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Just-noticeable difference as a behavioural foundation of the critical cost-efficiency index

Pawel Dziewulski[a]

[a]Department of Economics, University of Sussex, Brighton, UK

P.K.Dziewulski@sussex.ac.uk

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Just-noticeable difference as a behavioural foundation of the critical cost-efficiency index*

Paweł Dziewulski†

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1 Introduction

It is most common in the economic literature to assume that choices of a consumer are determined via maximisation of a utility function — to the extent of it being synonymous to rationality. The influential papers by Afriat (1967), Diewert (1973), and Varian (1982) investigate the testable restrictions of this hypothesis with the stipulation that the researcher can monitor only a finite number of expenditure data; i.e., where an observation consists of a consumption bundle chosen by the individual at the prevailing prices. The *generalised axiom of revealed preference* (GARP, for short), proposed in these works, exhausts all the observable implications of utility maximisation.

*I am grateful to Miguel Ballester, Ian Crawford, and John Quah for encouragement.
†Department of Economics, University of Sussex, Jubilee Building, Falmer, Brighton BN1 9SL, United Kingdom. Email: P.K.Dziewulski@sussex.ac.uk.

A problematic feature of GARP lies in its deterministic nature. A dataset either satisfies this restriction and, thus, is rationalisable, or it does not. In practice, it is desirable to evaluate how severe a violation is once it occurs. Afriat (1973) addresses this issue by introducing the *critical cost-efficiency index* (CCEI, for short). This measure of departures from rationality determines the minimal reduction of the consumer budget, uniform across all observations, that is sufficient for the data to be supportable with a utility maximisation. It evaluates the level of tolerance for wasted expenditure that is required for the choices to be rationalisable. Varian (1990) generalises this measure by allowing for adjustments of the budget to vary across observations.

Despite its ad hoc nature and criticism, CCEI remains the most commonly used measure of GARP violations.[1] We believe this is for two reasons: One, it has an appealing economic interpretation in terms of the share of wealth wasted by the consumer relatively to a fully rational one.[2] Two, it is convenient for empirical applications, since it can be evaluated using computationally efficient methods.[3]

In this paper we argue that, in addition, CCEI admits an intuitive behavioural interpretation in terms of the level of sensory discrimination. Specifically, we show that the index is equivalent to the *just-noticeable difference* — a measure of dissimilarity between alternatives that is sufficient for the agent to tell them apart.

The evidence from psychophysiology suggest that people can not discern between two physical stimuli unless their intensities are significantly (noticeably) different.[4] This idea was incorporated to choice theory by Armstrong (1950) and Luce (1956), who claimed that due to imperfect powers of discrimination of the human mind, consumers are unable to distinguish between goods/bundles that are similar. They postulated that any form of imperfect discrimination would require for indifferences to be non-transitive.[5] Hence, such a phenomenon is inconsistent with utility maximisation.

Following Fishburn (1970), we address the above issue by characterising preference of an individual with an *interval order*.[6] We postpone the formal definition of this notion

[1]See Sippel (1997), Harbaugh et al. (2001), Andreoni and Miller (2002), Choi et al. (2007), Fisman et al. (2007), Ahn et al. (2014), Choi et al. (2014), and Halevy et al. (2018) for empirical studies employing CCEI as a measure of revealed preference violations. For critical analysis of the index see Echenique et al. (2011), Apesteguia and Ballester (2015), Dean and Martin (2016), or Dziewulski (2017).

[2]Halevy et al. (2018) show that it is closely related to the money metric.

[3]See Smeulders et al. (2014) for details.

[4]See Laming (1997) or Algom (2001) for a comprehensive summary of this literature.

[5]Although it may be impossible to distinguish option x from y and y from z, alternatives x and z may be sufficiently different for the agent to strictly discriminate between them.

[6]Alternatively, Luce (1956), Scott and Suppes (1958), and Beja and Gilboa (1992) address the problem of noticeable differences by characterising preferences with *semiorders*.

until Section 2; however, an interval order can be thought of as a strict binary relation P for which there is a utility u and a positive threshold function δ that satisfy: xPy if and only if $u(x) > u(y) + \delta(x)$. Therefore, option x is strictly preferable to y if it yields a significantly higher utility, where the threshold is determined by the number $\delta(x)$. If neither xPy nor yPx, then x is hedonically indistinguishable from y, denoted by xIy. Notice that the latter constitutes a non-transitive indifference relation.

To capture the consumer's inability to differentiate among alternatives, we introduce a notion of a *noticeable difference* — defined as a number $\lambda > 1$ such that $\lambda' \geq \lambda$ implies $\big(\lambda' y\big)Py$, for all non-zero bundles y.[7] Therefore, it is the relative change in the size of a bundle that is sufficient for the agent to perceive the difference.[8] This concept is strongly inspired by the well-known Weber-Fechner law in psychophysics according to which people are unable to discriminate between two intensities of a physical stimulus unless the ratio of their magnitudes exceeds a particular value.[9] Ours is a natural extension of this idea to consumer choice over multi-dimensional domains.

We state our first main result in Section 3. In Theorem 1 we show that Afriat's critical cost-efficiency index is equal to the inverse of the (uniform) *just-noticeable difference*; i.e., the least noticeable difference $\lambda > 1$ for which there is an interval order P satisfying the aforementioned condition and rationalising the observed choices. Therefore, apart from being a measure of budgetary adjustments, CCEI can be interpreted in terms of the level of imperfect sensory discrimination. In Section 4 we extend the equivalence result to Varian's generalisation of this measure. Theorem 2 argues that the index coincides with the *variable* just-noticeable difference; a measure of cognitive ability to discriminate among alternatives that may fluctuate from one instance to another.

The proof of the two equivalence results is postponed until Section 5. We conclude the paper in Section 6 with a discussion on the revealed preference relations induced by the models of consumer choice presented in this paper.

Our results pertain to a broad class of choice problems introduced in Forges and Minelli (2009). In particular, none of the results depend on linearity of budget sets, as in Afriat (1973) or Varian (1990). We begin Section 2 by formalising our framework.

[7] This notion is analogous to the one introduced in Dziewulski (2017). However, unlike in that paper, we represent consumer preference with an interval order, rather than a semiorder.

[8] For example, for $\lambda = 1.01$ the option (λy) with 1% more of each commodity would be noticeably different and hedonically preferable to y. Note that, this imposes a weak form of monotonicity on P.

[9] Although *Stevens' power law* seems to better explain sensory discrimination than Weber-Fechner's law, the latter is considered to be the best first approximation. See, e.g., Algom (2001).

2 Preliminaries

In this section we introduce the main assumptions our analysis and revise the basic properties of interval orders that are crucial to our results.

2.1 Setup

Suppose a researcher monitors a finite number of observations $t \in T$, each consisting of a set of alternatives $B_t \subseteq \mathbb{R}^\ell_+$ available to the consumer and a bundle of ℓ goods $x_t \in B_t$ selected from the set. The *set of observations* is given by $\mathcal{O} = \big\{(B_t, x_t) : t \in T\big\}$. With a slight abuse of the notation, we denote cardinality of the set by T.

Throughout this paper we consider choices over *generalised budget sets* as in Forges and Minelli (2009). In particular, we assume that set B_t is compact and downward comprehensive, for all $t \in T$.[10] Moreover, there is some $y \in B_t$ such that $y \gg 0$.[11] Let the *upper bound* of the set B_t be denoted by

$$\partial B_t \;:=\; \Big\{y \in B_t : \text{ if } z \gg y \text{ then } z \notin B_t\Big\}$$

and suppose that, for any $y \in \partial B_t$ and scalar $\theta \in [0, 1)$, we have $\theta y \in B_t \setminus \partial B_t$. That is, for any non-zero $y \in \mathbb{R}^\ell_+$, ray $\big\{\theta y : \theta \geq 0\big\}$ intersects the boundary ∂B_t exactly once. Finally, we assume that in each observation at least one commodity is chosen in a strictly positive amount, i.e., $x_t \neq 0$. This is not without loss of generality, but it simplifies our analysis and is insignificant from the empirical point of view.

It is straightforward to verify that our framework includes linear budget sets, as in the original work of Afriat (1967), Diewert (1973), and Varian (1982). In such a case, a set of alternatives is given by $B_t := \big\{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t\big\}$, for some $p_t \in \mathbb{R}^\ell_{++}$, while the upper boundary ∂B_t of the set is the corresponding budget line.

