary assumptions regarding the functional specification

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† Department of Economics, University of Sussex, Jubilee Building, Falmer, Brighton BN1 9SL, United Kingdom. Email: P.K.Dziewulski@sussex.ac.uk.

of preferences and fully embrace heterogeneity among individual subjects. Moreover, the revealed preference theory establishes a general framework for testing models of consumer choice, estimating preferences from observable data, as well as making out-of-sample predictions and data-driven welfare comparisons. These methods have found a broad application to the empirical analysis of consumer demand, time preference, choices over risk and uncertainty, and multiperson household models, among others.[1]

At the same time, a significant number of empirical studies suggest that choices of individuals are *not* consistent enough to be congruent with utility maximisation.[2] Naturally, this poses the fundamental question whether the standard model of consumer choice is an appropriate description of human behaviour. In addition, from the practical standpoint, whenever the data are inconsistent with utility maximisation the standard revealed preference tools are no longer applicable, as they critically depend on the observations being accordant with the classic notion of rationality.

There are at least two ways of addressing this issue. One is to abandon the idea of utility maximisation entirely and explore other models of decision-making, either deterministic or stochastic. Such a radical departure from the standard approach has significant drawbacks. First of all, it would require an overhaul of a large and (arguably) fruitful part of the economic analysis built around deterministic utility maximisation. Moreover, the richness of data required to study choices of individuals with stochastic choice models is prohibitive in many empirical settings in which the researcher observes at most a few choices from a small number of menus.[3] The other way is to accept that utility maximisation (like other scientific models) serves merely as an approximation of individual behaviour, and develop methods for studying decision-making within the classic framework while accounting for some inconsistencies with the data.

In this paper we explore the second route and introduce a comprehensive method for

[1] For a handbook treatment of this literature see Chambers and Echenique (2016). For more recent results in the general revealed preference theory see Chambers et al. (2017), Halevy et al. (2017), Nishimura et al. (2017), Hu et al. (2020); time preference are discussed in Dziewulski (2018), Echenique et al. (2020), Blow et al. (2021); for applications to choice under risk and uncertainty see Echenique et al. (2018), Polisson et al. (2020); for multiperson household models see Cherchye et al. (2017, 2020).

[2] See Chapter 5 in Chambers and Echenique (2016) and more recently Halevy et al. (2018), Echenique et al. (2019), Feldman and Rehbeck (2020), Zrill (2020), Dembo et al. (2021), and Cappelen et al. (2021).

[3] Here we abstract from applying stochastic choice models to study choices in populations of heterogeneous agents. In fact, these models are very much in line with deterministic choice theory. They assume that decisions of individuals are deterministic. However, since the researcher can observe choices only in the aggregate, the data are evaluated as if they were stochastic.

studying *approximate utility maximisation*. We consider a model in which the decision-maker chooses an alternative from a set of available options only if its utility is not significantly lower then that of any other element in the menu. We answer the following question: Given a finite dataset of pairs (A, x), where $x \in A$ is an option selected from the menu A, under what conditions there is a utility u and a positive threshold function δ such that, for any observation (A, x) and $y \in A$, we have

$$u(x) + \delta(y) \geq u(y).$$ [4]

That is, the utility of the observed choice x is at most $\delta(y)$ utils lower than that of any other available option. In our main result (Theorem 1) we show that approximate utility maximisation characterises choices that violate transitivity of indifferences revealed in the data, but obey transitivity of revealed strict preference relations. Therefore, this natural extension of the standard model of consumer choice is derived from a meaningful, intuitive, and testable restriction on the observed choices.[5]

Our characterisation of approximate utility maximisation is instrumental in developing non-parametric tools for eliciting the "true" preferences of the decision-maker, that we identify with the utility function u, as well as making our-of-sample predictions and evaluating welfare. We address the first question in Theorem 2, where we estimate the unobserved preferences u that (together with some threshold function δ) approximately rationalise the set of observations as above. Although the individual may fail to optimise their utility, we show that it is possible to recover their preferences from the observations. Moreover, our estimates are tight, in the sense that improving them would necessarily exclude some preferences that could support the data.

Eliciting the true preferences allows us to develop a meaningful, data-driven welfare analysis. In Theorem 3 we characterise a criterion that is appropriate for our framework and allows us to compare the well-being of an individual who faces different menus or budget sets (not necessarily observed in the data). We say that a menu A' is *robustly preferable* to A if, for any utility u and a threshold function δ that are consistent with

[4] The reader may recognise that our model is analogous to the interval order representation of preferences proposed in Fishburn (1970). We address this in the Online supplement. Since our paper focuses on the utility u, we find the term *approximate utility maximisation* more appropriate.

[5] See also Allen and Rehbeck (2020a,b) who study approximate utility maximisation under the assumption that the utility u is quasilinear and the threshold function δ is constant. Alternatively, Aguiar et al. (2020) analyse datasets that violate transitivity of both the weak and strict preference. Beresteanu and Rigotti (2021) discuss similar issues in the stochastic choice framework.

the data in the approximate sense, any (possibly unobserved) choice from A' induced by such a model would be preferable to any choice from the set A, with respect to the true preferences u. This captures the idea that, although the decision-maker may fail to maximise their utility, the researcher evaluates their welfare using the true preferences, thus, separating the positive and normative aspects of the choice.[6]

We do not claim that approximate utility maximisation is the ultimate explanation for *any* deviation from the classic notion of rationality. Inevitably, some departures require a qualitatively different approach to modelling consumer choice. Rather, the point of this paper is to develop a comprehensive and meaningful framework for studying the canonical model of utility maximisation when the empirical data exhibit inconsistencies with the theory that could be attributed to non-transitive indifferences.

The idea of non-transitive indifferences was introduced to economics by Georgescu-Roegen (1936), Armstrong (1939, 1950) and Luce (1956). Inspired by the research in psychology and psychophysics, these papers acknowledge the inability of human beings to discern between close quantities of goods and claim that any descriptive theory of choice should allow for non-transitive indifferences that would result from "imperfect powers of discrimination of the human mind whereby inequalities become recognizable only when of sufficient magnitude" (Armstrong, 1950, page 122). Nevertheless, non-transitive indifferences go beyond limited perception and can be related to incommensurability, vagueness of judgement, and imprecise preferences — empirical phenomena in choice under risk reported in, e.g., Butler and Loomes (2007) and Cubitt et al. (2015). Finally, approximate utility maximisation may follow from the satisficing behaviour proposed in Simon (1947), where the subject fails to maximise their utility due to an unobserved mental or physical cost of switching from an inferior to a dominant alternative. In fact, we show that in many empirical settings the two models are observationally equivalent.

Throughout this paper we abstract away from specific economic environments and impose a very limited set of assumptions on the space of alternatives and the data available to the researcher. Because of this, our general results are applicable to a variety of empirical environments, including the classical consumer demand, state contingent consumption under risk and uncertainty, and choices over lotteries.

[6] A similar idea is discussed in Mandler (2005) and Nishimura (2018). See also Nishimura and Ok (2020) for a discussion on a broad class of related models.

Organisation of the paper In Section 2 we introduce our setup and the basic notation. Our first main result (Theorem 1) is presented in Section 3, where we characterise approximate utility maximisation in terms of non-transitive indifferences. We devote Section 4 to a discussion on imperfect discrimination and how this phenomenon can be studied using the toolkit developed in this paper. Theorem 2 is stated in Section 5, where we discuss the problem of eliciting the true preferences of approximate utility maximisers from their observable choices. We present the welfare analysis in Section 6, where we state our final main result (Theorem 3). Section 7 is devoted to some direct applications of our method. Specifically, in Section 7.1, we determine the class of empirical settings in which approximate utility maximisation is indistinguishable from the model of satisficing à la Simon (1947). In Section 7.2 we extend the results in Polisson et al. (2020) to study approximate utility maximisation within a broad class of models of choice under risk, that includes expected utility, rank dependent expected utility, and disappointment aversion. Finally, in Section 7.3 we extend the result in Dziewulski (2020) and show the tight relation between approximate utility maximisation and one of the most widespread measures of departures from rationality — the critical cost-efficiency index. Proofs of the main results are postponed until the Appendix. A number of related results and extensions are discussed in the Online supplement, including an alternative, constructive take on Theorem 1 that pertains to linear programming methods.

2 Preliminaries

We begin our discussion by introducing the notation and terminology.

2.1 The setup

Let X be the universal *consumption space*, i.e., the grand set of mutually exclusive choice alternatives. A *menu* is a non-empty subset A of X, and $\mathcal{A} = 2^X \setminus \{\emptyset\}$ denotes the set of all menus. A *dataset* (or a set of observations) $\mathcal{O}$ is a finite collection of pairs (A, x), where $x \in A$ is interpreted as a choice from the menu $A \in \mathcal{A}$. We often denote $\mathcal{O} = \big\{(A^t, x^t) : t \in T\big\}$, for some finite set T. Unless stated otherwise, we impose no additional assumptions on the space X or the dataset $\mathcal{O}$. To fix ideas, we present two commonly used empirical settings that fit our framework.

The first natural application is the classic consumer demand problem as in Afriat (1967), Diewert (1973), or Varian (1982), where an individual is choosing ℓ-commodity bundles $x^t \in X \subseteq \mathbb{R}^\ell_+$ from linear budget sets $A^t = \{y \in X : p^t \cdot y \leq m^t\}$, given some strictly positive prices $p^t \in \mathbb{R}^\ell_{++}$ and income $m^t \geq 0$, for all $t \in T$. Usually, it is assumed that the purchase exhausts the available budget, i.e., $m^t = p^t \cdot x^t$, for all $t \in T$, but this plays no role in our analysis. This framework can be extended to a general class of budget sets as in Forges and Minelli (2009), where the menu satisfies $A^t = \{y \in X : f^t(y) \leq 0\}$, for a well-defined, continuous, and strictly increasing function $f^t : X \to \mathbb{R}$, for all $t \in T$.[7] This embeds the setup of Afriat, where $f^t(y) = p^t \cdot y - m^t$, for all $t \in T$.

In numerous empirical studies, including the famous Allais experiment, researchers investigate choices of subjects over lotteries. In such a case, the space of alternatives X is the probability simplex $\{\pi \in \mathbb{R}^\ell_+ : \sum_{i=1}^\ell \pi_i = 1\}$, where π_i is a likelihood of the state/prize $i = 1, \ldots, \ell$ being realised. A menu can be given by an arbitrary (possibly finite) subset of X. One could also consider "budget sets" of lotteries as in Sopher and Narramore (2000) or Feldman and Rehbeck (2020).

In our main result, we characterise datasets that can be rationalised with a particular model of choice. Generally, a *choice correspondence* (or a model) is a set-valued mapping $c : \mathcal{A} \rightrightarrows X$ that assigns a menu $A \in \mathcal{A}$ to the set $c(A) \subseteq A$ of all possible choices from A.[8] The correspondence c *rationalises* the set of observations $\mathcal{O}$ if

$$(A, x) \in \mathcal{O} \;\text{ implies }\; x \in c(A), \tag{1}$$

i.e., the data are consistent with the model c. In the reminder of this paper we restrict our attention to rationalisation within specific classes of choice correspondences.

Our definitions of a choice model and rationalisation highlight two important aspects of the analysis. First of all, as the choice correspondence c is set-valued, we allow for the consumers to exhibit indifferences (or incomparabilities). Since our analysis is performed in general consumption spaces, this is inevitable, as shown in Nishimura and Ok (2014). On the other hand, we assume that the dataset available to the researcher is incomplete. That is, they monitor only some (usually, a single) elements of the set $c(A)$, for some menus $A \in \mathcal{A}$. These are natural assumptions for most empirical studies.

[7] Lemma 1 in Forges and Minelli (2009) specifies a large class of such menus.

[8] In principle, the set $c(A)$ may be empty, for some menu $A \in \mathcal{A}$. However, within the class of choice models discussed in this paper, one may assume that $c(A)$ is non-empty for any finite set A, without loss of generality. In Section 3 we discuss conditions under which $c(A)$ is non-empty for any compact A.

2.2 Choice monotonicity

Our notion of rationalisability is rather weak. In particular, any dataset is consistent with the identity correspondence $c(A) = A$, for all $A \in \mathcal{A}$. In order to make our main research question non-vacuous, we refer to a notion of choice monotonicity.[9] Given a correspondence $\Gamma : X \rightrightarrows X$, we say that a choice model $c : \mathcal{A} \rightrightarrows X$ is Γ-*monotone* if for any (possibly unobserved) alternative $x \in X$ and menu $A \in \mathcal{A}$, we have

$$\Gamma(x) \cap A \neq \emptyset \quad \text{implies} \quad x \notin c(A). \tag{2}$$

We interpret $\Gamma(x)$ as a set of alternatives that are *objectively* better than x; if any such option is available, then x may never be selected. Therefore, Γ summarises the additional restrictions imposed by the researcher on the choice model, that are separate from the data. Importantly, Γ-monotonicity is independent of any particular specification of the choice correspondence c and is itself testable. Moreover, as we show in the following section, this property arises naturally in the most canonical models.[10] In the remainder of the paper we shall refer to the pair $(\mathcal{O}, \Gamma)$ as a *choice environment*.

We discuss Γ-monotonicity thoroughly in Section 4. However, to fix ideas, consider two examples of the correspondence Γ. When analysing choices over consumption in an ℓ-dimensional commodity space $X \subseteq \mathbb{R}^\ell_+$, it may be sensible to require that bundles containing more of each good are objectively better. In such a case, the set $\Gamma(x)$ consists of all vectors $y \in X$ such that $y > x$ (or $y \gg x$), capturing the idea that "more is better."[11] Alternatively, when studying choices over lotteries, one may identify the set $\Gamma(x)$ with probability distributions that first order stochastically dominate x, thus, assuming affinity for gambles in which greater rewards are more likely.[12]

Throughout the analysis we impose minimal assumptions on the correspondence Γ. Specifically, it need not be well-defined, i.e., we allow for $\Gamma(x) = \emptyset$, for some $x \in X$. This is equivalent to imposing *no* ancillary assumptions on the relation between x and other

[9] In contrast, Balakrishnan et al. (2021) propose a method of estimating the entire set $c(A)$ and determining which comparisons represent indifferences (or indeterminacies) and which correspond to strict preferences. However, their approach requires rich datasets and applies only to finite menus.

[10] Alternatively, one could endow the consumption space X with a binary relation $\rhd$ that would determine an objective dominance ranking over X, as in Nishimura et al. (2017). Since it is always possible to define $\Gamma(x) := \{y \in X : y \rhd x\}$, the two notations are equivalent.

[11] We denote $x \geq y$ if $x_i \geq y_i$, for all $i = 1, \ldots, \ell$. The relation is *strict*, and denoted by $x > y$, if $x \geq y$ and $x \neq y$. Finally, we have $x \gg y$ if $x_i > y_i$, for all $i = 1, \ldots, \ell$.

[12] Lottery y *first order stochastically dominates* x, denoted by $y \geq_F x$, if for any increasing function $g : S \to \mathbb{R}$, we have $\sum_{s \in S} g(s)y(s) \geq \sum_{s \in S} g(s)x(s)$. We denote $y >_F x$ if $y \geq_F x$ and $y \neq x$.

elements in the domain. Nevertheless, we require the following basic properties.

Assumption 1 (Partial order)**.** The correspondence $\Gamma : X \rightrightarrows X$ satisfies:

(i) For any $x \in X$, we have $x \notin \Gamma(x)$.

(ii) For any $x, y \in X$, if $y \in \Gamma(x)$ then $\Gamma(y) \subseteq \Gamma(x)$.

The first condition guarantees that no alternative is objectively superior to itself. The second restriction imposes a form of transitivity on Γ.[13]

3 Revealing non-transitive indifferences

The goal of this section is to provide a revealed preference characterisation of approximate utility maximisation and discuss its relation to non-transitive indifferences. To give a better context for our analysis, we begin by presenting the observable implications of the classic model of the *exact* utility maximisation.

3.1 Classic revealed preference analysis

The principal question in the revealed preference analysis concerns the necessary and sufficient conditions under which a set of observations is rationalisable with *utility maximisation*. That is, when there is a function $u : X \to \mathbb{R}$ such that the correspondence

$$c(A) \ := \ \Big\{x \in A : u(x) \geq u(y), \text{ for all } y \in A\Big\} \tag{3}$$

rationalises the dataset $\mathcal{O}$ as in (1). As pointed out previously, without any further assumptions this question is vacuous, since any set of observations can be trivially rationalised with maximisation of a constant function u. In such a case, we have $x \in A = c(A)$, for any observation $(A, x) \in \mathcal{O}$. To make the problem interesting, it is common to impose additional restrictions on the utility u, e.g., an appropriate notion of strict monotonicity. Within the class of utility maximisation models, strict monotonicity of u is equivalent to Γ-monotonicity of the correspondence c, thus, motivating our definition.