2.2 Interval orders

Here we provide a formal definition and some basic properties or interval orders. For a comprehensive treatment of this topic, see Aleskerov et al. (2007).[12]

Following Wiener (1914) and Fishburn (1970), an *interval order* over a set X is a binary relation P that is (i) *irreflexive*, i.e., *not* xPx, for all $x \in X$, and (ii) satisfies the

[10] We endow $\mathbb{R}^\ell_+$ with the natural product order $\geq$, i.e., we have $x \geq y$ if $x^i \geq y^i$, for all $i = 1, 2, \ldots, \ell$. A set $B \subseteq \mathbb{R}^\ell_+$ is *downward comprehensive* whenever $y \in B$ and $y \geq z$ implies $z \in B$, for all $z \in \mathbb{R}^\ell_+$.

[11] We denote $x \gg y$ whenever $x^i > y^i$, for all $i = 1, 2, \ldots, \ell$.

[12] I am grateful to Ali Khan for pointing me to this publication.

interval order condition, i.e., if xPy and $x'Py'$ then either xPy' or $x'Py$, for all x, y, x', and y' in X. It is easy to verify that any interval order is asymmetric and transitive.

In the remainder of this paper, we associate P with the *strict preference*. If neither xPy nor yPx then alternative x is *indistinguishable* from y, which we denote by xIy. Observe that the latter is reflexive and symmetric, but not transitive.

An important characteristic of interval orders pertains to their utility representation. Following Fishburn (1970), for any interval order P defined over a countable set X there is a utility $u : X \to \mathbb{R}$ and a positive threshold function $\delta : X \to \mathbb{R}_+$ such that

$$xPy \quad \text{if and only if} \quad u(x) > u(y) + \delta(x). \tag{1}$$

Thus, option x is strictly preferred to y if and only if the utility of the former is sufficiently higher than that of the latter, where the threshold is determined by $\delta(x)$. Clearly, this implies that *not* yPx is equivalent to $u(x) + \delta(y) \geq u(y)$. Under some regularity conditions, this representation can be extended to interval orders over more general spaces. See Bridges (1985, 1986) and Chateauneuf (1987) for details.

Given that the representation of an interval order involves a utility function u, any such relation is inherently related to a weak order, i.e., a complete, reflexive, and transitive binary relation.[13] Consider a relation $\succeq$, with its strict part $\succ$ defined as

$$x \succ y \quad \text{if} \quad \textit{not } zPx \quad \text{and} \quad zPy, \text{ for some } z \in X.$$

In particular, since P is irreflexive, xPy implies $x \succ y$. Moreover, if neither $x \succ y$ nor $y \succ x$, then $x \sim y$. In the remainder of this paper we say that such a relation $\succeq$ is *induced by* the interval order P. Consider the following proposition.

Proposition 1. *Relation $\succeq$ induced by an interval order P is complete, transitive, and reflexive. In particular, its strict part $\succ$ is transitive and irreflexive.*

A proof of this result can be found in Aleskerov et al. (2007, p. 60). Nevertheless, as the argument is rather short, we state it below to keep this paper self-contained.

Proof of Proposition 1. Clearly, the relation $\succeq$ is complete. To prove the remainder of the result, it suffices to show that $\succ$ is asymmetric and negatively transitive. We prove the former by contradiction. Suppose that $x \succ y$ and $y \succ x$. Thus, there is some $z, z' \in X$ such that *not* zPx and zPy, as well as *not* $z'Py$ and $z'Px$. However, by the interval order condition, zPy and $z'Px$ imply either zPx or $z'Py$, which yields a contradiction.

[13]Weak orders are usually identified with *rational preferences*. We avoid the latter term purposely.

To show that $\succ$ is negatively transitive, assume that *not* $x \succ y$ and *not* $y \succ z$. This is true only if $z'Py$ implies $z'Px$, and $z'Pz$ implies $z'Py$, for all $z' \in X$. Thus, if $z'Pz$ then $z'Px$, and so *not* $x \succ z$, which completes the proof. □

One can easily verify that if the interval order P admits a representation as in (1), for some utility u and threshold δ, then $x \succ y$ implies $u(x) > u(y)$. Thus, the weak order induced by P is consistent with the ranking generated by the utility function u. Because of this, we interpret the relation $\succeq$ as the *"true" preference*, i.e., as if perfect discrimination were possible. In other words, if the consumer could hedonically distinguish between any two alternatives in X, or equivalently, if the threshold function δ were constantly equal to zero, the agent's preference would be characterised by $\succeq$.

3 Afriat's efficiency and noticeable differences

Before we proceed with the first main theorem of this paper, we define a generalised notion of the critical cost-efficiency index by Afriat (1973), that is appropriate to our framework. Then, we introduce the concept of the uniform just-noticeable difference.

3.1 Afriat's efficiency index

A set of observations $\mathcal{O} = \{(B_t, x_t) : t \in T\}$ is rationalisable with *efficiency parameter* $e \in [0, 1]$ if there is a locally non-satiated utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ that satisfies:

$$\text{if} \quad y \in eB_t \quad \text{then} \quad u(x_t) \geq u(y), \tag{2}$$

for all $t \in T$, where $eB_t := \{ey : y \in B_t\}$. This condition requires that the observed choice x_t is preferable to any other bundle in the adjusted set eB_t. In particular, whenever $e = 1$ this restriction coincides with utility maximisation.

Afriat's (critical cost-)*efficiency index*, denoted by e^*_A, is the supremum over all efficiency parameters $e \in [0, 1]$ for which the dataset $\mathcal{O}$ is rationalisable as in (2).

Whenever we consider linear budget sets $B_t := \{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq m_t\}$, for some strictly positive prices $p_t \in \mathbb{R}^\ell_{++}$ and wealth $m_t > 0$, for all $t \in T$, Afriat's efficiency index determines the minimal fraction $(1 - e^*_A)$ of wealth wasted by the consumer relatively to a fully rational one. Indeed, in such a case, we have $eB_t = \{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq em_t\}$, for any efficiency parameter e. See Figure 1 for a graphical interpretation.

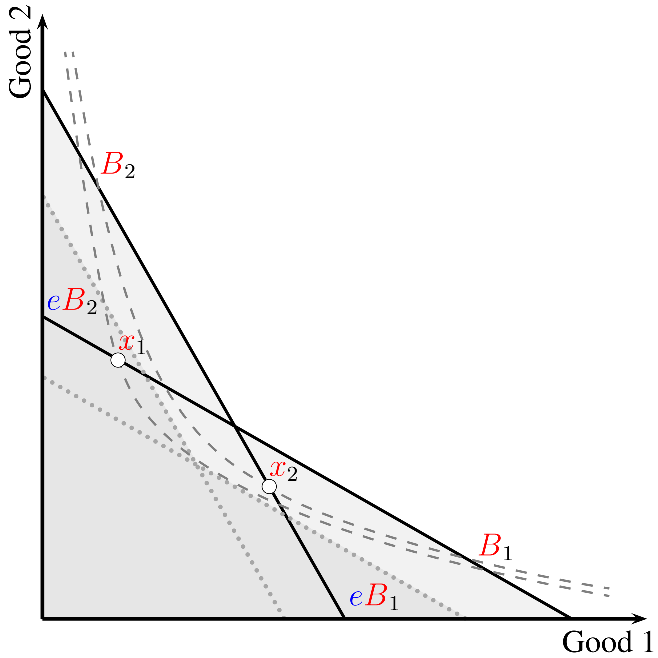

Figure 1: Dataset $\mathcal{O} = \{(B_1, x_1), (B_2, x_2)\}$ can not be rationalised with maximisation of a locally non-satiated utility u. Since x_1 is selected from B_1 with x_2 in the interior of the set, this would imply $u(x_1) > u(x_2)$. Analogously, we would have $u(x_2) > u(x_1)$, yielding a contradiction. Once we scale the two budget sets by a sufficiently large $e \in [0, 1]$ to the point where x_2 is excluded from eB_1, it is possible to rationalise the perturbed dataset as in (2).

3.2 Uniform just-noticeable difference

In this subsection we specify a model of consumer choice in which the agent is unable to perfectly discriminate among alternatives and her insensitivity to differences remains constant across observations. Formally, consider a consumer whose preference is represented by an interval order P. In each decision problem $t \in T$, her choice is determined by maximisation of the relation P over the set of available options B_t. We capture the agent's sensitivity to differences among alternatives with a noticeable difference. Specifically, we say that the relation P admits a *noticeable difference* $\lambda > 1$ whenever

$$\lambda' \geq \lambda \ \text{ implies } \ (\lambda' y) P y,$$

for all non-zero $y \in \mathbb{R}^{\ell}_{+}$. That is, the number λ determines by how much one should inflate a consumption bundle y in order to guarantee that the agent perceives the change. Roughly speaking, it is the relative change in sizes of two bundles that is sufficient for the agent to discern between them. This is inspired by the Weber-Fechner law in psychophysiology according to which people are unable to discriminate between two intensities of a physical stimulus unless their ratio exceeds a particular constant.

Set $\mathcal{O} = \{(B_t, x_t) : t \in T\}$ is rationalisable with a *uniform noticeable difference* $\lambda > 1$, if there is an interval order P that admits the noticeable difference λ and satisfies:

$$\text{if} \;\; y \in B_t \;\; \text{then} \;\; \textit{not}\; yPx_t, \tag{3}$$

for all $t \in T$. The latter condition requires that, for each observation $t \in T$, none of the available options is noticeably preferable to the selected bundle x_t. Therefore, this model assumes that the consumer choice is determined by maximisation of an interval order that admits the particular noticeable difference.[14]

Whenever we assume that P is representable by a utility u and threshold δ as in condition (1), the relation admits the noticeable difference λ if and only if

$$\lambda' \geq \lambda \;\; \text{implies} \;\; u(\lambda' y) > u(y) + \delta(\lambda' y),$$

for all non-zero $y \in \mathbb{R}^\ell_+$. Moreover, condition (3) is equivalent to

$$y \in B_t \;\; \text{implies} \;\; u(x_t) + \delta(y) \geq u(y),$$

for all $t \in T$. Therefore, the utility of bundle $(\lambda' y)$ must be sufficiently greater than that of y, where the threshold is determined by $\delta(\lambda' y)$. Similarly, for any observation $t \in T$, the utility of any available option $y \in B_t$ may not be significantly greater than the utility of the selected bundle x_t, i.e., value $u(y)$ can not exceed $u(x_t) + \delta(y)$.

The *uniform just-noticeable difference*, denoted by λ^*_U, is the infimum over all uniform noticeable differences $\lambda > 1$ for which the dataset is rationalisable.

3.3 The first equivalence result

Our first result regards the equivalence between Afriat's efficiency index and the uniform just-noticeable difference. Consider the following theorem.

Theorem 1. *For any set of observations $\mathcal{O}$, Afriat's efficiency index is equal to the inverse of the uniform just-noticeable difference, i.e., we have $e^*_A = 1/\lambda^*_U$.*

The above theorem is self-explanatory. It states that the minimal shift of the observable budget sets that is necessary to rationalise a set of observations with a locally non-satiated utility function is equivalent to the inverse of the least noticeable difference under which the choices can be rationalised as in (3).

[14] Our definition of rationalisation does not require for the order P to be representable as in (1).

We postpone the proof of this result until Section 5, however, our argument is based on the following observation. Whenever a dataset $\mathcal{O} = \{(B_t, x_t) : t \in T\}$ is rationalisable for some efficiency parameter e, it has to satisfy a particular generalisation of GARP. Specifically, for any cycle $\mathcal{C} := \{(a, b), (b, c), \ldots, (y, z), (z, a)\}$ in $T \times T$ such that $x_s \in eB_t$, for all $(t, s) \in \mathcal{C}$, it must be that $x_s \in \partial(eB_t)$, for all $(t, s) \in \mathcal{C}$. That is, for any cycle $\mathcal{C}$ in which the choice from observation s is available in the deflated budget set at time t, for all pairs $(t, s) \in \mathcal{C}$, each bundle along the sequence must belong to the upper bound of the corresponding adjusted set of alternatives.[15] Recall Figure 1.

In our argument, we show that whenever a set of observations satisfies the above property for some $e \in [0, 1]$, it can be rationalised with any uniform noticeable difference $\lambda > 1/e$.[16] This implies that Afriat's efficiency index is bounded from below by the inverse of the uniform just-noticeable difference. Conversely, whenever the set of observations is rationalisable with a noticeable difference $\lambda > 1$, it satisfies the condition stated above for all $e \geq 1/\lambda$. This excludes the possibility that $e^*_A > 1/\lambda^*_U$.

4 Varian's efficiency and noticeable differences

Here we extend Theorem 1 to the index proposed in Varian (1990). Specifically, we show that the measure coincides with a notion of the variable just-noticeable difference.

4.1 Varian's efficiency index

Afriat's efficiency index is constructed by deflating each budget set B_t by a uniform efficiency parameter e. In contrast, Varian (1990) considers a measure where each budget set B_t is being shifted by a different parameter e_t. A dataset $\mathcal{O} = \{(B_t, x_t) : t \in T\}$ is rationalisable with an *efficiency vector* $(e_t)_{t \in T}$, where $e_t \in [0, 1]$, for all $t \in T$, if there is a locally non-satiated utility $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ satisfying

$$\text{if} \quad y \in e_t B_t \quad \text{then} \quad u(x_t) \geq u(y), \tag{4}$$

for all $t \in T$, where $e_t B_t := \{e_t y : y \in B_t\}$. Analogously to Afriat's measure, this condition requires that, for all observations $t \in T$, the choice x_t is preferable to any other

[15] Clearly, for $e = 1$ the condition coincides with GARP.

[16] In particular, we show that any such set of observations is rationalisable by an interval order P that admits a representation as in (1). Moreover, the corresponding utility u and threshold function δ are continuous, with no loss of generality. Finally, under some additional conditions imposed on the budget sets, the utility u is quasiconcave. See Section 5 for details.

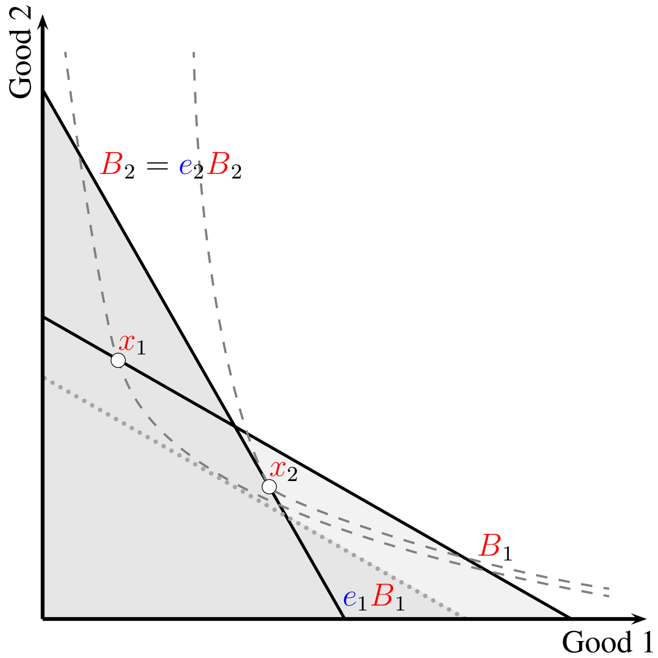

Figure 2: Recall the dataset from Figure 1. To rationalise the choices as in (4), it suffices to deflate the set B_1 by a sufficiently small $e_1 \in [0, 1]$, to the point where x_2 is excluded from $e_1 B_1$. At the same time, there is no need to perturb the set B_2, Thus, e_2 can be equal to 1.

bundle in the perturbed set $e_t B_t$. In particular, whenever $(e_t)_{t \in T}$ is the unit vector, it coincides with utility maximisation. See Figure 2.

Let $F : \mathbb{R}^T_+ \to \mathbb{R}$ be a well-defined, continuous, and increasing aggregator function that maps an efficiency vector $(e_t)_{t \in T}$ to a real number. *Varian's* (critical cost-)*efficiency index*, denoted by e^*_F, is the supremum of $F\big((e_t)_{t \in T}\big)$ with respect to all vectors $(e_t)_{t \in T}$ for which the dataset $\mathcal{O}$ is rationalisable as in (4).[17] Thus, this measure evaluates the minimal budget adjustments (with respect to the aggregator F) that are required for the data to be supported with utility maximisation. Equivalently, it determines the least money waste that is necessary for the set of observations to be rationalisable.

Whenever the aggregator function is given by $F\big((e_t)_{t \in T}\big) = \min\big\{e_t : t \in T\big\}$, then Varian's efficiency index is equivalent to the Afriat's measure of revealed preference violations. Thus, the latter is a special case of the former.

[17] In the original definition of Varian (1990), the aggregator function F is given by the mean of squares. However, other methods of aggregation may be considered. For alternative specifications, see Tsur (1989), Cox (1997), Alcantud et al. (2010), or Smeulders et al. (2014).

4.2 Variable just-noticeable difference

The model of consumer choice in Section 3.2 assumes that the agent's ability to discern among alternatives is constant across observations. It is not unreasonable to consider an instance in which the noticeable difference varies in time.[18] Formally, suppose that in each observation $t \in T$ the consumer is maximising an interval order P_t that admits a noticeable difference $\lambda_t > 1$, where P_t and λ_t may differ for each t.