Proposition 1. *For any correspondence $\Gamma : X \rightrightarrows X$, the choice model in (3) is Γ-monotone if, and only if, $y \in \Gamma(x)$ implies $u(y) > u(x)$.*

[13] Essentially, we require that the binary relation $\rhd := \big\{(y, x) : y \in \Gamma(x)\big\}$ is a strict partial order.

The proof is straightforward.[14] With the additional notation in place, we return to our initial question. Given a correspondence Γ, under what conditions can we rationalise the dataset $\mathcal{O}$ with a Γ-monotone utility maximisation; i.e., when there is a function u for which the choice model in (3) rationalises the observations *and* is Γ-monotone?

It is convenient to address this problem by referring to the revealed preference relations. An alternative x is *directly revealed preferred* to y, denoted by xR^*y, if there is an observation where both x and y were available and x was chosen. Formally,

$$xR^*y \quad \text{if} \quad (A, x) \in \mathcal{O} \text{ and } y \in A.$$

We think of this relation in terms of weak preference. Since x was selected when y was available, the former can be no worst than the latter.

To construct the strict counterpart of R^*, we employ the correspondence Γ. An alternative x is *directly revealed strictly preferred* to y, denoted by xP^*y, if there is an observation in which x was chosen over something objectively better than y, i.e.,

$$xP^*y \quad \text{if} \quad (A, x) \in \mathcal{O} \text{ and } \Gamma(y) \cap A \neq \emptyset.$$

Within the classic consumer choice framework of Afriat, the relations R^* and P^* coincide with the revealed preferences defined in Varian (1982). Suppose that $\Gamma(x)$ consists of all bundles $y \in \mathbb{R}^\ell_+$ such that $y > x$ (or $y \gg x$), for all $x \in X$. Whenever $A^t = \left\{y \in \mathbb{R}^\ell_+ : p^t \cdot y \leq p^t \cdot x^t\right\}$, for some prices $p^t \in \mathbb{R}^\ell_{++}$ and all $t \in T$, then $x^t R^* y$ if and only if $p^t \cdot y \leq p^t \cdot x^t$, and $x^t P^* y$ is equivalent to $p^t \cdot y < p^t \cdot x^t$.

One can easily show that, whenever the set $\mathcal{O}$ is rationalisable with a Γ-monotone utility maximisation, the relations R^*, P^* are consistent with the corresponding utility u, i.e., xR^*y implies $u(x) \geq u(y)$, and xP^*y implies $u(x) > u(y)$.[15] One testable restriction for utility maximisation immediately follows: For any sequence $z^1, z^2, \ldots, z^n$ of alternatives in X such that either $z^i R^* z^{i+1}$ or $z^i P^* z^{i+1}$, for all $i = 1, \ldots, (n-1)$, it may never be that $z^n P^* z^1$. That is, there can be no revealed preference cycle in which any two subsequent alternatives are ordered with the strict relation P^*. Otherwise, for any function u that rationalises the data, we would have $u(z^1) \geq u(z^2) \geq \ldots \geq u(z^n)$ and $u(z^n) > u(z^1)$,

[14] Indeed, if $y \in \Gamma(x)$ implies $u(y) > u(x)$ then the model in (3) must be Γ-monotone. Conversely, suppose that $y \in \Gamma(x)$. Whenever c is Γ-monotone, then $x \notin c\big(\{x, y\}\big)$, which requires $u(y) > u(x)$.

[15] The former follows directly from the definition of R^* and c. Suppose that xP^*y. By definition, we have $(A, x) \in \mathcal{O}$ and $\Gamma(y) \cap A \neq \emptyset$, for some $A \in \mathcal{A}$. Take any $z \in \Gamma(y) \cap A$. Thus, $z \in A$ implies $u(x) \geq u(z)$. Moreover, by Proposition 1, we have $u(z) > u(y)$, which suffices for $u(x) > u(y)$.

yielding a contradiction. Theorem 2 in Nishimura et al. (2017) states that, under Assumption 1 and some regularity conditions, this is also a sufficient condition for a dataset to be rationalisable in this sense.[16] Moreover, within the classic demand framework of Afriat, the above restriction coincides with the well-known *generalised axiom of revealed preference* (or GARP) introduced in Varian (1982).

Since transitive indifferences are critical for utility maximisation, the only revealed preference cycles admissible by this model are those induced by the weak relation R^* alone, i.e., where $z^i R^* z^{i+1}$, for all $i = 1, \dots, n-1$, and $z^n R^* z^1$. For any such sequence, each alternative in the cycle must be indifferent to all others. In the following subsection we investigate implications of non-transitive indifferences.

3.2 The main result

Once we relax transitivity of indifferences, it is possible to observe revealed preference cycles along which some alternatives are ordered with P^*. However, as we maintain transitivity of the strict preference, the directly revealed strict preference relation P^* must be *acyclic*. That is, there is no sequence $z^1, z^2, \dots, z^n$ in X such that

$$z^1 P^* z^2,\ z^2 P^* z^3,\ \dots\ ,\ z^{n-1} P^* z^n,\ \text{and}\ z^n P^* z^1. \tag{4}$$

This condition excludes any revealed preference cycles that are induced by the revealed strict relation P^* alone. Although acyclicity of P^* remains necessary for the dataset to be rationalisable with utility maximisation, it is no longer sufficient, as it allows for cycles that are generated by the weak R^* and the strict P^* relations jointly.

Before stating our main result, we impose one final assumption.

Assumption 2 (Weak separability)**.** There is a countable set $D \subseteq X$ such that $x \in \Gamma(y)$ implies $\Gamma(x) \subseteq \Gamma(z)$ and $z \in \Gamma(y)$, or $x \in \Gamma(z)$ and $\Gamma(z) \subseteq \Gamma(y)$, for some $z \in D$.

This condition holds trivially whenever X is countable, as one can always choose $D = X$ and set $z = x$ or $z = y$, for any $x, y \in X$ satisfying $x \in \Gamma(y)$. However, weak separability of Γ is indispensable when considering general spaces.

Theorem 1. *For an arbitrary dataset $\mathcal{O}$ and a correspondence $\Gamma : X \rightrightarrows X$ satisfying Assumptions 1 and 2, the following statements are equivalent.*

[16] Although our notation differs, this condition is equivalent to *cyclical $\trianglerighteq$-consistency* in Nishimura et al. (2017) for the relation (preorder) $\trianglerighteq := \{(y, x) : y \in \Gamma(x)\} \cup \{(x, x) : x \in X\}$.

(i) The directly revealed strict preference relation P^* is acyclic.

(ii) There is a utility $u : X \to \mathbb{R}$ and a positive threshold function $\delta : X \to \mathbb{R}_+$ such that the choice correspondence $c : \mathcal{A} \rightrightarrows X$, given by

$$c(A) \; := \; \Big\{x \in A : u(x) + \delta(y) \geq u(y), \text{ for all } y \in A\Big\}, \tag{5}$$

rationalises the set $\mathcal{O}$ as in (1) and is Γ-monotone as in (2). [Proof]

Approximate utility maximisation is a natural extension of the canonical model of choice. It posits that an alternative x is selected from a menu A if its utility is at most $\delta(y)$ utils lower than that of any other available option y. Therefore, it captures the behaviour of an individual who fails to optimise their true preferences exactly.

Our specification of approximate utility maximisation is not ad hoc, but is derived from acyclicity of the standard notion of the revealed strict preference relation P^*. Specifically, the model describes choices of individuals whose observable behaviour may violate transitivity of indifferences, but obeys transitivity of the strict preference. Thus, apart from an appealing utility-threshold representation, this model is fully characterised by an intuitive condition that can be verified using observable data.[17]

Our main theorem and the remaining results follow from the observation that the directly revealed strict preference relation P^* is consistent with *any* utility u that rationalises the data as in (5).[18] Indeed, suppose that xP^*y. By definition, we have $(A, x) \in \mathcal{O}$ and $\Gamma(y) \cap A \neq \emptyset$, for some $A \in \mathcal{A}$. Take any $z \in \Gamma(y) \cap A$. Since $z \in A$, it must be that $u(x) \geq u(z) - \delta(z)$, by definition of the correspondence c. In addition, given $z \in \Gamma(y)$ and Γ-monotonicity of c, we have $y \notin c\big(\{y, z\}\big)$, which requires that $u(z) - \delta(z) > u(y)$. Combining the two inequalities yields $u(x) > u(y)$.

Implication (ii) $\Rightarrow$ (i) follows directly from this observation. Clearly, whenever there is a sequence of alternatives $z^1, z^2, \ldots, z^n$ in X such that $z^i P^* z^{i+1}$, for all $i = 1, \ldots, (n-1)$, then $u(z^1) > u(z^n)$. Since $z^n P^* z^1$ implies $u(z^n) > u(z^1)$, P^* must be acyclic.

Showing the converse is more demanding and postponed until the Appendix. Our argument consists of two steps. First, we show that whenever the strict relation P^* is

[17] Although we find non-transitive indifferences to be a more natural interpretation of our condition, one could perceive the acyclicity of P^* as a particular relaxation of completeness. In our setup, the two notions are indistinguishable. We are grateful to Luca Rigotti for pointing this out.

[18] However, this is no longer true for the weak relation R^*. Similarly, it is not true that xP^*y implies $u(x) > u(y) + \delta(x)$. Specifically, it could be that xP^*y and $x, y \in c(A)$, for some menu $A \in \mathcal{A}$.

acyclic, there is a utility function u such that both $x \in \Gamma(y)$ and xP^*y imply $u(x) > u(y)$. This (and only this) part of the proof requires for Assumptions 1 and 2 to be satisfied. The second step is summarised in the following proposition.

Proposition 2. *For any correspondence $\Gamma : X \rightrightarrows X$, any set of observations $\mathcal{O}$, and any utility $u : X \to \mathbb{R}$ the following statements are equivalent.*

(i) If $x \in \Gamma(y)$ or xP^*y then $u(x) > u(y)$, for any $x, y \in X$.

(ii) There is a positive threshold function $\delta : X \to \mathbb{R}_+$ such that the correspondence $c : \mathcal{A} \rightrightarrows X$ in (5) rationalises the dataset $\mathcal{O}$ and is Γ-monotone. [Proof]

This proposition is of interest in itself. Theorem 1 specifies the necessary and sufficient condition under which there *exists* a utility u that approximately rationalises the data as in (5). However, in many applications the researcher is interested whether the choices of the individual are consistent with a particular function u. For example, whenever X is the space of ℓ-dimensional consumption bundles, it may be desirable to determine if there is a concave function u that rationalises the data. Alternatively, if X is the space of lotteries, one may be interested if u admits the expected utility specification. Proposition 2 stipulates that any such test is equivalent to verifying if the particular utility u is consistent with Γ and P^*. In Section 7.2 we employ this result to study preferences over state-contingent consumption under risk.

Determining acyclicity of the relation P^* can be done efficiently using, e.g., the well-established Warshall's algorithm (see Appendix II in Varian, 1982 for details). This is particularly important for practical applications. In fact, in order to verify whether the relation P^* is acyclic, it suffices to check if it generates no cycles over the observed choices $\{x^t\}_{t \in T}$ alone. This is analogous to the well-known result in Varian (1982).[19]

Due to the limited number of assumptions, it is difficult to determine any additional properties of the utility u that rationalises the data in the approximate sense. In the Online supplement we present an alternative, more constructive take on our main result using a linear programming approach similar to Afriat (1967) and Forges and Minelli (2009). By imposing additional structure on the consumption space X, we are able to identify properties of the model that are not falsifiable in certain environments.

Finally, approximate utility maximisation is tightly related to the notion of *interval orders* introduced in Wiener (1914) and Fishburn (1970). In the Online supplement we

[19] See also Forges and Minelli (2009) and Section II.E in Nishimura et al. (2017).

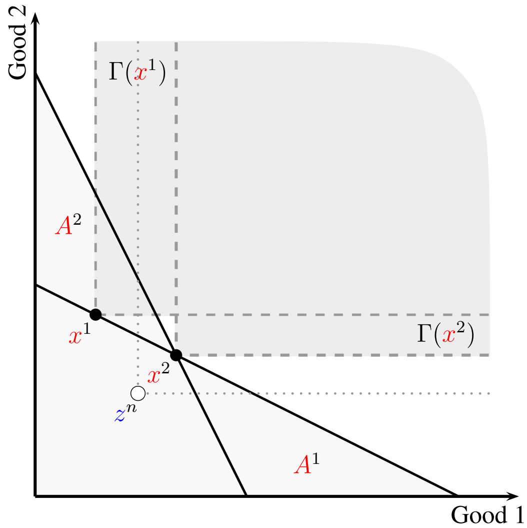

Figure 1: The dataset is *not* approximately rationalisable for a continuous utility u.

explore this connection further, and show that acyclicity of P^* is necessary and sufficient for a dataset to be rationalisable with an interval order maximisation.

3.3 Continuous approximate utility maximisation

Given the generality of our setup, Theorem 1 does not specify any particular properties of the function u that rationalises the data as in (5). Specifically, the utility need not be continuous, even in very well-behaved choice environments.

Consider the dataset $\mathcal{O} = \{(A^1, x^1), (A^2, x^2)\}$ depicted in Figure 1, where $X = \mathbb{R}^2_+$. Suppose that the correspondence Γ is given by $\Gamma(x) := \{y \in X : y \gg x\}$. One can easily verify that it satisfies Assumptions 1 and 2.[20] We claim that the dataset is rationalisable with approximate utility maximisation. Since $x^2 \in A^1$ and $\Gamma(x^1) \cap A^2 \neq \emptyset$ imply $x^1 R^* x^2$ and $x^2 P^* x^1$, respectively, the relation P^* is acyclic. Therefore, by Theorem 1, there is a utility u and a positive threshold δ that rationalise the data as in (5).[21] However, any such function u must be discontinuous at x^2.

Indeed, since $x^2 P^* x^1$ and $x^1 R^* x^2$ imply $u(x^2) > u(x^1)$ and $u(x^1) \geq u(x^2) - \delta(x^2)$, respectively, the two relations hold simultaneously only if $\delta(x^2) > 0$. Take any sequence of alternatives $\{z^n\}$ converging to x^2 such that $x^2 \in \Gamma(z^n)$, i.e., $x^2 \gg z^n$, for all n. Since Γ-monotonicity requires that $u(x^2) - u(z^n) > \delta(x^2)$, for all n, the utility function u would be continuous only if $\delta(x^2) = 0$, yielding a contradiction.

[20] Clearly, it obeys Assumption 1. By Lemma 4.1 in Peleg (1970), it satisfies Assumption 2.
[21] Since $x^1 R^* x^2 P^* x^1$, this set is *not* rationalisable with an *exact* utility maximisation.

Given the importance of continuity for establishing non-emptiness of the set $c(A)$ or eliciting preferences from limited data (see, e.g., Chambers et al., 2020), it is desirable to determine conditions under which there is a continuous rationalisation.

Assumption 3 (Continuity)**.** Suppose that X is a locally compact and separable metric space, and the correspondence $\Gamma : X \rightrightarrows X$ satisfies the following conditions:[22]

(i) The set $\{(y, x) : y \in \Gamma(x)\} \cup \{(x, x) : x \in X\}$ is closed.

(ii) For any compact set $Z \subseteq X$, the lower inverse $\Gamma^{\ell}(Z) := \{x \in X : \Gamma(x) \cap Z \neq \emptyset\}$ of the correspondence Γ is compact.

These continuity restrictions on X and Γ are sufficient to prove that acyclicity of P^* is equivalent to a *continuous* approximate rationalisation.

Proposition 3. *Let X and Γ satisfy Assumptions 1 and 3, and the menu A be compact, for all $(A, x) \in \mathcal{O}$. The relation P^* is acyclic if, and only if, there is a continuous utility u that rationalises $\mathcal{O}$ as in (5), for some positive threshold function δ.* [Proof]

The necessity part is immediate, since it is independent of any ancillary assumptions. To prove the converse, we apply Levin's Theorem (see Levin, 1983 or the appendix in Nishimura et al., 2017) to show that acyclicity of P^* is sufficient for existence of a *continuous* utility u such that both $x \in \Gamma(y)$ and xP^*y imply $u(x) > u(y)$, for any $x, y \in X$. The rest follows from Proposition 2.

The example in Figure 1 can not be rationalised with approximate utility maximisation for a continuous function u precisely because the correspondence Γ violates Assumption 3, specifically part (ii).[23] In the next section we apply our results to study imperfect discrimination and propose natural examples of Γ that obey this assumption.

[22] Whenever the space X is compact, any closed-valued and upper hemi-continuous correspondence Γ satisfies Assumption 3. In such a case, the set $\{(y, x) : y \in \Gamma(x)\}$ is closed by the Closed Graph Theorem (see Theorem 17.11 in Aliprantis and Border, 2006). Moreover, by Lemma 17.4 in Aliprantis and Border (2006), the lower inverse $\Gamma^{\ell}(Z)$ is closed, for any closed set Z. Since the space X is compact, this suffices for the second part of the assumption to be satisfied.