In order to capture changes in the agent's sensitivity to differences among alternatives, rather than variations in the underlying "true" preference, we restrict our attention to interval orders that are consistent with respect to the induced weak order. Let $\succeq_t$ denote the weak order induced by the interval order P_t, for $t \in T$. A profile $(P_t)_{t\in T}$ is *consistent* in the above sense whenever $\succeq_t = \succeq_s$, for all $t, s \in T$. This is to say that, even though the agent may be maximising a different ordering P_t in each observation, the underlying "true" preference relation, i.e., as if perfect discrimination were possible, remains unchanged. Roughly speaking, whenever interval orders $(P_t)_{t\in T}$ can be represented as in condition (1), there is a utility function u such that

$$xP_ty \quad \text{if and only if} \quad u(x) > u(y) + \delta_t(x),$$

for some (potentially different) threshold functions δ_t, for all $t \in T$.

A set of observations $\mathcal{O} = \big\{(B_t, x_t) : t \in T\big\}$ is rationalisable with *variable noticeable differences* $(\lambda_t)_{t\in T}$ if there is a profile $(P_t)_{t\in T}$ of consistent interval orders such that, for each $t \in T$, relation P_t admits the noticeable difference λ_t and

$$y \in B_t \quad \text{implies} \quad \textit{not } yP_tx_t, \tag{5}$$

i.e., the interval order P_t is maximised over B_t in observation t. This notion of rationalisation captures the idea that, even though the underlying "true" preference of the consumer remain unchanged throughout the experiment, the level of cognitive ability to discriminate among options varies from one instance to another.

Let $F : \mathbb{R}^T_+ \to \mathbb{R}$ be a well-defined, continuous, and increasing aggregator function. The *variable just-noticeable difference*, denoted by λ^*_F, is the infimum of $F\big((\lambda_t)_{t\in T}\big)$ over all noticeable differences $(\lambda_t)_{t\in T}$ that rationalise $\mathcal{O}$ as in (5). In other words, conditional on the criterion F, it measures the least noticeable differences that are sufficient to explain the observations with the model presented above. We show in the Section 5 that whenever $F\big((\lambda_t)_{t\in T}\big) := \max\big\{\lambda_t : t \in T\big\}$, the uniform and variable just-noticeable differences coincide. Thus, the former is a special case of the latter.

[18] In this paper we are agnostic about the source of such variations.

4.3 The second equivalence result

In our second theorem we argue that Varian's efficiency index coincides with the variable just-noticeable difference. Before we state the result, recall that $F : \mathbb{R}^T_+ \to \mathbb{R}$ denotes a well-defined, continuous, and increasing aggregator function.

Theorem 2. *For any set of observations $\mathcal{O}$ and aggregator function F, the variable just-noticeable difference λ^*_F equals the infimum of $F\big((1/e_t)_{t\in T}\big)$ with respect to all efficiency parameters $(e_t)_{t\in T}$ for which the dataset is rationalisable as in (4). Conversely, Varian's efficiency index e^*_F equals the supremum of $F\big((1/\lambda_t)_{t\in T}\big)$ with respect to all variable noticeable differences $(\lambda_t)_{t\in T}$ that rationalise the set as in (5).*

Evaluating Varian's efficiency index is equivalent to determining the corresponding variable just-noticeable difference. Thus, this measure can be interpreted in terms of the consumer's cognitive ability to differentiate among alternatives.

The proof of the above result is similar to the argument supporting Theorem 1. In fact, we show that any efficiency parameters $(e_t)_{t\in T}$ that rationalise the set of observations as in (4) are essentially equal to the inverses of variable noticeable differences $(\lambda_t)_{t\in T}$ supporting the data as in (5). First of all, a dataset $\mathcal{O} = \big\{(B_t, x_t) : t \in T\big\}$ is rationalisable for efficiency indices $(e_t)_{t\in T}$ only if, for any cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(y,z),(z,a)\big\}$ in $T\times T$ such that $x_s \in e_tB_t$, for all $(t,s)\in\mathcal{C}$, we have $x_s \in \partial(e_tB_t)$, for all $(t,s)\in\mathcal{C}$. Clearly, it is a generalisation of the condition discussed at the end of Section 3.

As shown in Section 5, a dataset obeys the above restriction only if it is rationalisable with any variables noticeable difference $(\lambda_t)_{t\in T}$ such that $\lambda_t \geq 1/e_t$, for all $t \in T$, and $\lambda_t > 1/e_t$, for some $t \in T$.[19] Given monotonicity and continuity of the aggregator F, the variable just-noticeable difference is bounded from below by the supremum of $F\big((1/e_t)_{t\in T}\big)$. Conversely, whenever the set $\mathcal{O}$ is rationalisable with some noticeable differences $(\lambda_t)_{t\in T}$, it can be supported with any efficiency parameters $(e_t)_{t\in T}$ that satisfy $e_t \geq 1/\lambda_t$, for all $t\in T$. This excludes the possibility that λ^*_F strictly greater than the supremum of $F\big((1/e_t)_{t\in T}\big)$. We prove the latter part of the result analogously.

In general, Varian's efficiency index is not equal to the inverse of the variable just-noticeable difference. However, for particular specifications of the aggregator function F,

[19] In particular, we show that the set of observations is rationalisable with a profile of consistent interval orders $(P_t)_{t\in T}$ that is, in addition, monotone with respect to the noticeable difference. This is to say that, if $\lambda_t \leq \lambda_s$ then $P_s \subseteq P_t$, for any t, $s \in T$. Therefore, whenever the agent is affected by a lower noticeable difference, she is able to differentiate among a greater number of alternatives. Moreover, analogously to the case in Section 3.2, each interval order P_t can be represented with a continuous utility u (uniform across observations) and some continuous threshold δ_t, for $t\in T$. Finally, under some additional conditions on budget sets B_t, the function u is quasiconcave, with no loss of generality.

the equivalence between the two measures holds in this stronger sense.

Corollary 1. *Suppose that the aggregator function $F : \mathbb{R}^T_+ \to \mathbb{R}$ is given by the geometric mean, i.e., $F\big((z_t)_{t\in T}\big) := \sqrt[T]{\prod_{t\in T} z_t}$. For any dataset $\mathcal{O}$, Varian's efficiency index is equal to the inverse of the variable just-noticeable difference, i.e., we have $e^*_F = 1/\lambda^*_F$.*

Proof. By Theorem 2, we obtain

$$e^*_F \;=\; \sup \sqrt[T]{\prod_{t\in T} \frac{1}{\lambda_t}} \;=\; \sup \frac{1}{\sqrt[T]{\prod_{t\in T} \lambda_t}} \;=\; \frac{1}{\inf \sqrt[T]{\prod_{t\in T} \lambda_t}} \;=\; \frac{1}{\lambda^*_F},$$

where the supremum and infimum are taken with respect to all noticeable differences $(\lambda_t)_{t\in T}$ under which the set of observations $\mathcal{O}$ is rationalisable as in (5). □

5 Proof of the equivalence theorems

We conduct the proof in four steps. First, we characterise datasets that can be rationalised with efficiency parameters, as in Section 4.1. Then, we discuss the necessary and sufficient conditions under which observations are rationalisable with variable noticeable differences, as in Section 4.2. In the third part, we employ these preliminary results to prove Theorem 2. Since Theorem 1 is implied by the former, we postpone its proof until the end of this section. Our arguments are constructive and provide a deeper understanding of the relationship between efficiency indices and noticeable differences. Therefore, we believe that the following auxiliary results are of interest in themselves.

Part 1: Rationalisation with efficiency vectors

Suppose that a set of observations $\mathcal{O}$ is rationalisable with a locally non-satiated utility function for some efficiency vector $(e_t)_{t\in T}$.

Axiom 1. *For any cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(y,z),(z,a)\big\}$ in $T \times T$, if $x_s \in e_t B_t$, for all $(t,s) \in \mathcal{C}$, then $x_s \in \partial(e_t B_t)$, for all $(t,s) \in \mathcal{C}$.*

This restriction requires that, for any cycle $\mathcal{C}$ induced by the dataset, in which the choice from observation s is an element of the perturbed budget from observation t, for all pairs $(t,s) \in \mathcal{C}$, all bundles belong to the upper bounds of their respective sets. Notice that, whenever $(e_t)_{t\in T}$ is the unit vector, Axiom 1 is equivalent to the generalised version of GARP in Forges and Minelli (2009).[20]

[20]Moreover, whenever sets B_t are linear, for all $t \in T$, this condition coincides with the original formulation of GARP as in Afriat (1967), Diewert (1973), and Varian (1982).

Necessity of Axiom 1 for this form of rationalisation follows immediately from the definition. Indeed, for any locally non-satiated utility u that rationalises $\mathcal{O}$, it must be that $x_s \in e_t B_t$ implies $u(x_t) \geq u(x_s)$, for any $t, s \in T$. Moreover, by local non-satiation of u, the latter inequality must be strict whenever x_s is not in the upper bound of $e_t B_t$. Therefore, for any cycle $\mathcal{C}$ specified as in the axiom, we obtain

$$u(x_a) \geq u(x_b) \geq u(x_c) \geq \ldots \geq u(x_y) \geq u(x_z) \geq u(x_a),$$

which can be satisfied only if all the inequalities are binding. However, this requires for x_s to be in the upper bound of $e_t B_t$, for all $t \in T$. In other words, Axiom 1 excludes the possibility of any strict cycles in the revealed preference relation. Below we argue that it is also a sufficient condition for the data to be rationalisable as in (4).

Proposition 2. *Set of observations $\mathcal{O}$ is rationalisable with an efficiency vector $(e_t)_{t \in T}$ if and only if it satisfies Axiom 1 for $(e_t)_{t \in T}$.*

The sufficiency part of the result can be supported with a simple modification of the argument in Forges and Minelli (2009, Section 1.2). Thus, we omit the proof. It follows immediately that a dataset is rationalisable as in (2) for some efficiency parameter $e \in [0, 1]$ if and only if it satisfies the axiom for $e_t = e$, for all $t \in T$.

Part 2: Rationalisation with noticeable differences

In this subsection we characterise sets of observations that can be rationalised with some variable noticeable differences $(\lambda_t)_{t \in T}$. We postpone our discussion on uniform noticeable difference until Part 4. Consider the following condition.

Axiom 2. *For an arbitrary cycle $\mathcal{C} = \big\{(a, b), (b, c), \ldots, (y, z), (z, a)\big\}$ in $T \times T$, we have $(\lambda_t x_s) \notin B_t$, for some $(t, s) \in \mathcal{C}$.*

This condition requires that there is no cycle $\mathcal{C}$ in the set of observations such that $(\lambda_t x_s) \in B_t$, or equivalently, $x_s \in (1/\lambda_t) B_t$, for all $(t, s) \in \mathcal{C}$. Therefore, for any sequence $\mathcal{C}$ with $x_s \in B_t$, for all $(t, s) \in \mathcal{C}$, at least one bundle x_s has to be sufficiently close to the upper bound of B_t. See Figure 3. In particular, the above condition refers to one element cycles $\mathcal{C} = \big\{(t, t)\big\}$. Hence, we have $(\lambda_t x_t) \notin B_t$, for all $t \in T$.

Showing that Axiom 2 is necessary for the set of observations to be rationalisable with variable noticeable differences $(\lambda_t)_{t \in T}$ is quite straightforward. Suppose there is a profile $(P_t)_{t \in T}$ of consistent interval orders that satisfy condition (5). In particular, whenever

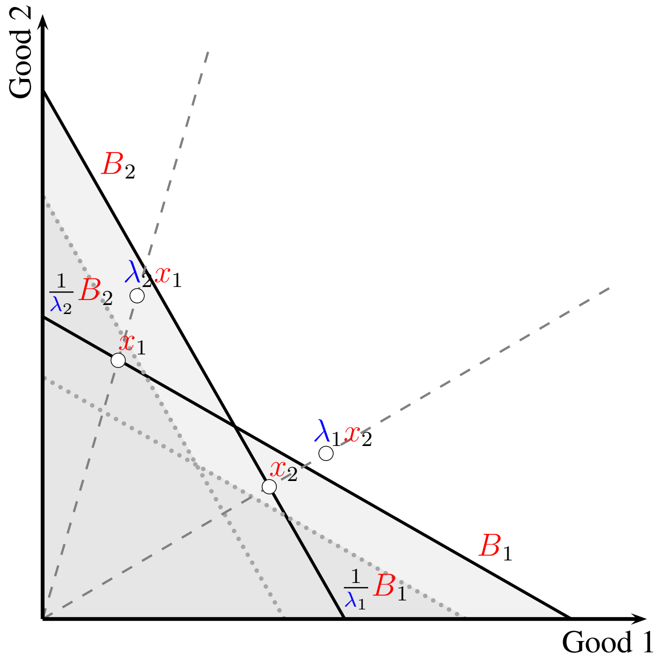

Figure 3: Recall the dataset form Figure 1. Although $(\lambda_2 x_1)$ belongs to B_2, bundle $(\lambda_1 x_2)$ is not an element of B_1. Equivalently, we have $x_1 \in (1/\lambda_2)B_2$ and $x_2 \notin (1/\lambda_1)B_1$. Therefore, these observations satisfy Axiom 2 for the particular values of λ_1, λ_2.

$(\lambda_t x_s) \in B_t$, it must be that *not* $(\lambda_t x_s)P_t x_t$ and $(\lambda_t x_s)P_t x_s$, for any $t, s \in T$. Hence, we have $x_t \succ_t x_s$, where $\succeq_t$ denotes the weak order induced by P_t. Since the interval orders are consistent, we can denote $\succeq_t = \succeq$, for all $t \in T$.

The above observation implies that, for any cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(y,z),(z,a)\big\}$ in $T \times T$ such that $(\lambda_t x_s) \in B_t$, for all $(t,s) \in \mathcal{C}$, we obtain

$$x_a \succ x_b \succ x_c \succ \ldots \succ x_y \succ x_z \succ x_a,$$

which yields $x_a \succ x_a$, by transitivity of $\succ$. However, this contradicts that the relation is irreflexive (recall Proposition 1). Therefore, no such cycle is admissible. Below we claim that the condition is also sufficient for this from of rationalisation.

Proposition 3. *Set of observations $\mathcal{O}$ is rationalisable with variable noticeable differences $(\lambda_t)_{t\in T}$ if and only if it satisfies Axiom 2 for $(\lambda_t)_{t\in T}$. In addition, with no loss of generality, the corresponding profile $(P_t)_{t\in T}$ is monotone with respect to the noticeable difference, i.e., if $\lambda_t \leq \lambda_s$ then $P_s \subseteq P_t$, for all $t, s \in T$.*

This proposition consists of two parts. First of all, it states that Axiom 2 is both necessary and sufficient for a dataset to be rationalisable with variable noticeable differences. In addition, whenever the set is rationalisable in the above sense with some consistent profile of interval orders $(P_t)_{t\in T}$, it can be rationalised with a profile that is monotone with respect to the noticeable difference. That is, for any two indices $t, s \in T$, if the noticeable difference admissible by P_t is lower than that by P_s, i.e., $\lambda_t \leq \lambda_s$, then the relation P_t is finer than P_s, or simply xP_sy implies xP_ty, for any bundles x, $y \in \mathbb{R}^\ell_+$. Therefore, whenever the agent is affected by a lower noticeable difference, she is able to discriminate among a greater number of alternatives.

The proof of the sufficiency part of the above proposition is rather extensive. For this reason we conduct the argument via several lemmas.

Lemma 1. *For all $t \in T$, there is a continuous function $h_t : \mathbb{R}^\ell_+ \setminus \{0\} \to \mathbb{R}$ such that*

(i) $h_t(\theta y) > h_t(y)$, for all $\theta > 1$ and non-zero $y \in \mathbb{R}^\ell_+$;

(ii) $h_t(\lambda' y) \geq h_t(y) + 1$, for all $\lambda' \geq \lambda_t$ and non-zero $y \in \mathbb{R}^\ell_+$;

(iii) $k_{ts} \geq h_t(x_s) > k_{ts} - 1$, where $k_{ts} := \inf\{k \in \mathbb{Z} : x_s \in \lambda_t^k B_t\}$, for all $s \in T$.

Proof. Define the gauge function $\gamma_t : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ by $\gamma_t(y) := \inf\{\theta > 0 : y \in \theta B_t\}$, for all $t \in T$. Following Lemma 1 in Forges and Minelli (2009), this function is continuous, homogeneous of degree one, and satisfies $\gamma_t(y) \leq 1$ if and only if $y \in B_t$. Moreover, observe that, we have $\gamma_t(y) = 0$ if and only if $y = 0$.