[23] Indeed, Assumption 3(ii) is critical. Suppose that the set $\Gamma(x)$ consists of all $y \in X$ such that $y > x$, rather than $y \gg x$. Although it obeys Assumption 3(i), it is not sufficient to rationalise the dataset in Figure 1 with a continuous approximate utility maximisation that is Γ-monotone. This can be shown by applying our previous argument. However, in the Online supplement we show that this dataset is approximately rationalisable with an upper semi-continuous utility.

4 Imperfect discrimination

Insofar our examples of Γ-monotonicity were focused on relatively strong forms of dominance. Whether the set $\Gamma(x)$ consisted of elements y that were strictly greater than x with respect to $>$ (or $\gg$) in an ℓ-dimensional commodity space, or first order stochastically dominant in the space of lotteries, the ranking imposed by the researcher required that the objectively better alternative was always chosen over the inferior one, even when the difference between the options was infinitesimal. Although such a property may be desirable from the theoretical and normative standpoint, there is growing empirical evidence suggesting that it may be too demanding.

Sippel (1997) conducted an experimental study of consumer choice within the standard Afriat-like framework, in which subjects were making purchases of different consumption goods subject to various budget constraints. Even though the individuals were incentivised to exhaust their budgets, a significant number of them failed to do so, thus, directly violating that "more is better".[24] More recently, Nielsen and Rehbeck (2020) reported direct violations of first order stochastic dominance in choices over lotteries. In their experimental study, 90% of subjects expressed the desire to obey first order stochastic dominance, yet 85% of those violated the condition at least once in the subsequent choice experiment. This is in line with Dembo et al. (2021), who find that violations of the expected utility theory are caused predominantly by inconsistencies of choice with first order stochastic dominance, rather than the independence axiom.

We do not postulate that, based on this evidence, one should abandon the idea of monotonicity entirely. Undoubtedly, strong forms of monotonicity have a great normative appeal. However, when studying choices that involve small stakes, like in experimental settings or day-to-day consumption decisions, it may be sensible to consider weaker forms of monotonicity that admit some level of insensitivity to small differences among alternatives and describe the observed behaviour more accurately.

One reason for which individuals may violate strong forms of monotonicity is imper-

[24] Unlike Sippel (1997), other experimental studies that employ an Afriat-like setup restrict choices to the budget line *only*. Therefore, the design makes it impossible to observe direct violations of strict monotonicity. See, e.g., Harbaugh et al. (2001), Andreoni and Miller (2002), Choi et al. (2007), Fisman et al. (2007), Andreoni and Sprenger (2012a,b), Ahn et al. (2014), Choi et al. (2014), Halevy et al. (2018), Echenique et al. (2019), Zrill (2020), Cappelen et al. (2021), and Dembo et al. (2021). Similarly, one could not violate stochastic dominance directly in Feldman and Rehbeck (2020).

fect discrimination studied extensively in the psychophysics literature.[25] Following the empirical evidence, individuals perceive differences between intensities of a physical stimulus (e.g., light, touch, sound) only if they are significantly (noticeably) different. The well-established Weber-Fechner law stipulates that people perceive the change whenever the ratio of intensities exceeds a particular constant, the so-called *just-noticeable difference*, that is specific to the stimulus. The same law applies to human perception of numerosities, which may be more relevant to Economics.[26] In the remainder of this section we employ this idea to consumer choice, by allowing the individual to imperfectly discriminate among bundles unless they are sufficiently different.

Example 1. Let $X \subseteq \mathbb{R}^\ell_+$ be the ℓ-dimensional commodity space. Following the idea of Weber and Fechner, one may define the correspondence Γ by

$$\Gamma(x) \ := \ \Big\{y \in X : y \geq x \text{ and } y_i \geq \lambda_i x_i + a_i, \text{ for some } i = 1, \dots, \ell\Big\},$$

for some numbers $\lambda_i \geq 1$ and $a_i > 0$, for all $i = 1, \dots, \ell$. Given our interpretation of Γ, this is to say that bundle y is objectively better than x is it contains more of each good, and *significantly* more of some good i. Here, the number a_i is the absolute change in the amount of good i that is sufficient to perceive the difference, while λ_i captures the relative change.[27] See also Figure 2 (left) on page 20.

A simplification of this idea is discussed in Dziewulski (2020), where the correspondence Γ is given by $\Gamma(x) := \big\{\lambda' x : \lambda' \geq \lambda\big\}$, for some $\lambda > 1$. That is, if the relative amount of *all* goods in x increases by at least λ then the enlarged bundle is always chosen over x. We discuss the importance of this formulation in Section 7.3.

Example 2. Let X be a space of probability measures over $S \subseteq \mathbb{R}$. In such a case, one could model imperfect discrimination with the correspondence

$$\Gamma(x) \ := \ \Big\{y \in X : y \geq_F x \text{ and } d(x, y) \geq \lambda\Big\},$$

for some $\lambda > 0$, where $\geq_F$ denotes the first order stochastic dominance and d is a metric on X. Hence, a lottery x is dominated by any probability distribution that dominates it

[25] See, e.g., Gescheider (1997) for a handbook treatment of this topic.
[26] See Dehaene (2008) for a survey of this literature.
[27] Setting $a_i > 0$ allows subjects to experience insensitivity between none and an infinitesimal amount of the good i. This follows the empirical evidence suggesting that the noticeable increase in the relative intensity of a physical stimulus is hyperbolic with respect to the initial intensity. See Gescheider (1997).

in the stochastic sense and is sufficiently distant from it. Alternatively, one could explore an idea based on Rubinstein (1988) and impose conditions on ratios of probabilities and prizes that are sufficient to distinguish between two lotteries.[28]

The examples presented above and the formulation in Dziewulski (2020) satisfy Assumptions 1 and 3.[29] By Proposition 3, for each of these correspondences and any dataset $\mathcal{O}$, acyclicity of the direct revealed strict preference relation P^* is necessary and sufficient for the data to be rationalisable with an approximate utility maximisation for a *continuous* function u and some threshold δ. This allows for a general analysis of choices that exhibit weaker forms of monotonicity and non-transitive indifferences resulting from imperfect discrimination, as suggested in Armstrong (1950).

The examples above require some comment. It is critical to point out that Weber-Fechner law is a *statistical property* attributed to distributions of choices. In contrast, Examples 1 and 2 specify imperfect discrimination in deterministic terms. Since our setup assumes limited choice data, it would be impossible to falsify any form of stochastic choice and, therefore, the statistical definition of just-noticeable difference would be vacuous. To obtain any testable implications, we have to interpret this law literally. As a result, the above examples specify an upper bound for the insensitivity of the subject, rather than the average just-noticeable difference studied in psychophysics.

It is imperative to remind the reader that the notion of Γ-monotonicity is imposed on the choice model c, rather than the corresponding utility u that we identify with the true preferences. Although the choices of the decision-maker may be subject to some degree of insensitivity to differences among alternatives, this does *not* preclude the function u from being increasing in a stronger sense, capturing the normative affinity for even the most infinitesimal increases in consumption or improvement of odds in a gamble. This contrasts with the exact utility maximisation, where monotonicity of choice and preferences

[28] Rubinstein (1988) discusses a model of choice over simple lotteries that assign a probability p to a monetary prize and $(1-p)$ to receiving nothing. Roughly speaking, the lotteries are considered to be distinguishable if the ratios of either prizes or probabilities exceed a particular constant.

[29] Note that the correspondence in Example 1 can be represented as $\Gamma(x) := \big\{y : F(x,y) \geq 0\big\}$, where $F(x,y) := \min\big\{\min\{y_i - x_i : i = 1,\dots,\ell\}, \max\{y_i - \lambda_i x_i - a_i : i = 1,\dots,\ell\}\big\}$ is a continuous function. Hence, its graph $\operatorname{Gr}\Gamma = F^{-1}\big([0,\infty)\big)$ is closed, which suffices for Assumption 3(i) to hold. Take any sequence $\{x^k\}$ in $\Gamma^\ell(Z)$ that converges to some x, and the corresponding sequence $\{y^k\}$ in Z such that $F(x^k, y^k) \geq 0$, for all k. Since Z is compact, we may assume that $\{y^k\}$ converges to some y, without loss. Clearly, we have $F(x,y) \geq 0$, and so $x \in \Gamma^\ell(Z)$. Thus, the set $\Gamma^\ell(Z)$ is closed. Given that $\Gamma^\ell(Z)$ is also bounded, it must be compact. Therefore, Assumption 3(ii) is satisfied. By an analogous argument one can show that the other examples satisfy the condition as well.

always coincide (recall Proposition 1). Although Γ-monotonicity of approximate utility maximisation requires that $x \in \Gamma(y)$ implies $u(x) > u(y)$, the converse is no longer true.[30] This separation of preferences and choice has a footing in empirical evidence. The aforementioned experiment in Nielsen and Rehbeck (2020) shows a systematic inconsistency between the decision-theoretic rules that individuals consider to be desirable (including first order stochastic dominance) and their actual choices. The authors conclude that "even though individuals may want to follow [stochastic dominance], this may not translate to them making choices consistent with it even when given an explanation of how the axiom applies to a decision problem."[31] The descriptive and normative aspects of consumer choice are disjoint, which is consistent with our model. Nevertheless, despite this separation, it is still possible to elicit the utility u from the observable data and make welfare statements, as we show in the following sections.

We do not deny that, with sufficient care and attention, individuals are capable of identifying which alternative is strictly greater in the particular sense, even when the difference between them is infinitesimal. Rather, we hypothesise that when it comes to every-day consumption or choices with small stakes in experimental settings, decisions may follow intuitive judgements based on approximate quantities involved.[32] Violations of strict monotonicity could also result from unobserved mental or physical costs of switching from an inferior to a dominant alternative, as subjects may not find the change worthwhile, unless it yields sufficiently more utility. This is in line with the idea of satisficing by Simon (1947). In Section 7.1 we discuss the close relationship between this model and approximate utility maximisation.

5 Recovering preferences from almost optimal choices

In Theorem 1 we established the necessary and sufficient conditions under which a dataset $\mathcal{O}$ is rationalisable with approximate utility maximisation. Here, we turn to an alternative question: Assuming that the observed choices are generated by such a model, how can

[30] In the Online supplement we discuss the distinction between Γ-monotonicity of approximate utility maximisation and strict monotonicity of the corresponding utility u.

[31] Nielsen and Rehbeck (2020, p. 19). Similar distinction between choice and preferences of a decision-maker are discussed in Mandler (2005), Nishimura (2018), and Nishimura and Ok (2020).

[32] See Dehaene (2008).

we estimate the true preferences u of the individual?[33]

Throughout this section we take a dataset $\mathcal{O}$ and a correspondence Γ as the premise. Moreover, we assume that $\mathcal{O}$ is rationalisable with a Γ-monotone approximate utility maximisation as in (5), for some unobserved utility u and threshold δ. By P^* we denote the directly revealed strict preference relation, defined in Section 3.

It is convenient to refer to the notion of the *revealed strict preference relation* P, i.e., the transitive closure of P^*. Formally, we have xPy whenever there is a sequence of alternatives $z^1, z^2, \ldots, z^n$ in X such that $z^1 = x$, $z^n = y$, and

$$z^1 P^* z^2,\ z^2 P^* z^3,\ \ldots\ ,\ z^{n-2} P^* z^{n-1},\ \text{and } z^{n-1} P^* z^n.$$

Obviously, the directly revealed relation P^* is acyclic if and only if its transitive closure P is irreflexive, i.e., we have *not* xPx, for all $x \in X$. Therefore, by Theorem 1, this is equivalent to the data being rationalisable as in (5).

We proceed with our discussion on recoverability of preferences. Take an arbitrary alternative $x \in X$, not necessarily observed in the dataset. First, we are interested in evaluating the set of all alternatives that are strictly inferior to x with respect to the latent utility u. Define the *revealed worst set* by

$$RW(x) \ := \ \Big\{ y \in X : xPy;\ \text{or } x \in \Gamma(y);\ \text{or } x \in \Gamma(z) \text{ and } zPy,\ \text{for some } z \in X \Big\}.$$

Consider the dataset depicted in Figure 2 (right), where the consumption space is $X = \mathbb{R}^2_+$ and each observed menu is given by $A^t = \big\{ y \in X : p^t \cdot y \leq 1 \big\}$, for some prices $p^t \in \mathbb{R}^2_{++}$ and $t = 1, 2, 3$. In addition, let the correspondence Γ be given as in Example 1, for some $\lambda_1, \lambda_2 \geq 1$ and $a_1, a_2 > 0$. The lower gray area represents the set $RW(x)$ for the bundle x. Indeed, the set contains all elements y such that $x \in \Gamma(y)$. In particular, this includes x^1. Thus, any alternative that is revealed strictly inferior to x^1 must also belong to $RW(x)$, i.e., any $y \in X$ such that $x^1 P^* y$ or $x^1 P^* x^2 P^* y$.

Analogously, one can define the *revealed preferred set* as

$$RP(x) \ := \ \Big\{ y \in X : yPx;\ \text{or } y \in \Gamma(x);\ \text{or } y \in \Gamma(z) \text{ and } zPx,\ \text{for some } z \in X \Big\}.$$

Revisit Figure 2, where the revealed preferred set is represented by the top shaded area. Clearly, the set includes every element y that belongs to $\Gamma(x)$. Moreover, since

[33] Our question is analogous to the one discussed in Varian (1982), Halevy et al. (2017), or Nishimura et al. (2017) regarding the exact utility maximisation.

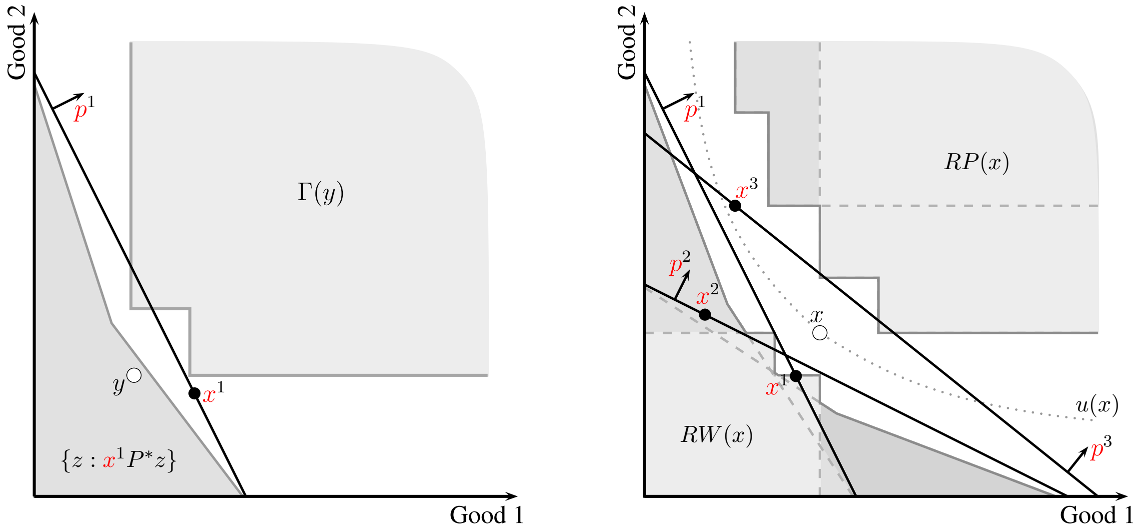

Figure 2: The set of alternatives that are directly revealed strictly inferior to x^1 (left), and the revealed worst and revealed preferred sets for x (right).

$\Gamma(x) \cap A^3 \neq \emptyset$, we have $x^3 P^* x$. Hence, both x^3 and any $y \in \Gamma(x^3)$ are also in the set. The next result follows immediately from the definitions of the two sets.

Corollary 1. *For any $x, y \in X$, we have $y \in RW(x)$ if and only if $x \in RP(y)$.*

The two sets are essential in estimating the unobserved preferences from the data. In particular, $RP(x)$ provides a tight bound for the set of all alternatives that are strictly preferable to x with respect to the unobserved utility u. We formalise this below. Before stating the result, define $V_u(x) := \{y \in X : u(y) > u(x)\}$, for any utility function $u : X \to \mathbb{R}$. Moreover, let $NRW(x)$ be the complement of $RW(x)$.