Take any continuous and strictly increasing function $f_t : [1, \lambda_t] \to [0, 1]$ that satisfies $f_t(1) = 0$ and $f_t(\lambda_t) = 1$. Let $g_t : \mathbb{R}_{++} \to \mathbb{R}$ be an extension of f_t to $\mathbb{R}_{++}$ given by

$$g_t(z) \;:=\; \sum_{k\in\mathbb{Z}} \Big[f\big(z/\lambda_t^{k-1}\big) + k - 1\Big]\chi_{A_k}(z),$$

where χ_{A_k} is the indicator function and $A_k := \big(\lambda_t^{k-1}, \lambda_t^k\big]$, for all $k \in \mathbb{Z}$. It can be easily verified that the function is continuous and strictly increasing. We argue that if $\lambda' \geq \lambda_t$ and $z > 0$ then $g_t(\lambda' z) \geq g_t(z) + 1$. Since $z \in A_k$ implies $(\lambda_t z) \in A_{k+1}$, we have

$$g_t(\lambda' z) - g_t(z) \;\geq\; g_t(\lambda_t z) - g_t(z) \;=\; \Big[f_t\big(\lambda_t z/\lambda_t^k\big) + k\Big] - \Big[f_t\big(z/\lambda_t^{k-1}\big) + k - 1\Big] \;=\; 1,$$

where the first inequality follows from monotonicity of g_t. In addition, by construction of the function g_t, we know that $z \in A_k$ implies $k \geq g_t(z) > k - 1$.[21]

Define $h_t : \mathbb{R}^\ell_+ \setminus \{0\} \to \mathbb{R}$ by $h_t(y) := (g_t \circ \gamma_t)(y)$, which is a continuous function. To show that it satisfies property (i), take any $\theta > 1$ and a non-zero $y \in \mathbb{R}^\ell_+$. By homogeneity of the gauge function γ_t and strict monotonicity of g_t, we obtain

$$h_t(\theta y) \;=\; g_t\big(\gamma_t(\theta y)\big) \;=\; g_t\big(\theta\gamma_t(y)\big) \;>\; g_t\big(\gamma_t(y)\big) \;=\; h_t(y).$$

[21] This is because $z \in A_k$ implies $k = g(\lambda^k) \geq g_t(z) > g(\lambda^{k-1}) = k - 1$.

In order to prove (ii), take any $\lambda' \geq \lambda_t$ and a non-zero $y \in \mathbb{R}^\ell_+$. Since function γ_t is homogeneous and $g_t(\lambda' z) \geq g_t(z) + 1$, for all $z > 0$, we conclude that

$$h_t(\lambda' y) \;=\; g_t\big(\gamma_t(\lambda' y)\big) \;=\; g_t\big(\lambda' \gamma_t(y)\big) \;\geq\; g_t\big(\gamma_t(y)\big) + 1 \;=\; h_t(y) + 1.$$

Finally, take any $t,\, s \in T$. By construction of k_{ts}, we have $\lambda_t^{k_{ts}} \geq \gamma_t(x_s) > \lambda_t^{k_{ts}-1}$, or equivalently $\gamma_t(x_s) \in A_{k_{ts}}$. This implies $k_{ts} \geq g_t\big(\gamma_t(x_s)\big) > k_{ts} - 1$, which proves (iii). $\square$

The proof of Lemma 1 does not require for Axiom 2 to be satisfied. In fact, given our framework, existence of such functions is independent of that restriction. Next, we claim that the axiom implies existence of a solution to a particular linear system.

Lemma 2. *Suppose that a set of observations $\mathcal{O}$ obeys Axiom 2 for some $(\lambda_t)_{t\in T}$. For any functions $(h_t)_{t\in T}$ specified in Lemma 1, there are numbers $(\phi_t)_{t\in T}$ and strictly positive numbers $(\mu_t)_{t\in T}$ such that $\phi_s < \phi_t + \mu_t\big[h_t(x_s) + 1\big]$, for all $t, s \in T$.*

The system of inequalities specified in the above lemma is very similar to the well-known *Afriat's inequalities*. However, unlike in the classical case, we require that each of the inequalities is strict. Nevertheless, the result can be proven in a relatively standard fashion. Below, we modify the approach by Fostel et al. (2004).

Proof of Lemma 2. Denote $q_{ts} := \big[h_t(x_s) + 1\big]$, for all $t,\, s \in T$. First, we claim that whenever set $\mathcal{O}$ obeys Axiom 2 then for any cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(y,z),(z,a)\big\}$ in $T \times T$, we have $q_{ts} > 0$, for some $(t,s) \in \mathcal{C}$. Define $k_{ts} := \inf\big\{k \in \mathbb{Z} : x_s \in \lambda_t^k B_t\big\}$, for any $t, s \in T$. If Axiom 2 holds, we have $(\lambda_t x_s) \notin B_t$, or equivalently $k_{ts} \geq 0$, for some $(t,s) \in \mathcal{C}$. By Lemma 1(iii), this implies $q_{ts} = h_t(x_s) + 1 > k_{ts} - 1 + 1 \geq 0$.

Next, we argue that there is some $t \in T$ satisfying $q_{ts} > 0$, for all $s \in T$. Otherwise, it would be possible to find indices a, b in T such that $q_{ab} \leq 0$. Similarly, there would be some $c \in T$ such that $q_{bc} \leq 0$, and so on. Eventually, we would construct a cycle $\mathcal{C} := \big\{(a,b),(b,c),\ldots,(y,z),(z,a)\big\}$ with $q_{ts} \leq 0$, for all $(t,s) \in \mathcal{C}$, violating Axiom 2.

We conduct the remainder of the proof by induction on the size of the set of observations. Whenever set $\mathcal{O} = \big\{(B_t, x_t)\big\}$ is a singleton, it must be that $q_{tt} > 0$. Clearly, this guarantees that for any numbers ϕ_t and $\mu_t > 0$, we have $\phi_t < \phi_t + \mu_t q_{tt}$.

To show the inductive step, suppose that the claim in the lemma holds for any set of size $(T-1)$. Take any $t \in T$ such that $q_{ts} > 0$, for all $s \in T$. By our earlier claim, such an index exists. Denote $T' := T \setminus \{t\}$. Clearly, the sub-dataset $\mathcal{O}' = \big\{(B_s, x_s) : s \in T'\big\}$ satisfies Axiom 2 and has the cardinality of $(T-1)$. Thus, there exist numbers $(\phi_s)_{s\in T'}$ and strictly positive numbers $(\mu_s)_{s\in T'}$ such that $\phi_s < \phi_r + \mu_r q_{rs}$, for all $s,\, r \in T'$. Take any

ϕ_t satisfying $\phi_t < \phi_s + \mu_s q_{st}$, for all $s \in T'$. Finally, choose $\mu_t > 0$ such that $\phi_s < \phi_t + \mu_t q_{ts}$, for all $s \in T$. Since $q_{ts} > 0$, for all $s \in T$, it is always possible. $\square$

In the next step, we show that whenever there is a solution to the above system of inequalities there exist particular functions u and v_t, for $t \in T$. These are instrumental for constructing interval orders that rationalise the data.

Lemma 3. *Suppose that the system of inequalities in Lemma 2 has a solution. There are continuous functions $u : \mathbb{R}^\ell_+ \setminus \{0\} \to \mathbb{R}$ and $v_t : \mathbb{R}^\ell_+ \setminus \{0\} \to \mathbb{R}$, for all $t \in T$, such that*

(i) $u(y) \geq v_t(y)$, for all non-zero $y \in \mathbb{R}^\ell_+$;

(ii) $\lambda' \geq \lambda_t$ implies $v_t(\lambda' y) > u(y)$, for all non-zero $y \in \mathbb{R}^\ell_+$;

(iii) $y \in B_t$ implies $u(x_t) \geq v_t(y)$, for all non-zero $y \in \mathbb{R}^\ell_+$;

(iv) $v_t(y) \geq v_s(y)$, for all non-zero $y \in \mathbb{R}^\ell_+$ and t, $s \in T$ such that $\lambda_t \leq \lambda_s$.

Proof. Take any functions $(h_t)_{t\in T}$ specified as in Lemma 1 and numbers $(\phi_t)_{t\in T}$, $(\mu_t)_{t\in T}$ that solve the inequalities in Lemma 2. Define function $u : \mathbb{R}^\ell_+ \setminus \{0\} \to \mathbb{R}$ by

$$u(y) \;:=\; \min\Big\{\phi_s + \mu_s\big[h_s(y) + 1\big] : s \in T\Big\}.$$

Clearly, it is continuous and satisfies $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero $y \in \mathbb{R}^\ell_+$.[22] In addition, following Lemma 2, it must be that $\phi_t < u(x_t)$, for all $t \in T$.

For any $t \in T$, define the continuous function $w_t : \mathbb{R}^\ell_+ \setminus \{0\} \to \mathbb{R}$ by

$$w_t(y) \;:=\; \min\Big\{u(x_s) + \mu_s h_s(y) : s \in T \text{ such that } \lambda_s \leq \lambda_t\Big\}.$$

Note that, for any t, $s \in T$ such that $\lambda_s \leq \lambda_t$, we have $w_s(y) \geq w_t(y)$, for $y \in \mathbb{R}^\ell_+ \setminus \{0\}$. Moreover, by Lemma 1(ii), for any $t \in T$, number $\lambda' \geq \lambda_t$, and a non-zero $y \in \mathbb{R}^\ell_+$,

$$\begin{aligned} u(y) \;&:=\; \min\Big\{\phi_s + \mu_s\big[h_s(y) + 1\big] : s \in T\Big\} \\ &\leq\; \min\Big\{\phi_s + \mu_s\big[h_s(y) + 1\big] : s \in T \text{ such that } \lambda_s \leq \lambda_t\Big\} \\ &<\; \min\Big\{u(x_s) + \mu_s h_s(\lambda' y) : s \in T \text{ such that } \lambda_s \leq \lambda_t\Big\} \\ &=:\; w_t(\lambda' y), \end{aligned}$$

where the strict inequality is implied by $\phi_s < u(x_s)$ and $h_s(y) + 1 \leq h_s(\lambda' y)$, for all $s \in T$ such that $\lambda' \geq \lambda_t \geq \lambda_s$. Finally, following Lemma 1(iii), whenever $y \in B_t$ then $h_t(y) \leq 0$. In particular, this implies that, for any $y \in B_t$, we have

$$w_t(y) \;\leq\; u(x_t) + \mu_t h_t(y) \;\leq\; u(x_t).$$

[22] The later follows from $h_t(\theta y) > h_t(y)$, for all $\theta > 1$ and non-zero y, and $\mu_t > 0$, for all $t \in T$.

For each $t \in T$, let $v_t : \mathbb{R}^\ell_+ \setminus \{0\} \to \mathbb{R}$ be given by $v_t(y) := \min\big\{u(y), w_t(y)\big\}$, which is continuous and satisfies $u(y) \geq v_t(y)$, for all non-zero $y \in \mathbb{R}^\ell_+$. Thus, condition (i) holds. Next, take any $\lambda' \geq \lambda_t$ and a non-zero $y \in \mathbb{R}^\ell_+$. By our previous observations,

$$u(y) \;<\; \min\big\{u(\lambda' y), w_t(\lambda' y)\big\} \;=\; v_t(\lambda' y),$$

since $\lambda' > 1$. This proves property (ii). In addition, we have $v_t(y) \leq w_t(y) \leq u(x_t)$, for any non-zero $y \in B_t$, which implies (iii). Finally, notice that if $\lambda_t \leq \lambda_s$ then

$$w_t(y) \geq w_s(y) \;\text{ implies }\; v_t(y) \geq v_s(y),$$

for any $y \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $t,\, s \in T$. Therefore, condition (iv) holds as well. □

In order to complete the proof, take any functions u and v_t specified in Lemma 3, and define a continuous function $\delta_t : \mathbb{R}^\ell_+ \setminus \{0\} \to \mathbb{R}$ by $\delta_t(y) := \big[u(y) - v_t(y)\big]$, for all $t \in T$. By Lemma 3(i), it is positive. Moreover, Lemma 3(ii) implies

$$u(y) + \delta_t(\lambda' y) \;<\; v_t(\lambda' y) + \delta_t(\lambda' y) \;=\; u(\lambda' y),$$

for any $\lambda' \geq \lambda_t$ and $y \in \mathbb{R}^\ell_+ \setminus \{0\}$, while Lemma 3(iii) guarantees that

$$u(x_t) \;\geq\; v_t(y) \;=\; u(y) - \delta_t(y),$$

for all non-zero $y \in B_t$. Finally, by property (iv) in Lemma 3, we obtain $\delta_t(y) \leq \delta_s(y)$, for all vectors $y \in \mathbb{R}^\ell_+ \setminus \{0\}$ and indices $t,\, s \in T$ that satisfy $\lambda_t \leq \lambda_s$.

For each $t \in T$, construct a binary relation P_t by

$$xP_t y \quad \text{if and only if} \quad u(x) \geq u(y) + \delta_t(x),$$

with $yP_t 0$, for all non-zero $y \in \mathbb{R}^\ell_+$. Clearly, relation P_t is an interval order satisfying $(\lambda' y)P_t y$, for all $\lambda' \geq \lambda_t$ and non-zero $y \in \mathbb{R}^\ell_+$. Moreover, if $y \in B_t$ then *not* $yP_t x_t$.

Next, we show that the profile $(P_t)_{t\in T}$ is consistent with respect to the induced weak order. Take any $t,\, s \in T$. We need to show that $x \succ_t y$ if and only if $x \succ_s y$. By construction, we have $y \succ_t 0$, for all non-zero $y \in \mathbb{R}^\ell_+$ and $t \in T$. Otherwise, if $x \succ_t y$ then $u(x) > u(y)$. Given that $u(\lambda_s x) - \delta_s(\lambda_s x) > u(x)$ and $u(y) \geq u(y) - \delta_s(y)$, by continuity of u and δ_s there is some z such that $u(x) \geq u(z) - \delta_s(z) > u(y)$. By definition of P_s, this implies *not* $zP_s x$ and $zP_s y$, which is equivalent to $x \succ_s y$.

Finally, to show that the profile is monotone with respect to the noticeable difference, take any $t,\, s \in T$ such that $\lambda_t \leq \lambda_s$ and suppose that $xP_s y$. By definition of P_s,

$$u(x) \;>\; u(y) + \delta_s(x) \;\geq\; u(y) + \delta_t(x),$$

since $\delta_s(x) \geq \delta_t(x)$. Clearly, this implies xP_ty, which completes our proof.

Apart from proving that Axiom 2 is necessary and sufficient for a set of observations to be rationalisable with variable noticeable differences, the above argument shows that, without loss of generality, any rationalisable set can be supported by a profile of interval orders that is monotone with respect to the noticeable difference. Therefore, this form of rationalisation captures precisely the ability of the consumer to differentiate between alternatives. In addition, any such profile can be represented with a continuous utility u and threshold functions δ_t as in (1). The above properties come "for free".

Finally, given the construction of the utility u in the proof of Lemma 3, it is easy to show that whenever the complement of set B_t is convex, for all $t \in T$, then the function u is quasiconcave. Therefore, in such a case, the hypothesis regarding convexity of the "true" preference relation $\succeq$ is irrefutable with the observable data.

5.1 Part 3: Proof of Theorem 2

The second equivalence result is implied by the following two lemmas. First, we show that any noticeable differences rationalising the set of observations are equal to inverses of the corresponding efficiency parameters. Consider the following result.

Lemma 4. *If set $\mathcal{O}$ is rationalisable with variable noticeable differences $(\lambda_t)_{t\in T}$ then it is rationalisable with any efficiency vector $(e_t)_{t\in T}$ such that $e_t \leq 1/\lambda_t$, for all $t \in T$.*

Proof. Let dataset $\mathcal{O}$ satisfy Axiom 2 for some noticeable differences $(\lambda_t)_{t\in T}$. For any cycle $\mathcal{C} = \{(a,b),(b,c),\ldots,(y,z),(z,a)\}$ in $T \times T$, we have $(\lambda_t x_s) \notin B_t$, or equivalently $x_s \notin (1/\lambda_t)B_t$, for some $(t,s) \in \mathcal{C}$. This suffices for Axiom 1 to be satisfied for any efficiency parameters $(e_t)_{t\in T}$ such that $e_t \leq 1/\lambda_t$, for all $t \in T$. □

The second result is the converse to Lemma 4. We show that efficiency parameters are essentially equivalent to inverses of noticeable differences.