Theorem 2. *For any alternative $x \in X$ and utility u that rationalises the dataset $\mathcal{O}$ as in (5) for some threshold δ, we have $RP(x) \subseteq V_u(x) \subseteq NRW(x)$. Moreover, under Assumptions 1 and 2, we have $y \notin RP(x)$ and $y \neq x$ only if $\mathcal{O}$ is rationalisable as in (5) for a utility u satisfying $u(x) > u(y)$ and some threshold δ.* [Proof]

Following Proposition 2, it is clear that $y \in RP(x)$ implies $u(y) > u(x)$, for any utility that rationalises $\mathcal{O}$ as in (5). Analogously, since $y \in RW(x)$ implies $u(y) < u(x)$, the set $V_u(x)$ must be nested in $NRW(x)$. Thus, the sets $RW(x)$ and $RP(x)$ are indeed bounds for the true preferences, since any indifference curve intersecting x must belong to the complement of $RW(x) \cup RP(x)$. In addition, the bounds are *tight* in the following sense: If y is not in $RP(x)$, one can always rationalise the data with an approximate maximisation of some utility u that ranks x strictly above y. Thus, it is impossible to improve the estimate without excluding some preferences that could explain the data. By Corollary 1, an analogous result holds for the set $RW(x)$.

Remark 1. Theorem 2 can be extended to the class of *continuous* utility functions u. Suppose that the menu A is compact, for all observations $(A, x) \in \mathcal{O}$. Through a combination of the arguments supporting Proposition 3 and Theorem 2, one can show that under Assumptions 1 and 3, we have $y \notin RP(x)$ and $y \neq x$ only if the dataset is rationalisable as in (5) with a *continuous* utility u satisfying $u(x) > u(y)$.

6 Robust welfare comparisons

Theorem 2 shows how to estimate the unobserved utility u that rationalises the data as in (5). This allows us to partially rank alternatives in X with respect to the true yet unobserved preferences of the individual. However, when performing welfare analysis, it is much more natural to compare sets of alternatives rather than particular options. For example, when evaluating different tax structures, one is interested in ranking budget sets the consumer would face under each regime. Here, we introduce and characterise an intuitive ordering over menus that allows us to make meaningful, data-driven welfare statements under approximate utility maximisation.

The main difficulty in evaluating welfare within our framework follows from the separation of choice — guided by the model in (5), from the agent's well-being — summarised by the utility u. Suppose that choices of the consumer are determined with the correspondence $c(A) := \{x \in A : u(x) + \delta(y) \geq u(y), \text{ for all } y \in A\}$, for some utility u and threshold δ. For the time being, we assume that the two functions are known. For any two menus $A, A' \subseteq X$, the set A' is *preferred* to A if any choice from A' is strictly preferable to any choice from A with respect to the utility u. Formally, for any $x \in c(A')$ and $y \in c(A)$, we have $u(x) > u(y)$.[34] Although we accept that the agent may choose options that are not maximising their utility u exactly, due to imperfect discrimination, imprecision, or satisficing, we identify welfare with their true preferences.

Since the dataset $\mathcal{O}$ is finite and incomplete, it can be supported by multiple functions u, δ. We address this issue by focusing on a robust comparison over menus. As in the

[34] Unlike for the exact utility maximisation, for any menu A the elements of the choice set $c(A)$ may be assigned different values of the utility u under approximate utility maximisation.

previous section, we take a dataset $\mathcal{O}$ and a correspondence Γ as the premise. Moreover, we assume that the set $\mathcal{O}$ is rationalisable as in (5), for some unobserved utility u and threshold δ. For any two menus $A, A' \in \mathcal{A}$ (not necessarily observed in the data), we say that A' is *robustly preferred* to A, if for *any* functions u, δ that rationalise $\mathcal{O}$ as in (5), the set A' is preferred to A in the sense defined above. Therefore, for any model of approximate utility maximisation that is consistent with the the data, any choice from A' has to be superior to any choice from A — with respect to the utility u.

In this section we characterise this robust ordering over menus and show how to determine these comparisons using the revealed preference relations P^* and P defined earlier. First, we need to introduce some notation. For an arbitrary menu A (not necessarily observed in the dataset), we identify the set of all possible choices from A that would be consistent with the set of observations $\mathcal{O}$. For any menu $A \in \mathcal{A}$, let

$$S(A) \;:=\; \Big\{ y \in A : \Gamma(y) \cap A = \emptyset; \text{ and } xPy \text{ implies } \Gamma(x) \cap A = \emptyset, \text{ for all } x \in X \Big\},$$

where P denotes the revealed strict preference relation induced by $\mathcal{O}$, i.e., it is the transitive closure of the directly revealed strict preference relation P^*.

Proposition 4. *Under Assumptions 1 and 2, for any $A \in \mathcal{A}$ and $y \in A$, the hypothetical dataset $\mathcal{O} \cup \big\{(A, y)\big\}$ is rationalisable as in (5) if, and only if, $y \in S(A)$.* [Proof]

The above result is of interest in itself. It states that the set $S(A)$ contains all (both within and out-of-sample) choices from the menu A that are consistent with the dataset $\mathcal{O}$. Hence, it contains all predictions consistent with the data. This is particularly useful when performing a counterfactual analysis.

Given the generality of our setup, we can not guarantee that the set $S(A)$ is non-empty, for all $A \in \mathcal{A}$. However, it is easy to show that this is always true when A is finite. Similarly, whenever the dataset $\mathcal{O}$ is rationalisable as in (5) for a continuous function u, then $S(A)$ is non-empty for any compact A. See Remark 2 below.

Consider the main result of this section.

Theorem 3. *Under Assumptions 1 and 2, for any menus $A, A' \in \mathcal{A}$, the set A' is robustly preferable to A if, and only if, for any $y \in A$, either* (i) $\Gamma(y) \cap A' \neq \emptyset$; *or* (ii) $\Gamma(z) \cap A' \neq \emptyset$ *and zPy, for some $z \in X$; or* (iii) xPy, *for all $x \in S(A')$.* [Proof]

The robust comparison over menus is partial and, in general, does not rank any two sets of alternatives. In fact, unlike for the exact utility maximisation, it is possible that two menus A, A' are unordered, even when A is a subset of A'. Since choices are not necessarily maximising the utility u, there may be alternatives in A that are strictly preferable to some options selected from the set A'. However, once A' dominates A in the robust sense, any alternative that would be selected from A is inferior to any alternative chosen from A', even when the individual fails to maximise their utility.

Remark 2. Suppose that Assumptions 1 and 3 are satisfied, and the menu A is compact, for each observation $(A, x) \in \mathcal{O}$. Since, by assumption, the set $\mathcal{O}$ is rationalisable with a Γ-monotone approximate utility maximisation, Proposition 3 guarantees that the corresponding utility u is continuous, without loss. In particular, this suffices for the set $S(A)$ to be non-empty, for any compact menu A. By combining the arguments supporting Proposition 3 and Theorem 3, one can also prove the following result: *For any compact menus $A, A' \in \mathcal{A}$, the set A' is robustly preferable to A if, and only if, for any* continuous *utility u and some threshold δ that rationalise $\mathcal{O}$ as in (5), the set A' is preferred to A in the sense defined at the beginning of this section.*

7 Applications

We conclude this paper with a few applications of our main results.

7.1 Satisficing

As it was pointed out in Section 4, approximate utility maximisation can be interpreted in terms of satisficing à la Simon (1947), where the individual selects alternatives that are "good enough" with respect to some criterion. Formally, a choice correspondence $c : \mathcal{A} \rightrightarrows X$ represents the *satisficing* behaviour if there is a utility $u : X \to \mathbb{R}$ such that $x \in c(A)$ and $u(y) \geq u(x)$ implies $y \in c(A)$, for any $y \in A$ and $A \in \mathcal{A}$

One can easily verify that approximate utility maximisation is a special case of satisficing. Indeed, suppose that $c(A) = \{x \in A : u(x) + \delta(y) \geq u(y), \text{ for all } y \in A\}$, for some utility u and threshold function δ. Since $x \in c(A)$ implies $u(x) \geq u(z) - \delta(z)$, for all $z \in A$, then $u(y) \geq u(x)$ only if $y \in c(A)$, for any $y \in A$. By Theorem 1, it immediately follows that, for any correspondence Γ satisfying Assumptions 1, 2 and any dataset $\mathcal{O}$, acyclicity of the revealed preference relation P^* is sufficient for the observations to be rationalisable with a Γ-monotone satisficing behaviour.

The converse is not true. Suppose that $X = \{a, b, c, d\}$ and the correspondence Γ is given by $\Gamma(a) = \{b\}$, $\Gamma(c) = \{d\}$, and $\Gamma(b) = \Gamma(d) = \emptyset$, which satisfies Assumptions 1 and 2, since X is finite. Consider the dataset $\mathcal{O}$ consisting of observations $(\{a, d\}, a)$ and $(\{b, c\}, c)$. Given that both $\Gamma(c) \cap \{a, d\}$ and $\Gamma(a) \cap \{b, c\}$ are non-empty, we have aP^*c and cP^*a. Therefore, the set $\mathcal{O}$ is not rationalisable as in (5). Nevertheless, it is consistent with a Γ-monotone satisficing behaviour.[35]

Although, in general, the testable implications of the two models differ, there is an important class of choice environments in which they are indistinguishable.

Proposition 5. *Take a dataset $\mathcal{O}$ and a correspondence Γ obeying Assumptions 1 and 2, and suppose that $\Gamma(y) \cap A \neq \emptyset$ implies $y \in A$, for all $(A, x) \in \mathcal{O}$ and $y \in X$. Then, the set $\mathcal{O}$ is rationalisable with a Γ-monotone approximate utility maximisation as in (5) if, and only if, it is rationalisable with a Γ-monotone satisficing behaviour.* [Proof]

The additional assumption in the proposition is satisfied in various choice environments. Suppose that $X = \mathbb{R}^\ell_+$ and the menu A is downward comprehensive, for each observation $(A, x) \in \mathcal{O}$.[36] Specifically, this holds within the classical consumer demand setting à la Afriat and in the general framework of Forges and Minelli (2009). In addition, if $x \in \Gamma(y)$ implies $y \geq x$, for all $x, y \in X$, then the assumption in Proposition 5 is always satisfied. Importantly, this class of correspondences contains the examples discussed in the previous sections, including $\Gamma(x)$ that consists of elements $y \in X$ such that $y > x$ (or $y \gg x$); the correspondence Γ in Example 1; or the specification in Dziewulski (2020) given by $\Gamma(x) := \{\lambda' x : \lambda' \geq \lambda\}$, for some $\lambda > 1$. Therefore, in each of these cases, the testable implications of approximate utility maximisation and satisficing are equivalent. We explore this observation further in Section 7.3.

7.2 State-contingent consumption under risk

Proposition 2 implies that any utility u that is consistent with the correspondence Γ and the directly revealed strict preference relation P^* can rationalise the set of observations

[35] For example, take any utility u such that $u(b) > u(a) > u(d) > u(c)$, and a Γ-monotone correspondence c satisfying $c(\{a, d\}) = \{a\}$ and $c(\{b, c\}) = \{b, c\}$.

[36] A set $A \subseteq X \subseteq \mathbb{R}^\ell$ is *downward comprehensive* if $x \in A$ and $y \leq x$ implies $y \in A$, for all $y \in X$, where $\geq$ denotes the coordinate-wise ordering. Recall footnote 11.

as in (5), for some threshold function δ. Since, in general, the correspondence Γ and relation P^* induce an infinite number of binary comparisons, verifying whether a utility u is consistent with both of them may be difficult. In this subsection we apply Proposition 2 to an important class of preferences over state-contingent consumption under risk. We extend the method of *generalised restriction of infinite domains* (GRID) by Polisson et al. (2020) to show that within a broad class of models checking for consistency with Γ and P^* can be restricted to a finite number of comparisons.

Suppose there is a finite set of states $S = \{1, 2, \ldots, \ell\}$ and the probability π_s of each state $s \in S$ is known to the consumer and the observer. The contingent consumption space is $X = \mathbb{R}^\ell_+$, where the s'th entry x_s of the vector $x \in X$ denotes the consumption level in the state $s \in S$. As previously, a set of observations is given by $\mathcal{O} = \big\{(A^t, x^t) : t \in T\big\}$, where $x^t \in A^t$ denotes the state-contingent consumption bundle selected from the menu A^t. Here we require that A^t is bounded, for all $t \in T$.

Choices over contingent consumption were studied in, e.g., Choi et al. (2007, 2014) Ahn et al. (2014), Halevy et al. (2018), Zrill (2020), Cappelen et al. (2021), and Dembo et al. (2021).[37] In these particular experiments, the subjects were making multiple choices from budget lines $A^t = \big\{y \in \mathbb{R}^\ell_+ : p^t \cdot y = 1\big\}$, given some state-contingent prices $p^t \in \mathbb{R}^\ell_{++}$, for all $t \in T$, making it similar to the classic Afriat-like setup. Nevertheless, the following approach is applicable to arbitrary bounded menus.

In this subsection we employ Theorem 1 and Proposition 2 to provide an easy-to-apply test for approximate utility maximisation as in (5), where the corresponding function u is given by a particular formulation of preference under risk. Many such utilities can be represented as $u(y) := F\big(v(y_1), v(y_2), \ldots, v(y_\ell)\big)$, where $v : \mathbb{R}_+ \to \mathbb{R}_+$ is a Bernoulli function and $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is an aggregator. For example, given the state probabilities π_s, for all $s \in S$, the expected utility formulation is

$$u(y) \;=\; F\big(v(y_1), v(y_2), \ldots, v(y_\ell)\big) \;=\; \sum_{s=1}^{\ell} \pi_s v(y_s), \tag{6}$$

where the aggregator F takes the form $F(z) = \sum_{s=1}^{\ell} \pi_s z_s$, for any $z \in \mathbb{R}^\ell_+$. Similarly, the model of rank dependent expected utility in Quiggin (1982) and disappointment aversion preferences in Gul (1991) admit such a representation for a particular aggregator F. See Section I.D in Polisson et al. (2020) for details.

[37] See also Gneezy and Potters (1997) and Hey and Pace (2014).

For simplicity, we focus on the case where the aggregator F is the same across all observations $t \in T$. Clearly, this is not without loss of generality. For example, when studying the expected utility as in (6), this would require that state probabilities π_s remain constant across all observations. Nevertheless, our result can be easily generalised to accommodate a variable aggregator F, as we show in the Online supplement. Below we extend Theorem 1 in Polisson et al. (2020) to approximate utility maximisation over state-contingent consumption. Let $\mathcal{X} := \{0\} \cup \big\{x_i^t : \text{ for some } i = 1, \dots, \ell \text{ and } t \in T\big\}$ be the finite set of all consumption levels observed in the dataset and 0.

Proposition 6. *For any dataset $\mathcal{O} = \big\{(A^t, x^t) : t \in T\big\}$ with bounded menus A^t, for all $t \in T$, a continuous and strictly increasing aggregator F,*[38] *and a correspondence Γ such that $y \in \Gamma(x)$ implies $y > x$, for any $x, y \in X$, the following statements are equivalent.*

(i) There is a strictly increasing Bernoulli function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mathcal{O}$ is rationalisable as in (5) for the utility $u(y) := F\big(v(y_1), v(y_2), \dots, v(y_\ell)\big)$ and some threshold function δ. Moreover, v is upper-semicontinuous without loss.[39]

(ii) There is a strictly increasing function $\bar{v} : \mathcal{X} \to \mathbb{R}_+$ satisfying

$$F\big(\bar{v}(x_1), \bar{v}(x_2), \dots, \bar{v}(x_\ell)\big) \;>\; F\big(\bar{v}(y_1), \bar{v}(y_2), \dots, \bar{v}(y_\ell)\big),$$

for any $x, y \in \mathcal{X}^\ell$ such that xP^*z and $z \geq y$, for some $z \in X$.

We postpone the proof until the Online supplement. In order to verify if the data is rationalisable as in (5) for a utility $u(y) := F\big(v(y_1), v(y_2), \dots, v(y_\ell)\big)$, for some Bernoulli function v, it suffices to check if it is rationalisable over the finite grid $\mathcal{X}^\ell$. This simplifies the test significantly and, in the case of expected utility, rank dependent expected utility, and disappointment aversion, reduces it to a linear programming problem.[40]

Proposition 6 crucially depends on the assumption that $y \in \Gamma(x)$ implies $y > x$, for all $x, y \in X$. Clearly, this is satisfied by the mappings in Example 1 as well as the correspondence discussed in Dziewulski (2020). Otherwise, we impose no restrictions on Γ. In particular, neither of the assumptions presented in Section 3 are required for this result to hold. Whenever the condition is violated, Proposition 6 is not applicable, and consistency of the function u with Γ and P^* has to be verified differently.

[38] A function $F : X \to \mathbb{R}$ defined over $X \subseteq \mathbb{R}^\ell$ is *strictly increasing* if $x > y$ implies $F(x) > F(y)$.
[39] The function v is *upper semi-continuous* if the set $\big\{y \in \mathbb{R}_+ : v(y) \geq a\big\}$ is closed, for any number a.
[40] This can be shown by re-purposing the approach in Sections I.B and I.D in Polisson et al. (2020).

As it was pointed out in Section 3, it is not always possible to approximately rationalise a set of observations with a continuous function u. Similarly, Proposition 6 does not guarantee that the Bernoulli function v and, thus, $y \to F\big(v(y_1), v(y_2), \ldots, v(y_\ell)\big)$ are continuous. In the Online supplement, we show that whenever the menu A^t is compact, for each observation $t \in T$, and the correspondence Γ satisfies Assumption 3(ii), one can assume that the function v is continuous, without loss of generality.

7.3 A universal measure of departures from rationality

It is a common observation in numerous empirical studies that choices of individuals are not consistent enough to be congruent with the exact utility maximisation. As a result, a significant part of the revealed preference literature is devoted to measures that evaluate how severely the data departs from the classic notion of rationality. Arguably, the most common of them all is the *critical cost-efficiency index* (CCEI, also known as *Afriat's efficiency index*), introduced in Afriat (1973) to evaluate violations of utility maximisation within the standard consumer demand framework.[41]

Throughout this subsection, let $X = \mathbb{R}^\ell_+$ and, for any observation $t \in T$, the corresponding menu be given by $A^t = \big\{y \in \mathbb{R}^\ell_+ : p^t \cdot y \leq p^t \cdot x^t\big\}$, for some prices $p^t \in \mathbb{R}^\ell_{++}$. The dataset $\mathcal{O} = \big\{(A^t, x^t) : t \in T\big\}$ is rationalisable for an *efficiency parameter* $e \in [0,1]$ (a number) if there is a strictly increasing utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that

$$e(p^t \cdot x^t) \geq p^t \cdot y \ \text{ implies } \ u(x^t) \geq u(y),$$

for all $t \in T$. That is, the observed bundle x^t is preferable to all alternatives that are cheaper than the fraction e of x^t, given prices p^t, for all $t \in T$. Clearly, for $e = 1$, this coincides with the exact utility maximisation. CCEI is equal to the supremum over all efficiency parameters e for which the above condition holds.

Dziewulski (2020) provides a behavioural foundation of this measure. Namely, CCEI is the reciprocal of the infimum over all numbers $\lambda > 1$ for which the dataset is rationalisable as in (5), for a strictly increasing utility u and threshold δ, when the correspondence Γ is given by $\Gamma(x) := \big\{\lambda' x : \lambda' \geq \lambda\big\}$. Therefore, CCEI attributes violations of the exact utility maximisation to the particular form of imperfect discrimination. This equivalence

[41] Among others, CCEI was employed in Sippel (1997), Harbaugh et al. (2001), Andreoni and Miller (2002), Choi et al. (2007), Fisman et al. (2007), Ahn et al. (2014), Choi et al. (2014), Cherchye et al. (2017), Echenique et al. (2019), Cherchye et al. (2020), Dembo et al. (2021), and Cappelen et al. (2021).

result is established for the general specification of the utility function u. However, in some applications CCEI is used to measure departures from a specific formulation of the utility u. For example, Cherchye et al. (2017, 2020) apply an analogous measure to a multiperson household model; Polisson et al. (2020) evaluate CCEI for departures from expected utility, rank dependent utility, and disappointment aversion; Cappelen et al. (2021) and Dembo et al. (2021) employ it to estimate deviations from the model of probabilistic sophistication and expected utility maximisation. We apply Proposition 2 to extend the equivalence result to an arbitrary sub-class of utilities.

Proposition 7. *For any dataset $\mathcal{O}$, any strictly increasing utility $u : \mathbb{R}^\ell_+ \to \mathbb{R}$, and any number $e^* \in (0, 1]$, the following statements are equivalent.*

(i) For any $e < e^*$, if $e(p^t \cdot x^t) \geq p^t \cdot y$ then $u(x^t) \geq u(y)$, for any $y \in \mathbb{R}^\ell_+$ and $t \in T$.

(ii) For any $\lambda > 1/e^*$, the dataset $\mathcal{O}$ is rationalisable as in (5) for the correspondence $\Gamma(x) := \{\lambda' x : \lambda' \geq \lambda\}$, the utility u, and some threshold δ. [Proof]

It immediately follows that for *any* strictly increasing utility u, the CCEI under which the function supports the data is equal to the reciprocal of the infimum over all λs for which approximate maximisation of the same utility rationalises the data as in (5), for the correspondence Γ. In addition, Proposition 5 implies the following corollary.

Corollary 2. *For any dataset $\mathcal{O}$ the corresponding CCEI is equal to the the infimum over all numbers $\lambda > 1$ for which the observations are rationalisable with a Γ-monotone model of satisficing, where $\Gamma(x) := \{\lambda' x : \lambda' \geq \lambda\}$.*

Most measures in the existing literature focus on departures from rationality within the classic consumer demand framework à la Afriat. This includes Afriat (1973), Varian (1990), Echenique et al. (2011), Dean and Martin (2016), Echenique et al. (2018, 2020), Allen and Rehbeck (2020b, 2021), and de Clippel and Rozen (2021). In addition, the measure developed in Echenique et al. (2018, 2020) is applicable only to a particular class of additively separable models of time preference and choice under risk and uncertainty. Allen and Rehbeck (2021) focus solely on departures from quasilinear utility maximisation. In contrast, Apesteguia and Ballester (2015) develop an index that is suitable for environments beyond Afriat's, but their approach is applicable only to choices over finite domains. Finally, a versatile take on this issue was proposed in Houtman and Maks (1985), yet it lacks an appealing economic interpretation.

Our observations allow for a natural extension of CCEI not only to arbitrary utilities u, but also to empirical settings beyond the classic demand framework à la Afriat. Given any dataset $\mathcal{O}$ with arbitrary menus A, one can establish the severity of departures from rationality with the least $\lambda > 1$ for which the data can be rationalised as in (5) for the correspondence $\Gamma(x) := \{\lambda' x : \lambda' \geq \lambda\}$. Unlike the original interpretation, our take on CCEI does not depend on the linear specification of the budget sets and allows for a meaningful comparison across different choice environments. Moreover, given the results in Section 5 and 6, it permits not only to measure departures from rationality, but also to estimate the true preferences of the individual, make out-of sample predictions, and evaluate welfare when the data is not consistent with utility maximisation.

A Appendix

Here we present proofs that were omitted in the main body of the paper. Before stating the argument supporting Theorem 1, it is convenient to prove Proposition 2.

A.1 Proof of Proposition 2

To prove implication (ii) $\Rightarrow$ (i), suppose that the function u rationalises the observations as in (5), for some threshold δ. If $x \in \Gamma(y)$ then $u(x) > u(y) + \delta(x) \geq u(y)$, where the first inequality follows from Γ-monotonicity of c, and the second is implied by $\delta(x) \geq 0$. Whenever xP^*y, there is some menu A such that $(A, x) \in \mathcal{O}$ and $\Gamma(y) \cap A \neq \emptyset$. In particular, we have $u(x) \geq u(z) - \delta(z) > u(y)$, for any $z \in \Gamma(y) \cap A$.

To prove the converse, take any utility u specified as in the proposition and define the function δ as follows: If $y \in A^t$, for some $t \in T$, then

$$\delta(y) \;:=\; \max\Big\{ \max\big\{u(y) - u(x^t), 0\big\} : t \in T \text{ and } y \in A^t \Big\}.$$

Otherwise, let $\delta(y) = 0$. Hence, the function is well-defined and positive.

First, we claim that the resulting choice correspondence c is Γ-monotone. Take any menu A and $x \in c(A)$. Towards contradiction, suppose there is some $y \in \Gamma(x) \cap A \neq \emptyset$. By assumption, this implies that $u(y) > u(x)$. If $\delta(y) = 0$, then $x \in c(A)$ implies $u(x) \geq u(y) - \delta(y) > u(x)$, yielding a contradiction. Alternatively, suppose that $\delta(y) > 0$. By construction, this holds only if $y \in \Gamma(x) \cap A^t$ or, equivalently, $x^t P^* x$, for some $t \in T$.

Thus, we have $u(x^t) > u(x)$, for any such $t \in T$. In particular, for some $t \in T$,

$$u(x) + \delta(y) \;=\; u(x) + u(y) - u(x^t) \;<\; u(y),$$

which contradicts that $x \in c(A)$, and so the correspondence c is Γ-monotone.

To prove that c rationalises $\mathcal{O}$, take any observation (A^t, x^t) and $y \in A^t$. By construction of the threshold δ, we have $\delta(y) \geq \max\{u(y) - u(x^t), 0\} \geq u(y) - u(x^t)$. This suffices for x^t to be an element of $c(A^t)$, which concludes the proof.

A.2 Proof of Theorem 1

We prove that statement (i) implies (ii). Given Proposition 2, it suffices to show that there is a utility $u : X \to \mathbb{R}$ such that if $x \in \Gamma(y)$ or xP^*y then $u(x) > u(y)$. Before we proceed with our argument, we introduce an auxiliary result.

Lemma A.1. *Let $\succ$ be an irreflexive, transitive binary relation, and $D \subseteq X$ be a countable set $D \subseteq X$ such that $x \succ y$ implies either $z \not\succ x$ and $z \succ y$, or $x \succ z$ and $y \not\succ z$, for some $z \in D$. Then, there is a function $u : X \to \mathbb{R}$ such that $x \succ y$ implies $u(x) > u(y)$.*

Proof. Take any countable set D specified as in the proposition and enumerate its elements so that $D = \{z^k\}_{k=1}^{\infty}$. For any $x \in X$ define the set $M(x) := \{k : x \succ z^k\}$ and $N(x) := \{k : z^k \succ x\}$. One can easily show that $x \succ y$ implies $M(y) \subseteq M(x)$ and $N(y) \supseteq N(x)$, for any $x, y \in X$. Moreover, at least one of the set inclusions must be strict. Indeed, if $x \succ z$ and $y \not\succ z$, for some $z \in D$, then $M(y) \subset M(x)$, while $z \not\succ x$ and $z \succ y$ implies $N(y) \supset N(x)$. Define the function $u : X \to \mathbb{R}$ by

$$u(x) \;:=\; \sum_{k \in M(x)} 2^{-k} - \sum_{k \in N(x)} 2^{-k},$$

which is well-defined and, by our previous observation, consistent with $\succ$. $\square$

We continue with the main proof. We assume throughout that the relation P^* is acyclic, thus, the revealed preference relation P is irreflexive.[42] To simplify the notation, define the binary relation $\rhd := \{(x, y) : x \in \Gamma(y)\}$, which is the graph of the correspondence Γ. Under Assumption 1, it is irreflexive and transitive.

Lemma A.2. *If xPy and $\Gamma(y) \subseteq \Gamma(z)$ then xPz, for any $x, y, z \in X$.*

[42] Recall that P is the transitive closure of P.

Proof. Suppose that xPy. By definition, there is some $t \in T$ such that $x^t P^* y$ or, equivalently, $\Gamma(y) \cap A^t \neq \emptyset$. Since $\Gamma(y) \subseteq \Gamma(z)$ implies $\Gamma(z) \cap A^t \neq \emptyset$, we have $x^t P^* z$. If $x = x^t$, we are done. Otherwise, we have xPx^t and $x^t P^* z$, which implies xPz. □

The next lemma is an immediate corollary to the previous result.

Lemma A.3. *Under Assumption 1, if $x \; P \; y \vartriangleright z$ then xPz, for any $x, y, z \in X$.*[43]

Indeed, by definition, we have $y \vartriangleright z$ if and only if $y \in \Gamma(z)$. By Assumption 1(ii), this implies $\Gamma(y) \subseteq \Gamma(z)$ and, thus, xPz (by Lemma A.2). Let $\succ$ denote the transitive closure of $P \cup \vartriangleright$. The next lemma is critical to our arguments.

Lemma A.4. *Under Assumption 1, the relation $\succ$ is equal to $P \cup \vartriangleright \cup (\vartriangleright \circ P)$.*[44]

Proof. Clearly, $P \cup \vartriangleright \cup (\vartriangleright \circ P)$ is a subset of $\succ$. To prove the converse, suppose that $x \succ y$. Since P and $\vartriangleright$ are transitive, this holds in four instances: Either (i) xPy or (ii) $x \vartriangleright y$. Alternatively, (iii) there are elements $z^1, z^2, \ldots, z^n$ in X such that

$$x \;=\; z^1 \; P \; z^2 \;\vartriangleright\; z^3 \; P \; z^4 \;\vartriangleright\; \ldots \;\vartriangleright\; z^{n-2} \; P \; z^{n-1} \;\vartriangleright\; z^n \;=\; y.$$

By Lemma A.3 and transitivity of P, this implies xPy. Finally, (iv) we have

$$x \;=\; z^1 \;\vartriangleright\; z^2 \; P \; z^3 \;\vartriangleright\; z^4 \; P \; \ldots \; P \; z^{n-2} \;\vartriangleright\; z^{n-1} \; P \; z^n \;=\; y,$$

for some alternatives $z^1, z^2, \ldots, z^n$ in X. Similarly, by Lemma A.3 and transitivity of P this implies that $x \vartriangleright z^2 \; P \; y$. It is straightforward to show that any other case can be reduced to one of the four above. This concludes our proof. □

Lemma A.5. *Under Assumption 1, the transitive closure $\succ$ of $P \cup \vartriangleright$ is irreflexive.*

Proof. Given Lemma A.4 and the fact that P and $\vartriangleright$ are irreflexive, it suffices to show that $\vartriangleright \circ P$ is irreflexive. Suppose that $x \vartriangleright z \; P \; x$, for some $x, z \in X$. Since this is equivalent to $z \; P \; x \vartriangleright z$, and so zPz (by Lemma A.3), it contradicts that P is irreflexive. □

Below we present a useful extension of Lemma A.2.

Lemma A.6. *Under Assumption 1, if $x \succ y$ and $\Gamma(y) \subseteq \Gamma(z)$ then $x \succ z$, for any $x, y, z \in X$.*

[43] Throughout, we denote $x \; P \; y \vartriangleright z$ in place of xPy and $y \vartriangleright z$, for any $x, y, z \in X$.
[44] We denote $(\vartriangleright \circ P) := \{(x, y) : x \vartriangleright z \; P \; y, \text{ for some } z \in X\}$.

Proof. Suppose that $x \succ y$. By Lemma A.4, this holds is three instances. If xPy, then $\Gamma(y) \subseteq \Gamma(z)$ implies xPz, by Lemma A.2. Following the same argument, if $x \rhd z' \; P \; y$, for some $z' \in X$, and $\Gamma(y) \subseteq \Gamma(z)$ then $x \rhd z' \; P \; z$. Finally, we have $x \rhd y$ only if $x \in \Gamma(y) \subseteq \Gamma(z)$, which implies $x \rhd z$. Either way, we obtain $x \succ z$. □

Consider the final auxiliary result.

Lemma A.7. *Under Assumptions 1 and 2, there is a countable set $D \subseteq X$ such that $x \succ y$ implies either $z \not\succ x$ and $z \succ y$, or $x \succ z$ and $y \not\succ z$, for some $z \in D$.*

Proof. Take any set $D \subseteq X$ specified as in Assumption 2 and define $D' := D \cup \{x^t\}_{t \in T}$, which is countable (since T is finite). Suppose that $x \succ y$. By Lemma A.4, it suffices to consider three instances. If xPy then $z \not\succ x$ and $z \succ y$, for $z = x \in D'$. Whenever $x \rhd z \; P \; y$, for some $z \in X$, then $z \not\succ x$ and $z \succ y$, where $z \in D'$.

Finally, suppose that $x \rhd y$. By Assumption 2, there is some $z \in D$ such that either (i) $\Gamma(x) \subseteq \Gamma(z)$ and $z \in \Gamma(y)$, or (ii) $x \in \Gamma(z)$ and $\Gamma(z) \subseteq \Gamma(y)$. Suppose that (i) is true. Clearly, we have $z \succ y$. We show that $z \not\succ x$ by contradiction. By Lemma A.6, if $z \succ x$ and $\Gamma(x) \subseteq \Gamma(z)$ then $z \succ z$, which contradicts that $\succ$ is irreflexive. Analogously, we show that condition (ii) implies $x \succ z$ and $y \not\succ z$. □

By Lemmas A.5 and A.7, the relation $\succ$ is irreflexive, transitive, and satisfies the separability condition. By Lemma A.1, there is a utility $u : X \to \mathbb{R}$ such that $x \succ y$ implies $u(x) > u(y)$. In particular, if $x \in \Gamma(y)$ or xP^*y then $u(x) > u(y)$. By Proposition 2, there is a threshold δ for which the dataset $\mathcal{O}$ is rationalisable as in (5).