Lemma 5. *Whenever a set of observations $\mathcal{O}$ is rationalisable with some efficiency parameters $(e_t)_{t\in T}$, then it is rationalisable with any noticeable differences $(\lambda_t)_{t\in T}$ such that $\lambda_t \geq 1/e_t$, for all $t \in T$, and $\lambda_t > 1/e_t$, for some $t \in T$.*

Proof. Take any numbers $(\lambda_t)_{t\in T}$ specified above. It suffices to verify that for any cycle $\mathcal{C} = \{(a,b),(b,c),\ldots,(y,z),(z,a)\}$ in $T \times T$, we have $(\lambda_t x_s) \notin B_t$, for some $(t,s) \in \mathcal{C}$. If not, there is a cycle $\mathcal{C}$ with $(\lambda_t x_s) \in B_t$, or equivalently $x_s \in (1/\lambda_t)B_t \subseteq e_tB_t$, for all $(t,s) \in \mathcal{C}$. Since $e_t > 1/\lambda_t$, for some $t \in T$, this implies $x_s \in (e_tB_t) \setminus \partial(e_tB_t)$, for some pair $(t,s) \in \mathcal{C}$, which contradicts that the dataset satisfies Axiom 1. □

Given the two lemmas, we proceed with the proof of Theorem 2. First, we argue that the variable just-noticeable difference λ^*_F is equal to the infimum of $F\big((1/e_t)_{t\in T}\big)$ with respect to all efficiency vectors $(e_t)_{t\in T}$ for which the dataset is rationalisable. Denote the latter value by v^*_F. To show that $\lambda^*_F \geq v^*_F$, take any noticeable differences $(\lambda_t)_{t\in T}$ that rationalise the data as in (5). By Lemma 4, there exists an efficiency vector $(e_t)_{t\in T}$ that rationalise the data in the appropriate sense and satisfies

$$F\big((\lambda_t)_{t\in T}\big) \;=\; F\big((1/e_t)_{t\in T}\big) \;\geq\; v^*_F.$$

By taking the infimum over the left hand side, we obtain $\lambda^*_F \geq v^*_F$. Suppose that $\lambda^*_F > v^*_F$. Given monotonicity and continuity of the aggregator F, along with Lemma 5, there are vectors $(\lambda_t)_{t\in T}$ and $(e_t)_{t\in T}$ that rationalise the data in the respective sense and satisfy

$$\lambda^*_F \;>\; F\big((\lambda_t)_{t\in T}\big) \;\geq\; F\big((1/e_t)_{t\in T}\big) \;\geq\; v^*_F.$$

However, this contradicts that λ^*_F is the variable just-noticeable difference. We prove the second part of the theorem analogously. This concludes our argument.

5.2 Part 4: Proof of Theorem 1

This result follows from Proposition 3 and Theorem 2. First of all, notice that a set of observations $\mathcal{O}$ is rationalisable with a uniform noticeable difference $\lambda > 1$ if and only if it satisfies Axiom 2 for $\lambda_t = \lambda$, for all $t \in T$. Following the argument in Part 1, the condition is necessary. To show that it is sufficient, recall that (by Proposition 3) whenever the axiom is satisfied, there is a profile $(P_t)_{t\in T}$ of consistent interval orders that rationalise the data. Moreover, the profile is monotone with respect to the noticeable difference, hence, $\lambda_t = \lambda_s = \lambda$ implies $P_t = P_s = P$, for all $t, s \in T$.

Given the above observation, it is easy to show that the uniform and the variable just-noticeable differences coincide whenever $F\big((\lambda_t)_{t\in T}\big) = \max\big\{\lambda_t : t \in T\big\}$.

To prove Theorem 1, recall that Afriat's efficiency index is equivalent to the Varian's measure whenever $F\big((e_t)_{t\in T}\big) = \min\big\{e_t : t \in T\big\}$. By Theorem 2, we obtain

$$e^*_A \;=\; \sup\Big\{\min\big\{1/\lambda_t : t\in T\big\}\Big\} \;=\; \frac{1}{\inf\Big\{\max\big\{\lambda_t : t\in T\big\}\Big\}} \;=\; \frac{1}{\lambda^*_U},$$

where the supremum and infimum are taken with respect to noticeable differences $(\lambda_t)_{t\in T}$ for which the set of observations is rationalisable as in (5).

6 On revealed preference

An important feature of revealed preference analysis is that it allows to infer consumer preference form observable choices. In this section we investigate the properties of revealed relations induced by the models discussed in Sections 4.1 and 4.2.

First, we focus on implications of the model with efficiency parameters that was introduced in Section 4.1. Take any set of observations $\mathcal{O} = \big\{(B_t, x_t) : t \in T\big\}$ and some efficiency parameters $e = (e_t)_{t\in T}$. Conditional on the vector e, we define the *directly revealed preference* relation $\succeq_e^{**}$ as follows:

$$x \succeq_e^{**} y \quad \text{whenever} \quad x = x_t \text{ and } y \in e_t B_t, \text{ for some } t \in T.$$

The relation is *strict* and denoted by $x \succ_e^{**} y$ if $x = x_t$ and $y \in e_t B_t \setminus \partial(e_t B_t)$, for some $t \in T$, i.e., whenever bundle y is in the interior of $e_t B_t$. The *revealed preference* relation $\succeq_e^{*}$ is the transitive closure of $\succeq_e^{**}$. That is, we have $x \succeq_e^{*} y$ whenever there is a sequence $\{z_k\}_{k=1}^{K}$ such that $z_1 = x$, $z_K = y$, and $z_k \succeq_e^{**} z_{k+1}$, for all $k = 1, \ldots, (K-1)$. The relation is *strict*, and denoted by $x \succ_e^{*} y$, if $z_k \succ_e^{**} z_{k+1}$, for some k.

Proposition 4. *Set $\mathcal{O}$ is rationalisable for efficiency parameters $e = (e_t)_{t\in T}$ if and only if the strict revealed preference relation $\succ_e^{*}$ is irreflexive.*

It is straightforward to verify that the relation $\succeq_e^{*}$ is consistent with the ordering induced by *any* locally non-satiated utility function u that rationalises the set of observations for $e = (e_t)_{t\in T}$. This is to say that: if $x \succeq_e^{*} y$ then $u(x) \geq u(y)$, and $x \succ_e^{*} y$ implies $u(x) > u(y)$. Therefore, the relation recovers preferences of the agent. This obviously implies that set $\mathcal{O}$ is rationalisable in this sense only if $\succ_e^{*}$ is irreflexive.

To show the converse, notice that we have $x \succ_e^{*} x$, for some $x \in \mathbb{R}_+^{\ell}$, only if Axiom 1 is violated. Hence, by Proposition 2, it suffices for $\succ_e^{*}$ to be irreflexive for the set of observations to be rationalisable with efficiency indices $e = (e_t)_{t\in T}$.

Next, we turn to the revealed preference relation induced by the model of consumer choice with noticeable differences. Conditionally on noticeable differences $\lambda = (\lambda_t)_{t\in T}$, we define the *directly revealed strict preference* relation $\succ_\lambda^{**}$, as follows:

$$x \succ_\lambda^{**} y \quad \text{whenever} \quad x = x_t \text{ and } (\lambda_t y) \in B_t, \text{ for some } t \in T.$$

The *revealed strict preference* relation $\succ_\lambda^{*}$ is defined as the transitive closure of $\succ_\lambda^{**}$. That is, for any x and y in $\mathbb{R}_+^{\ell}$, we have $x \succ_\lambda^{*} y$ if there is a sequence $\{z_k\}_{k=1}^{K}$ such that $z_1 = x$, $z_K = y$, and $z_k \succ_\lambda^{**} z_{k+1}$, for all $k = 1, \ldots, (K-1)$. Notice that, we do not define the weak counterpart of $\succ_\lambda^{**}$ or $\succ_\lambda^{*}$.

Proposition 5. *Set $\mathcal{O}$ is rationalisable with noticeable differences $\lambda = (\lambda_t)_{t\in T}$ if and only if the strict revealed preference relation $\succ^*_\lambda$ is irreflexive.*

We omit the proof. Suppose that set $\mathcal{O}$ is rationalisable with a profile $(P_t)_{t\in T}$ of consistent interval orders, where P_t admits the noticeable difference λ_t, for each $t \in T$. It can be shown that the revealed preference relation $\succ^*_\lambda$ is consistent with the strict part of the weak order $\succeq$ induced by the profile. That is, if $x \succ^*_\lambda y$ then $x \succ y$. Given our discussion in Section 4.2, this allows us to recover the "true" preferences of the consumer, i.e., as if perfect discrimination were possible. Once the interval orders admit a representation as in (1), i.e, there is a utility u and a threshold function δ_t such that xP_ty if and only if $u(x) > u(y) + \delta_t(y)$, for each $t \in T$, the relation $\succ^*_\lambda$ is consistent with the ordering induced by u, i.e., if $x \succ^*_\lambda y$ then $u(x) > u(y)$. Below we summarise the relationship among the above notions of revealed preference.

Proposition 6. *For any efficiency vector $e = (e_t)_{t\in T}$ and variable noticeable differences $\lambda = (\lambda_t)_{t\in T}$ such that $e_t = 1/\lambda_t$, for all $t \in T$, we have $\succ^*_e \subset \succ^*_\lambda = \succeq^*_e$.*

This observation follows directly from the definition of the revealed relations, hence, we skip the proof. The proposition states that the revealed preference induced by the two models are essentially equivalent. The only distinction pertains to revealed indifferences. In particular, whenever a set of observations is rationalisable with efficiency parameters $e = (e_t)_{t\in T}$ but fails to satisfy Axiom 2 for $\lambda_t = 1/e_t$, for all $t \in T$, it must be that $x \succeq^*_e x$, for some bundle $x \in \mathbb{R}^\ell_+$, i.e., a bundle is revealed indifferent to itself. For this reason, efficiency parameters and noticeable differences coincide only in the limit.

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