A.3 Proof of Proposition 3

Implication ($\Leftarrow$) follows from Theorem 1, since it is true independently of ancillary assumptions. To show the converse, suppose that X is a locally compact and separable metric space. Moreover, for any $(A, x) \in \mathcal{O}$, let the menu A be compact. Finally, the directly revealed strict preference relation P^* is acyclic, thus, its transitive closure P is irreflexive. Define $\rhd := \big\{(x, y) : x \in \Gamma(y)\big\}$, which is irreflexive and transitive whenever Assumption 1 holds. By $\succ$ we denote the transitive closure of $P \cup \rhd$.

To prove the result, we show that $\succeq := \succ \cup \big\{(x, x) : x \in X\big\}$ is a closed-continuous preorder, i.e., a closed, reflexive, and transitive binary relation. We then apply Levin's

Theorem to prove that there is a continuous function $u : X \to \mathbb{R}$ that extends $\succ$, i.e., $x \succ y$ implies $u(x) > u(y)$. See the original result in Levin (1983), or the appendix in Nishimura et al. (2017). The rest follows from Proposition 2.

We proceed with the proof. It is straightforward to show that $\succeq$ is a preorder. We show that it is closed-continuous via two lemmas.

Lemma A.8. *Under Assumption 3, the revealed strict preference relation P is compact.*

Proof. We begin the proof by showing that the *directly* revealed strict preference relation P^* is compact. Indeed, we have $P^* = \bigcup_{t \in T} \big\{(x^t, y) : \Gamma(y) \cap A^t \neq \emptyset\big\}$. Since the menu A^t is compact, for each $t \in T$, Assumption 3 implies that so is $\big\{(x^t, y) : \Gamma(y) \cap A^t \neq \emptyset\big\}$. Given that T is finite, the relation P^* is compact as well.

We show that P is compact by induction. Let $E^0 = P^*$ and

$$E^n := \bigcup_{t \in T} \Big\{(x^t, y) : x^t E^{n-1} x^s \text{ and } x^s P^* y, \text{ for some } s \in T\Big\},$$

for any $n \geq 1$. Since E^0 and P^* are compact, the set E^n is a finite union of compact sets, thus, itself compact, for any $n \geq 1$. Hence, the set $P = \bigcup_{n=0}^{|T|} E^n$ is compact. $\square$

The above result implies the following observation.

Lemma A.9. *Under Assumptions 1 and 3, the relation $\succeq$ is closed.*

Proof. By Lemma A.3, it suffices to show that $P \cup \rhd \cup (\rhd \circ P) \cup \big\{(x, x) : x \in X\big\}$ is closed. By Assumption 3, the union $\rhd^* := \rhd \cup \big\{(x, x) : x \in X\big\}$ is closed. Moreover, Lemma A.10 implies that P is compact. Following Lemma C in Nishimura et al. (2017), the relation $\rhd^* \circ P = (\rhd \circ P) \cup P$ is closed, thus, so is $(\rhd \circ P) \cup P \cup \rhd^* = P \cup \rhd \cup (\rhd \circ P) \cup \big\{(x, x) : x \in X\big\}$. This completes the proof. $\square$

Since $\succeq$ is a closed-continuous preorder, Levin's Theorem guarantees that there is a continuous function $u : X \to \mathbb{R}$ such that $x \succ y$ implies $u(x) > u(y)$. In particular, both $x \in \Gamma(y)$ and $x P^* y$ imply $u(x) > u(y)$. The rest follows from Proposition 2.

A.4 Proof of Theorem 2

We prove only the second part. As previously, denote $\rhd := \big\{(x, y) : x \in \Gamma(y)\big\}$, which is transitive and irreflexive under Assumption 1, and let $\succ$ be the transitive closure of $P \cup \rhd$. By Lemmas A.4 and A.5, $\succ$ is irreflexive and equal to $P \cup \rhd \cup (\rhd \circ P)$.

Let $\hat{\succ}$ denote the transitive closure of $\succ \cup \{(x, y)\}$.

Lemma A.10. *Under Assumption 1, the binary relation $\hat{\succ}$ is irreflexive.*

Proof. Since $y \notin RW(x)$ and $\succ = P \cup \rhd \cup (\rhd \circ P)$, we have $y \not\succ x$, by definition of $RW(x)$. We consider two cases. If $x \succ y$ then $\hat{\succ} = \succ$, which is irreflexive. Otherwise, the relation $\hat{\succ}$ fails to be irreflexive only if $z \succ x$ and $y \succ z$, for some $z \in X$. However, this implies $y \succ x$, which contradicts our initial claim. $\square$

The following lemma shows that $\hat{\succ}$ satisfies the separability condition.

Lemma A.11. *Under Assumptions 1 and 2, there is a countable set $D \subseteq X$ such that $z' \hat{\succ} z$ implies either $z' \hat{\succ} z''$ and $z \hat{\not\succ} z''$, or $z'' \hat{\not\succ} z'$ and $z'' \hat{\succ} z$, for some $z \in D$*

Proof. Take any set D specified in Assumption 2 and define $D' := D \cup \{x^t\}_{t \in T} \cup \{x, y\}$, which is countable. Suppose that $z' \hat{\succ} z$. If $z' \not\succ z$, then either $z' = x$, $z = y$, or $z' \succ x$ and $y \succ z$. Clearly, the required condition is satisfied for $z'' = x$ or $z'' = y$.

Alternatively, suppose that $z' \succ z$. By Lemma A.4, this holds in three instances. If $z' P z$, let $z'' = z' \in D'$. Since $\hat{\succ}$ is irreflexive, it must be that $z'' \hat{\not\succ} z'$ and $z'' \hat{\succ} z'$. Similarly, if $z' \rhd z'' \, P \, z$, for some $z'' \in X$, then $z'' \hat{\not\succ} z'$ and $z'' \hat{\succ} z'$, where $z'' \in D'$.

Finally, suppose that $z' \rhd z$. By Assumption 2, either (i) $\Gamma(z') \subseteq \Gamma(z'')$ and $z'' \in \Gamma(z)$, or (ii) $z' \in \Gamma(z'')$ and $\Gamma(z'') \subseteq \Gamma(z)$, for some $z'' \in D$. Whenever (i) is true, then $z'' \rhd z$, and so $z'' \hat{\succ} z$. Towards contradiction, suppose that $z'' \hat{\succ} z'$. If $z'' \succ z'$, then $\Gamma(z') \subseteq \Gamma(z'')$ implies $z'' \succ z''$ (by Lemma A.6), yielding a contradiction. Similarly, if $z'' \succ x$ and $y \succ z'$, then $\Gamma(z') \subseteq \Gamma(z'')$ implies $y \succ z'' \succ x$, contradicting that $y \not\succ x$. Thus, we have $z'' \hat{\not\succ} z'$ and $z'' \hat{\succ} z$. Analogously, if (ii) holds, then $z' \hat{\succ} z''$ and $z \hat{\not\succ} z''$, for some $z'' \in D$. $\square$

By Lemmas A.10, A.11, and A.1, there is utility $u : X \to \mathbb{R}$ such that $z' \hat{\succ} z$ implies $u(z') > u(z)$. Therefore, both $z' \in \Gamma(z)$ and $z' P^* z$ imply $u(z') > u(z)$, as well as $u(x) > u(y)$. The rest follows from Proposition 2.

A.5 Proof of Proposition 4

Denote $\tilde{\mathcal{O}} = \mathcal{O} \cup \{(A, y)\}$ and let $\tilde{P}^*, \tilde{P}$ be the revealed relations induced by $\tilde{\mathcal{O}}$. In particular, we have $P \subseteq \tilde{P}$. Clearly, the set $\tilde{\mathcal{O}}$ is rationalisable only if $y \in S(A)$. Otherwise, $\Gamma(y) \cap A \neq \emptyset$ would imply $y \tilde{P}^* y$, while $x P y$ and $\Gamma(x) \cap A \neq \emptyset$ would imply $x \tilde{P} x$. Either way, this would contradict that the relation $\tilde{P}$ is irreflexive.

We prove the converse by contradiction. Suppose that $y \in S(A)$, but the set $\tilde{\mathcal{O}}$ is not rationalisable. Given that $\mathcal{O}$ is rationalisable by assumption and, thus, the relation P is irreflexive, this holds only if $y\tilde{P}y$, which can take place in two instances: If (i) $y\tilde{P}^*y$ then $\Gamma(y) \cap A \neq \emptyset$; if (ii) $y\tilde{P}^*x$ and xPy, then $\Gamma(x) \cap A \neq \emptyset$ and xPy, for some $x \in X$. Either way, this contradicts that $y \in S(A)$ and completes our proof.

A.6 Proof of Theorem 3

Implication ($\Leftarrow$) is straightforward. Indeed, for any u, δ that rationalise $\mathcal{O}$ as in (5), and any $x \in c(A')$, each of the conditions (i)–(iii) would imply $u(x) > u(y)$, for all $y \in A$.

We prove the converse by contradiction. Suppose that A' is robustly preferred to A, but there is some $y \in A$ that violates each of the conditions (i)–(iii). In particular, there is some $x \in S(A')$ such that *not* xPy. Take any such x and denote $\tilde{\mathcal{O}} := \mathcal{O} \cup \big\{(A', x)\big\}$. By Proposition 4, the set $\tilde{\mathcal{O}}$ is rationalisable as in (5). Let $\tilde{P}$ denote the revealed strict preference relation induced by $\tilde{\mathcal{O}}$, and $R\tilde{W}(x)$ be the corresponding revealed worst set for x. We claim that $y \notin R\tilde{W}(x)$. Indeed, it can not be that $x\tilde{P}y$, since this would imply one of the conditions (i)–(iii). Similarly, if $x \in \Gamma(y)$ then $\Gamma(y) \cap A' \neq \emptyset$. Finally, suppose that $x \in \Gamma(z)$ and $z\tilde{P}y$, for some $z \in X$. Since $x \in \Gamma(z)$ implies $\Gamma(z) \cap A' \neq \emptyset$, we have $x\tilde{P}^*z$. Moreover, if $z\tilde{P}y$, then either zPy, or zPx and $x\tilde{P}y$. However, this implies that either y satisfies condition (ii), or $x\tilde{P}x$, contradicting that $x \in S(A')$.

Since $y \notin R\tilde{W}(x)$, Theorem 2 guarantees that there are functions u, δ that rationalise $\tilde{\mathcal{O}}$ as in (5) and $u(y) > u(x)$. This contradicts that A' is robustly preferred to A.

A.7 Proof of Proposition 5

We only prove the "if" part. Suppose that the set $\mathcal{O}$ is rationalisable with a Γ-monotone model c of satisficing behaviour. There is a function $u : X \to \mathbb{R}$ such that $x \in c(A)$ and $u(y) \geq u(x)$ implies $y \in c(A)$, for any $A \in \mathcal{A}$ and $y \in A$.

We claim that xP^*y implies $u(x) > u(y)$. By definition, we have $\Gamma(y) \cap A \neq \emptyset$, for some $(A, x) \in \mathcal{O}$. By assumption, this implies $y \in A$. Since c is Γ-monotone and rationalises the data, it must be that $u(x) > u(y)$. If not, then $x \in c(A)$ and $u(y) \geq u(x)$ would imply $y \in c(A)$, contradicting that c is Γ-monotone.

By the above observation, the directly revealed strict preference relation P^* must be acyclic. Therefore, Theorem 1 guarantees that the dataset $\mathcal{O}$ is rationalisable with a

Γ-monotone approximate utility maximisation as in (5).

A.8 Proof of Proposition 7

To show that (i) implies (ii), take any $\lambda > 1/e^*$. Following (i), there is some e such that $\lambda > 1/e > 1/e^*$, and $e(p^t \cdot x^t) \geq p^t \cdot y$ implies $u(x^t) \geq u(y)$. By monotonicity of u, this guarantees that $e(p^t \cdot x^t) > p^t \cdot y$ only if $u(x^t) > u(y)$. Note that, $\Gamma(y) \cap A^t \neq \emptyset$, and so $x^t P^* y$, if and only if $p^t \cdot x^t \geq p^t \cdot (\lambda y)$. Since $1/\lambda < e$, this suffices for $x^t P^* y$ to imply $u(x^t) > u(y)$. Moreover, monotonicity of u implies $u(\lambda y) > u(y)$, for any $\lambda > 1$. By Proposition 2, the data is rationalisable as in (5) for the utility u.

To show the converse, take any $e < e^*$. By (ii), there is some number λ such that $e < 1/\lambda < e^*$. By the argument above and Proposition 2, we know that $p^t \cdot x^t \geq p^t \cdot (\lambda y)$ implies $u(x^t) > u(y)$. Since $e < 1/\lambda$, this suffices for (i) to hold.

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Online supplement to: "A comprehensive revealed preference approach to approximate utility maximisation"

Paweł Dziewulski*

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Abstract

This supplement contains additional results related to Dziewulski (2021). These notes should be read in conjunction with the main paper.

Here we include results that complement the findings presented in the main paper. In Section B.1 we discuss an alternative, more constructive take on Theorem 1 based on liner programming methods. Specifically, this allows us to determine properties of the utility function u that are not testable in certain choice environments. In Section B.2, we explore the relation between approximate utility maximisation and interval orders. Finally, in Section B.3 we state proofs of the results presented in Section 7.2 of the main paper, regarding choice over state-contingent consumption under risk.

Throughout this supplement we employ the notation introduced in the main paper. In order to keep our exposition compact, we say that a dataset $\mathcal{O}$ is *approximately rationalisable*, if there is a utility u and threshold function δ that rationalise the observations in the sense specified in Theorem 1, given a correspondence Γ.

* Department of Economics, University of Sussex, Jubilee Building, Falmer, Brighton BN1 9SL, United Kingdom. Email: P.K.Dziewulski@sussex.ac.uk.

B.1 The constructive approach

Theorem 1 and Proposition 3 in the main paper establish equivalence between acyclic direct revealed strict preference P^* and approximate utility maximisation in a general setting. However, the lack of a trackable constructive argument makes it difficult to establish any properties of the functions u and δ that rationalise the data. Here, we impose additional structure on our framework to present an alternative take on our results.

We assume throughout that the Euclidean consumption space $X = \mathbb{R}^\ell_+$ is endowed with the natural product order $\geq$.[1] We focus on choices from *generalised budget sets*, as in Forges and Minelli (2009). That is, for any observation $(A^t, x^t) \in \mathcal{O}$, there is a well-defined and strictly increasing function $f^t : X \to \mathbb{R}$ such that

$$A^t \;=\; \Big\{y \in X : f^t(y) \leq 0\Big\}.$$[2]

As pointed out in Section 2 of the main paper, this includes the classic consumer choice setup discussed in Afriat (1967), Diewert (1973), and Varian (1982). Finally, we impose the following assumption on the correspondence Γ.

Assumption B.1. *For all $x \in X$, the set $\Gamma(x)$ is non-empty. Moreover, if $y \in \Gamma(x)$ and z is in the closure of $\Gamma(y)$ then $z' < z$, for some $z' \in \Gamma(x)$.*

It is critical for our constructive argument that the correspondence Γ is well-defined. The second part of the assumption imposes a specific form of monotonicity on the correspondence. In particular, the condition implies $x \notin \Gamma(x)$, for all $x \in X$.[3]

Remark B.1. It will become clear from our exposition that all the results presented in this section can be generalised to any space X that is endowed with some preorder $\geq_X$, and where X is either finite or bounded from below with respect to the ordering $\geq_X$, i.e., there is some $y \in X$ such that $x \in X$ implies $x \geq_X y$. This includes the space of probability distributions over $S = \mathbb{R}_+$, endowed with the first order stochastic dominance.

[1] We denote $x \geq y$ if $x_i \geq y_i$, for all $i = 1, \ldots, \ell$, then $x \geq y$. The relation is *strict*, and denoted by $x > y$, if $x \geq y$ and $x \neq y$. Finally, we have $x \gg y$ if $x_i > y_i$, for all $i = 1, \ldots, \ell$.

[2] If A^t can be represented as $A^t = \big\{y \in X : f_i^t(y) \leq 0,$ for all $i = 1, \ldots, n\big\}$ for multiple well-defined and strictly increasing functions $f_i^t : X \to \mathbb{R}$, for all $i = 1, \ldots, n$, then $A^t = \big\{y \in X : f^t(y) \leq 0\big\}$, where the function $f^t(y) := \max\big\{f_i^t(y) : i = 1, \ldots, \ell\big\}$ is well-defined and strictly increasing.

[3] Clearly, if $x \in \Gamma(x)$ then, for any z' in the closure of $\Gamma(x)$, there would have to be some $z \in \Gamma(x)$ such that $z' > z$, which yields a contradiction.

B.1.1 Constructive rationalisation

Given our discussion in Section 3 of the main paper, it is clear that whenever the set of observations $\mathcal{O}$ is rationalisable with approximate utility maximisation then the corresponding directly revealed strict preference relation P^* is acyclic. This observation follows directly from the definition of the relation and is independent of ancillary assumptions. In this subsection we provide a constructive argument supporting the converse. We propose a utility u and a threshold δ that rationalise the data in this sense.

We begin our construction by defining the function $g^t : X \to \mathbb{R}$ as

$$g^t(x) \;:=\; \begin{cases} f^t(x) & \text{if } f^t(x) \leq 0; \\ f^t(x) + \epsilon & \text{otherwise;} \end{cases} \tag{B.1}$$

for some $\epsilon > 0$, where f^t is the well-defined and strictly increasing function that represents the menu A^t, for all $t \in T$. Thus, the function g^t is also well-defined and strictly increasing. Moreover, we have $g^t(y) \leq 0$ if and only if $y \in A^t$, for all $t \in T$. Define function $h^t : X \to \mathbb{R}$ as $h^t(x) := \inf\left\{g^t(y) : y \in \Gamma(x)\right\}$, for all $t \in T$.

Lemma B.1. *For all $t \in T$, we have $h^t(x) \leq 0$ if, and only if, $\Gamma(x) \cap A^t \neq \emptyset$.*

Proof. If $y \in \Gamma(x) \cap A^t \neq \emptyset$ then $0 \geq f^t(y) = g^t(y) \geq h^t(x)$. To show the converse, suppose that $h^t(x) \leq 0$ and $\Gamma(x) \cap A^t = \emptyset$. In particular, for any $y \in \Gamma(x)$, we have $g^t(y) = f^t(y) + \epsilon > \epsilon$. This implies $h^t(x) \geq \epsilon > 0$, yielding a contradiction. □

It is easy to show that the revealed relation P^* is acyclic if and only if, for any cycle $\mathcal{C} = \left\{(a, b), (b, c), \ldots, (z, a)\right\}$ in $T \times T$, we have $\Gamma(x^s) \cap A^t = \emptyset$, for some $(t, s) \in \mathcal{C}$. By the lemma above, this is equivalent to $h^t(x^s) > 0$, for some $(t, s) \in \mathcal{C}$. In the following result we show that this suffices to solve a particular linear system.

Lemma B.2. *The relation P^* is acyclic only if there are numbers $(\phi^t)_{t\in T}$ and strictly positive numbers $(\mu^t)_{t\in T}$ such that $\phi^s < \phi^t + \mu^t h^t(x^s)$, for all $t, s \in T$.*

The system of inequalities presented in this lemma is very similar to the so-called *Afriat inequalities*. However, it requires for all the inequalities to be strict. The result itself is analogous to Lemma A.2 in Dziewulski (2020) and can be proven using an argument based on the one in Section 2 of Fostel et al. (2004). Before stating the main result of this section, we introduce one final lemma.

Lemma B.3. *Under Assumption B.1, if* $y \in \Gamma(x)$ *then* $h^t(y) > h^t(x)$*, for any* $t \in T$*.*

Proof. By monotonicity of g^t and definition of h^t, there is some z in the closure of $\Gamma(y)$ such that $h^t(y) \geq g^t(z)$. Following Assumption B.1, there is $z' \in \Gamma(x)$ satisfying $z' < z$. Since g^t is strictly increasing, we obtain $h^t(y) \geq g^t(z) > g^t(z') \geq h^t(x)$. □

The main theorem of this section presents a particular utility u and a threshold function δ that approximately rationalise the set of observations $\mathcal{O}$.

Theorem B.1. *Under Assumption B.1, the dataset* $\mathcal{O}$ *is approximately rationalisable with the utility* $u : X \to \mathbb{R}$*, given by* $u(y) := \min\left\{\phi^t + \mu^t h^t(y) : t \in T\right\}$*, and the threshold* $\delta : X \to \mathbb{R}_+$*, given by* $\delta(y) := \max\left\{0; \max\{u(y) - \mu^t g^t(y) - u(x^t) : t \in T\}\right\}$*, for any numbers* $(\phi^t)_{t\in T}$ *and strictly positive numbers* $(\mu^t)_{t\in T}$ *specified in Lemma B.2.*

Proof. Clearly, both u and δ are well-defined. Let the function $v : X \to \mathbb{R}$ be given by $v(y) := \min\left\{u(y); \min\{u(x^t) + \mu^t g^t(y) : t \in T\}\right\}$. Thus, $u(y) \geq v(y)$, for all $y \in X$.

We claim that $y \in \Gamma(x)$ implies $v(y) > u(x)$. Indeed, by Lemma B.3, we have

$$u(x) \;=\; \min\left\{\phi^t + \mu^t h^t(x) : t \in T\right\} \;<\; \min\left\{\phi^t + \mu^t h^t(y) : t \in T\right\} \;=\; u(y),$$

since μ^t is strictly positive, for all $t \in T$. On the other hand, by construction of the numbers $(\phi^t)_{t\in T}$, $(\mu^t)_{t\in T}$, we have $\phi^t < u(x^t)$, for all $t \in T$. This implies

$$u(x) \;=\; \min\left\{\phi^t + \mu^t h^t(x) : t \in T\right\} \;<\; \min\left\{u(x^t) + \mu^t g^t(y) : t \in T\right\},$$

since $y \in \Gamma(x)$ implies $h^t(x) \leq g^t(y)$. The two observations guarantee $u(x) < v(y)$.

Since $g^t(y) = f^t(y) \leq 0$ implies $v(y) \leq u(x^t) + \mu^t f^t(y) \leq u(x^t)$, we have $u(x^t) \geq v(y)$, for all $y \in A^t$ and $t \in T$. Given that $v(y) = u(y) - \delta(y)$, the proof is complete. □

The next corollary follows immediately from the above construction.

Corollary B.1. *Suppose that the function* f^t *representing the menu* A^t *is continuous, for each observation* $t \in T$*. Moreover, let the correspondence* $\Gamma : X \to X$ *be given by* $\Gamma(x) := \left\{y \in X : y > x\right\}$*.*[4] *Then, the dataset* $\mathcal{O}$ *is approximately rationalisable for an upper semi-continuous utility* u *and some threshold* δ*, without loss of generality.*[5]

[4] Clearly, the same result holds for $\Gamma(x) := \left\{y \in X : y \gg x\right\}$.

[5] The function u is *upper semi-continuous* if the set $\left\{x \in X : u(x) \geq a\right\}$ is closed, for any number a.

Proof. Since Γ satisfies Assumption B.1, Theorem B.1 guarantees that the dataset $\mathcal{O}$ is rationalisable with the utility function $u(y) := \min\{\phi^t + \mu^t h^t(y) : t \in T\}$, and a threshold δ. By strict monotonicity of f^t, the function h^t is equal to $h^t(y) = f^t(y)$, if $f^t(y) < 0$, and $h^t(y) = f^t(y) + \epsilon$ otherwise, for some $\epsilon > 0$. Clearly, it is upper semi-continuous. In particular, the function $y \to [\phi^t + \mu^t h^t(y)]$ is upper semi-continuous, for any number ϕ^t and strictly positive number μ^t, for all $t \in T$. Since the *min* operator preserves upper semi-continuity, the function u is upper semi-continuous. □

Recall the example in Figure 1 in the main paper. As we pointed out earlier, the directly revealed strict preference relation induced by these choices is acyclic. Given that linear budget sets can be represented with a strictly increasing and continuous function f^t, by the above corollary, this particular dataset can be approximately rationalised with an upper semi-continuous utility function u, without loss of generality.

B.1.2 Limits to testability

Proposition 3 of the main paper specifies conditions, under which the utility u that approximately rationalises the data is continuous, without loss of generality. Hence, in such environments, continuity is *not* testable. The construction of the utility u and the threshold δ in Theorem B.1 allow us to further investigate properties of these functions and identify choice environments $(\mathcal{O}, \Gamma)$ for which they are not falsifiable.

Throughout this subsection, we take the dataset $\mathcal{O}$ and correspondence Γ as the premise. In addition, we assume that the set of observations is approximately rationalisable. We begin our discussion by presenting sufficient conditions under which the data can be explained with continuous functions u and δ.

Assumption B.2. *The* lower bound *correspondence* $\partial\Gamma_\downarrow : X \rightrightarrows X$*, given by*

$$\partial\Gamma_\downarrow(x) \;:=\; \Big\{y \in \Gamma(x) : z < y \textit{ implies } z \notin \Gamma(x), \textit{ for all } z \in X\Big\},$$

is well-defined, compact-valued, and continuous.[6]

Recall the correspondence $\Gamma(x) := \{y \in X : y > x\}$. In this case, the lower bound $\partial\Gamma_\downarrow(x)$ is empty, for all $x \in X$, which violates the above assumption.[7]

[6] See Definition 17.4 in Aliprantis and Border (2006) for a definition of a *continuous* correspondence.
[7] The same applies to the correspondence $\Gamma(x) := \{y \in X : y \gg x\}$.

Proposition B.1. *Under Assumptions B.1 and B.2, if the function f^t representing the menu A^t is continuous, for all $t \in T$, then the dataset $\mathcal{O}$ is approximately rationalisable, for a continuous utility u and a continuous threshold δ, without loss of generality.*

Proof. Define function $g^t : X \to \mathbb{R}$ as $g^t(x) := f^t(x)$, which is well-defined, strictly increasing, and continuous, for all $t \in T$. Moreover, let the function $h^t : X \to \mathbb{R}$ be given as in Section B.1.1, for all $t \in T$. By continuity and strict monotonicity of g^t, and compactness of $\partial\Gamma_\downarrow(x)$, we have $h^t(x) = \min\{g^t(y) : y \in \partial\Gamma_\downarrow(x)\}$. Since the function g^t and the correspondence $\partial\Gamma_\downarrow$ are continuous, Berge Maximum Theorem guarantees that h^t is continuous (see, e.g., Theorem 17.31 in Aliprantis and Border, 2006).

We claim that $h^t(x) \leq 0$ if, and only if, $\Gamma(x) \cap A^t \neq \emptyset$. Clearly, if $y \in \Gamma(x) \cap A^t$ then $0 \geq f^t(y) = g^t(y) \geq h^t(x)$. Conversely, if $h^t(x) \leq 0$ then $g^t(y) = f^t(y) \leq 0$, for some $y \in \partial\Gamma_\downarrow(x) \subseteq \Gamma(x)$, which can be satisfied only if $\Gamma(x) \cap A^t \neq \emptyset$.

This observation guarantees that the dataset $\mathcal{O}$ is rationalisable if and only if, for any cycle $\mathcal{C} = \{(a, b), (b, c), \ldots, (z, a)\}$ in $T \times T$, we have $h^t(x^s) > 0$, for some $(t, s) \in \mathcal{C}$. Following the argument in Section B.1.1, this suffices for the set of observations to be rationalisable with the functions u and δ specified in Theorem B.1. Since g^t and h^t are continuous, for all $t \in T$, so is the utility u and the threshold δ. $\square$

In relation to Proposition 3 in the main paper, this result introduces alternative assumptions under which a dataset is rationalisable with a continuous utility. Moreover, the same conditions guarantee a continuous threshold δ.

Next, we address the question of strong monotonicity of the utility u. As stated in Proposition 2, any function u that rationalises the data with approximate utility maximisation, must be consistent with the correspondence Γ, i.e., if $y \in \Gamma(x)$ then $u(y) > u(x)$. However, unlike for the exact utility maximisation, the utility u can satisfy a stronger notion of monotonicity and still rationalise the observed choices.

A correspondence Γ is *increasing* if, for any $x' \geq x$ and $y' \in \Gamma(x')$, there is some $y \in \Gamma(x)$ such that $y' \geq y$. The correspondence is *strictly increasing* if, for any $x' > x'$ and y' in the closure of $\Gamma(x')$, there is some $y \in \Gamma(x)$ such that $y' > y$.

Proposition B.2. *Under Assumption B.1, if Γ is (strictly) increasing, then the dataset $\mathcal{O}$ is approximately rationalisable with a (strictly) increasing utility u.*[8]

[8] The function u is (strictly) *increasing* if $x\,(>) \geq y$ implies $u(x)\,(>) \geq u(y)$.

Proof. Define functions g^t and h^t as in Section B.1.1, for all $t \in T$. First, we show the result outside the brackets. Whenever Γ is increasing, for any $x' \geq x$ and $y' \in \Gamma(x')$, there is some $y \in \Gamma(x)$ such that $y' \geq y$. Since g^t is increasing, this implies $g^t(y') \geq g^t(y) \geq h^t(x)$, and so $h^t(x') \geq h^t(x)$. Hence, the function h^t is increasing, for all $t \in T$. This suffices to show that the utility u in Theorem B.1 is also increasing.

To prove the result within the brackets, take any $x' > x$ and y' in the closure of $\Gamma(x')$ satisfying $h^t(x') \geq g^t(y')$. By assumption, there is some $y \in \Gamma(x)$ such that $y' > y$, and so strict monotonicity of g^t implies $h^t(x') \geq g^t(y') > g^t(y) \geq h^t(x)$. Therefore, the function h^t is strictly increasing, which suffices for the utility u to be strictly increasing. □

This result highlights the distinction between monotonicity of choice and the utility u under approximate utility maximisation, discussed on page 18 in Section 4. Preferences u of the individual can be strictly monotone, yet this need not translate to the choice. For example, since the correspondence Γ in Example 1 is strictly increasing, any dataset that is approximately rationalisable can be supported with a strictly increasing utility u, without loss, even though the choice itself admits a degree of insensitivity to differences among alternatives. Although the agent may agree that more is better from the normative standpoint, they may fail to follow this rule due to imperfect discrimination or imprecision, similarly to the observation in Nielsen and Rehbeck (2020).

We conclude this section by addressing convexity of preferences. It is well-known since Afriat (1967), Diewert (1973), and Varian (1982) that, within the classic consumer choice framework, any dataset $\mathcal{O}$ that is rationalisable with exact maximisation of a strictly increasing utility, can be supported in this sense with a concave utility u, without loss. We extend this result to approximate utility maximisation.

We say the correspondence Γ is *quasiconcave* whenever, for any $x, x' \in X$, $\alpha \in [0,1]$, and $y \in \Gamma\big(\alpha x + (1-\alpha)x'\big)$ there is $z \in \Gamma(x)$, $z' \in \Gamma(x')$, and $\beta \in [0,1]$ such that $y \geq \beta z + (1-\beta)z'$. The correspondence is *concave* if this condition holds for $\beta = \alpha$. Note that, this definition does not require for the values of Γ to be convex.

Proposition B.3. *Under Assumption B.1, if the function f^t representing the menu A^t is quasiconcave, for all $t \in T$, and the correspondence Γ is quasiconcave, then the dataset $\mathcal{O}$ is approximately rationalisable for a quasiconcave utility u.*[9]

[9] A function $f : X \to \mathbb{R}$, defined over a convex domain X, is *quasiconcave* if, for any $x, x' \in X$ and $\alpha \in [0,1]$, we have $f\big(\alpha x + (1-\alpha)x'\big) \geq \min\big\{f(x), f(x')\big\}$.

Proof. Define functions g^t and h^t as in Section B.1.1, for all $t \in T$. Since the function f^t is strictly increasing and quasiconcave, so is g^t, for all $t \in T$. Take any $x, x' \in X$, $\alpha \in [0,1]$, and $z \in \Gamma\big(\alpha x + (1-\alpha)x'\big)$. By assumption, there is some $z \in \Gamma(x)$, $z' \in \Gamma(x')$, and $\beta \in [0,1]$ such that $y \geq \beta z + (1-\beta)z'$. This implies that

$$g^t(y) \;\geq\; g^t\big(\beta z + (1-\beta)z'\big) \;\geq\; \min\big\{g^t(z), g^t(z')\big\} \;\geq\; \min\big\{h^t(x), h^t(x')\big\},$$

where the inequalities follow from monotonicity of g^t, quasiconcavity of g^t, and the definition of h^t, respectively. By taking the infimum over the left hand-side, we conclude that $h^t\big(\alpha x + (1-\alpha)x'\big) \geq \min\big\{h^t(x), h^t(x')\big\}$. Hence, the function h^t is quasiconcave, for all $t \in T$. Given that quasiconcavity is preserved by the *min* operator, this suffices for the utility u specified in Theorem B.1 to be quasiconcave. □

Under some additional assumptions, we can guarantee that the utility u that approximately rationalises the observations is concave, without loss of generality.

Proposition B.4. *Under Assumptions B.1 and B.2, if the function f^t representing the menu A^t is continuous and concave, for all $t \in T$, and the correspondence Γ is concave, then the dataset $\mathcal{O}$ is approximately rationalisable for a concave utility u.*[10]

Proof. Define function h^t as in the proof of Proposition B.1, for all $t \in T$. Take any $x, x' \in X$, $\alpha \in [0,1]$, and $z \in \Gamma\big(\alpha x + (1-\alpha)x'\big)$. By assumption, there is some $z \in \Gamma(x)$, $z' \in \Gamma(x')$ such that $y \geq \alpha z + (1-\alpha)z'$. By monotonicity and concavity of f^t, we obtain

$$f^t(y) \;\geq\; f^t\big(\alpha z + (1-\alpha)z'\big) \;\geq\; \alpha f^t(z) + (1-\alpha)f^t(z') \;\geq\; \alpha h^t(x) + (1-\alpha)h^t(x').$$

Once we take the infimum over the left hand-side of the inequality, we conclude that the function h^t is concave, for all $t \in T$. Since the *min* operator preserves concavity, this suffices to show that the utility u specified in Theorem B.1 is concave. □

The correspondence $\Gamma(x) := \big\{y \in X : y > x\big\}$, the mapping introduced in Example 1, and the one studied in Dziewulski (2020) are all concave.[11] However, since the first one violates Assumption B.2, rationalising the data with a concave utility may be impossible in such a case. This is because concavity implies continuity, which is not guaranteed

[10] A function $f : X \to \mathbb{R}$, defined over a convex domain X, is *concave* if, for any $x, x' \in X$ and $\alpha \in [0,1]$, we have $f\big(\alpha x + (1-\alpha)x'\big) \geq \alpha f(x) + (1-\alpha)f(x')$.

[11] The correspondence $\Gamma(x) := \big\{y \in X : y \gg x\big\}$ is also concave.

for this correspondence, as shown in Section 3 of the main paper. Nevertheless, by Proposition B.3, one can rationalise such datasets with a quasiconcave utility.

Propositions B.3 and B.4 can be applied directly to the setup of Afriat (1967), Diewert (1973), and Varian (1982). Since the original framework assumes that the budget set A^t can be represented with the function $f^t(y) := p^t \cdot y - m^t$, for some prices p^t and income m^t, for all $t \in T$, the requirements of the two results are satisfied.

B.2 Relation to interval orders

Approximate utility maximisation is tightly related to the notion of interval orders introduced in Wiener (1914) and Fishburn (1970). An *interval order* is a binary relation $\succ_I$ over the consumption space X that is irreflexive, i.e., $x \not\succ_I x$, for all $x \in X$, and satisfies the interval order condition (or Ferrer's property), i.e., if $x \succ_I y$ and $x' \succ_I y'$ then either $x \succ_I y'$ or $x' \succ_I y$, for any $x, x', y, y' \in X$. Fishburn (1970) shows that any interval order defined on a *countable* space X can be represented by a utility u and a positive threshold δ as follows: $x \succ_I y$ if and only if $u(x) + \delta(y) > u(y)$.[12]

It is straightforward to show that whenever a set of observations is rationalisable with a Γ-monotone approximate utility maximisation for some functions u, δ, there is an interval order $\succ_I$ such that the correspondence $c : \mathcal{A} \rightrightarrows X$, given by

$$c(A) := \Big\{x \in A : y \not\succ_I x, \text{ for all } y \in A\Big\}, \tag{B.2}$$

rationalises the data and is Γ-monotone, as in (1) and (2) in the main paper. Therefore, under the assumptions specified in either Theorem 1 or Proposition 3, acyclicity of P^* is sufficient for the data to be rationalisable with an interval order maximisation. In the following proposition we state that this condition is also necessary.

Proposition 1. *For any dataset $\mathcal{O}$ and correspondence Γ, there is an interval order $\succ_I$ such that the correspondence c in (B.2) rationalises $\mathcal{O}$ and is Γ-monotone only if the directly revealed strict preference relation P^* is acyclic.*

Proof. Suppose that the correspondence $c(A) := \big\{x \in A : y \not\succ_I x, \text{ for all } y \in A\big\}$ rationalises the set of observations, where $\succ_I$ is an interval order. We show that the directly revealed strict preference relation P^* is acyclic.

[12] Fishburn (1973), Bridges (1985, 1986), and Chateauneuf (1987) specify conditions under which interval orders admit such a representation over a general space X.

First, define a binary relation P_I as: xP_Iy if $z \succ_I y$ and $z \not\succ_I x$, for some $z \in X$. Following Lemma 3.1 in Aleskerov et al. (2007), P_I is asymmetric and negatively transitive.[13] Given that the correspondence c is Γ-monotone, it must be that $x \in \Gamma(y)$ implies $x \succ_I y$. Otherwise, we would have $y \in c(\{x, y\})$, contradicting that c is Γ-monotone. We claim that xP^*y implies xP_Iy. Take any observation $(A, x) \in \mathcal{O}$ and $z \in \Gamma(y) \cap A$. Clearly, it must be that $z \succ_I y$ and $z \not\succ_I x$, which implies xP_Iy.

To show that P^* is acyclic, take any sequence $z^1, z^2, \ldots, z^n$ in X such that $z^iP^*z^{i+1}$, for all $i = 1, \ldots, (n-1)$. Thus, given the observation above, we obtain $z^iP_Iz^{i+1}$ or, equivalently, *not* $z^{i+1}P_Iz^i$, for all $i = 1, \ldots, (n-1)$ (by asymmetry of P_I). By negative transitivity of P_I, it must be that *not* $z^nP_Iz^1$, and so *not* $z^nP^*z^1$. □

This result complements Fishburn (1975) that characterises choice correspondences generated by an interval order maximisation under the assumption that the researcher observes the *entire* set $c(A)$ for *all* possible menus $A \in \mathcal{A}$. In contrast, we assume that the data are incomplete. Our result requires no assumptions on the space X, dataset $\mathcal{O}$, or the correspondence Γ. In particular, since we allow for the consumption space X to be uncountable, the interval order $\succ_I$ may not have a representation as in Fishburn (1970). Moreover, there is no direct relation between the revealed preference P^* and the interval order $\succ_I$ supporting the data. Specifically, xP^*y does *not* imply $x \succ_I y$.

B.3 State-contingent consumption under risk

Here we revisit the results in Section 7.2 of the main paper regarding choice over state-contingent consumption under risk. First, we state the proof of Proposition 6. Then we discuss some additional properties of these models.

B.3.1 Proof of Proposition 6

We prove implication (i) ⇒ (ii). Take any strictly increasing Bernoulli function v such that $u(y) := F(v(y_1), v(y_2), \ldots, v(y_\ell))$ approximately rationalises the data. By Proposition 2 in the main paper, if xP^*z then $u(x) > u(z)$, for any $x, z \in X$. Moreover, since u is

[13] A relation R on X is *asymmetric* if xRy implies *not* yRx. The relation *negatively transitive* if *not* xRy and *not* yRz implies *not* xRz.

strictly increasing, we have $u(x) > u(y)$, for any $x, y, z \in X$ such that xP^*z and $z \geq y$. In particular, the latter must be true for any $x, y \in \mathcal{X}^\ell$.

To show the converse, let $\mathcal{X} = \{z_1, z_2, \ldots, z_K\}$, where $0 = z_1 < z_2 < \ldots < z_K$. Take any strictly increasing function $\bar{v} : \mathcal{X} \to \mathbb{R}_+$ specified in statement (ii) and any strictly positive number $a \leq [\bar{v}(z_{k+1}) - \bar{v}(z_k)]/(z_{k+1} - z_k)$, for all $k = 1, \ldots, (K-1)$, define an upper semi-continuous and strictly increasing extension $v_a : \mathbb{R}_+ \to \mathbb{R}_+$ of $\bar{v}$ by

$$v_a(z) \;:=\; \sum_{k=1}^{K} \big[\bar{v}(z_k) + a(z - z_k)\big] \chi_{B_k}(y),$$

where $B_k = [z_k, z_{k+1})$, for all $k = 1, \ldots (K-1)$, and $B_K = [z_K, \infty)$.

For any set $Z \subseteq \mathbb{R}^\ell_+$, let $\underline{Z} := \big\{y' \in \mathbb{R}^\ell_+ : y' \leq y, \text{ for some } y \in Z\big\}$ be its downward comprehensive hull. Take any $\bar{z} \in \mathbb{R}_+$ such that $\bar{\mathbf{z}} := (\bar{z}, \bar{z}, \ldots, \bar{z}) \geq y$, for all $y \in \bigcup_{t \in T} \underline{A}^t$. Since the menus A^t are bounded, for all $t \in T$, such a number exists and $\bar{z} \geq z_K$. Without loss of generality, suppose that $\bar{z} - z_K \geq z_{k+1} - z_k$, for all $k = 1, \ldots, (K-1)$. By construction of the function v_a, for any $\epsilon > 0$ there is a sufficiently small $a > 0$ such that $\epsilon \geq v_a(z) - \bar{v}(z_k) \geq 0$, for any $z \in [0, \bar{z}]$, where $z_k = \max\big\{z' \in \mathcal{X} : z' \leq z\big\}$.[14]

Recall that $x^t P^* y$ if and only if $\Gamma(y) \cap A^t \neq \emptyset$, for any $t \in T$. Equivalently, this is to say that y belongs to the lower inverse $\Gamma^\ell(A^t)$. Since $\Gamma(x) \subseteq \big\{y \in \mathbb{R}^\ell_+ : y \geq x\big\}$, for $x \in X$, we have $\underline{\Gamma^\ell(A^t)} \subseteq \bigcup_{t \in T} \underline{A}^t$, and so $\bar{\mathbf{z}} \geq y$, for all $y \in \underline{\Gamma^\ell(A^t)}$. Moreover, for any $y \in \underline{\Gamma^\ell(A^t)}$, there is some $x \in \mathcal{X}^\ell \cap \underline{\Gamma^\ell(A^t)}$ such that $x_i = \max\big\{z \in \mathcal{X} : z \leq y_i\big\}$, for all $i = 1, \ldots, \ell$. By by our previous observation, there are numbers $\epsilon, a > 0$ such that

$$F\big(\mathbf{v}_a(x^t)\big) \;=\; F\big(\bar{\mathbf{v}}(x^t)\big) \;>\; F\big(\bar{\mathbf{v}}(y') + \epsilon\mathbf{1}\big) \;\geq\; F\big(\mathbf{v}_a(y)\big),$$

for any $y \in \underline{\Gamma^\ell(A^t)}$ and some $y' \in \mathcal{X}^\ell \cap \underline{\Gamma^\ell(A^t)}$, where $\mathbf{v}(y) := \big(v(y_1), v(y_2), \ldots, v(y_\ell)\big)$, for any function v, and $\mathbf{1}$ is the ℓ-dimensional unit vector.

For each $t \in T$, take any such a and denote it a_t. Define an upper semi-continuous and strictly increasing function $v : \mathbb{R}_+ \to \mathbb{R}_+$ by $v(z) := \min\big\{v_{a_t}(z) : t \in T\big\}$. Moreover, let $u(y) := F\big(\mathbf{v}(y)\big)$, which is strictly increasing and satisfies $u(x^t) > u(y)$, for all $y \in \Gamma^\ell(A^t)$ and $t \in T$. Since $\Gamma(x) \subseteq \big\{y \in \mathbb{R}^\ell_+ : y > x\big\}$, for all $x \in X$, this suffices to show that both $x \in \Gamma(y)$ and xP^*y imply $u(x) > u(y)$. By Proposition 2, there is a threshold function δ such that u approximately rationalises the data.

[14] It suffices to take any strictly positive $a \leq \epsilon/(\bar{z} - z_K)$

B.3.2 Related results

Continuity First, we address the question of continuity of the Bernoulli function v. Suppose that the menu A^t is compact, for all $t \in T$, and the correspondence Γ satisfies Assumption 3(ii). We claim that this suffices for the Bernoulli function v specified in Proposition 6 to be continuous, without loss of generality.

Indeed, in such a case, the lower inverse $\Gamma^\ell(A^t)$ is compact, for all $t \in T$, as is its downward comprehensive hull $\underline{\Gamma^\ell(A^t)}$. Since $\mathcal{X}^\ell$ is finite, there is a closed neighbourhood V of $\underline{\Gamma^\ell(A^t)}$ such that $\mathcal{X}^\ell \cap \underline{\Gamma}^\ell(A^t) = \mathcal{X}^\ell \cap V$. Denote $\underline{B}^t := V \cup \{y \in \mathbb{R}^\ell_+ : y \le x^t\}$, which is compact and contains $\underline{\Gamma^\ell(A^t)}$ in its interior. Moreover, for any strictly increasing function $\bar{v}$ specified as in statement (ii) of Proposition 6, we have $F\big(\bar{\mathbf{v}}(x^t)\big) > F\big(\bar{\mathbf{v}}(y)\big)$, for all $y \in (\underline{B}^t \cap \mathcal{X}^\ell) \setminus \{x^t\}$. By Theorem 1 in Polisson et al. (2020), there is a continuous and strictly increasing extension v of $\bar{v}$ such that $F\big(\mathbf{v}(x^t)\big) > F\big(\mathbf{v}(y)\big)$, for all $y \in \underline{\Gamma^\ell(A^t)}$. The rest of the result follows from Proposition 2 in the main paper.

Variable aggregator Proposition 6 can be extended to the case where the aggregator function varies across observations. Formally, consider a collection of continuous and strictly increasing functions $F_t : \mathbb{R}^\ell_+ \to \mathbb{R}$, for all $t \in T$. We claim that there is a strictly increasing Bernoulli function $v : \mathbb{R}_+ \to \mathbb{R}_+$ and a threshold δ_t such that

$$y \in A^t \ \text{ implies } \ F_t\big(\mathbf{v}(x^t)\big) + \delta_t(y) \ge F_t\big(\mathbf{v}(y)\big),$$

for all $t \in T$, if and only if there is a function $\bar{v} : \mathcal{X} \to \mathbb{R}_+$ such that $F_t\big(\bar{\mathbf{v}}(x^t)\big) > F_t\big(\bar{\mathbf{v}}(y)\big)$, for any $t \in T$ and $y \in \mathcal{X}$ satisfying $x^t P^* z$ and $z \ge y$, for some $z \in X$.

Indeed, partition the set T into disjoint subsets $T_1, T_2, \dots, T_K$ such that $t, t' \in T_k$ implies $F_t = F_{t'}$, for all $k = 1, \dots, K$. By Proposition 6, our claim is true for any sub-dataset $\mathcal{O}_k = \{(A^t, x^t) : t \in T_k\}$, for all k. One can show that this holds for the entire dataset $\mathcal{O}$ for the Bernoulli function $v(z) := \min\{v_k(z) : k = 1, \dots, K\}$.

Preference symmetry In some cases, the utility u of the agent may depend only on the distribution of consumption in a portfolio x, rather than the precise allocation of consumption to each state. Formally, we say that such a utility function is *symmetric*. That is, for any bundle $x \in X$ and permutation σ on $\{1, 2, \dots, \ell\}$, we have $u(x) = u(x_\sigma)$, where we denote $x_\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(\ell)})$. For example, this is true when u takes the expected utility formulation when all states $s \in S$ are equally probable, i.e., we have $u(x) = \sum_{s=1}^{\ell}(1/\ell)v(x_s)$, for some Bernoulli function v.

Whenever a dataset $\mathcal{O}$ is approximately rationalisable with a symmetric utility u, one would expect the corresponding threshold function δ to be symmetric as well. That is, the agent should be equally imprecise regarding a bundle x as with its permutation x_σ. This is indeed true, without loss of generality.

Proposition 2. *Suppose that* $\Gamma(x_\sigma) = \{y_\sigma : y \in \Gamma(x)\}$*, for any* $x \in X$ *and any permutation* σ*. If the dataset* $\mathcal{O}$ *is approximately rationalisable for a symmetric utility* u *and some threshold* δ*, then the function* δ *is symmetric, without loss of generality.*

Proof. Suppose that the dataset $\mathcal{O}$ is rationalisable with a symmetric utility u and some threshold δ', and define $\delta(y) := \max\{\delta'(y_\sigma) : \text{for some } \sigma\}$, which is well-defined and symmetric. We claim that u, δ approximately rationalise $\mathcal{O}$. First, we show that the model is Γ-monotone. Take any $y \in \Gamma(x)$. By assumption, we have $y_\sigma \in \Gamma(x_\sigma)$. Since u, δ' rationalise the data, there is some permutation σ such that $u(y) - \delta(y) = u(y_\sigma) - \delta'(y_\sigma) > u(x_\sigma) = u(x)$. To show that the model rationalises the data, take any $t \in T$ and $y \in A^t$. Then, $u(x^t) \geq u(y) - \delta'(y) \geq u(y) - \delta(y)$. □

The additional restriction on the correspondence Γ imposes a weak form of symmetry on the monotonicity of choice. Clearly, the condition holds for $\Gamma(x) := \{y \in X : y > x\}$. Similarly, it applies to Example 1, as long as $\lambda_s = \lambda_{s'}$ and $a_s = a_{s'}$, for all $s, s' = 1, \dots, \ell$. Finally, the mapping $\Gamma(x) = \{\lambda' x : \lambda' \geq \lambda\}$, for some $\lambda > 1$, studied in Dziewulski (2020), satisfies this form of symmetry as well.

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