Rotating quantum states

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Quantum field theory on curved space-time
Quantum field theory (QFT) on curved space-time

- Semi-classical limit of quantum gravity
- Background geometry fixed and classical
- Quantum field propagating on this background

Semi-classical Einstein equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} \]

- Renormalized stress-energy tensor \( \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} \)
Quantum field theory (QFT) on curved space-time

- Semi-classical limit of quantum gravity
- Background geometry fixed and classical
- Quantum field propagating on this background

Semi-classical Einstein equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} \]

- Renormalized stress-energy tensor \( \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} \)
Steps in quantum field theory on curved space-time

1. Classical Lagrangian describing the field
2. Canonical quantization
3. Definition of quantum states
   (a) Orthonormal basis of field modes
   (b) Choice of positive frequency
4. Physical interpretation of states and observables
Quantum scalar field $\Phi$

**Step 1 - classical Lagrangian**

- Classical Lagrangian density for a scalar field of mass $M$

  \[
  \mathcal{L}_\Phi = \nabla_\mu \Phi \nabla^\mu \Phi + (M^2 + \xi R) \Phi^2
  \]

- Klein-Gordon equation

  \[
  \left[ \Box - M^2 - \xi R \right] \Phi = 0
  \]

- Klein-Gordon inner product

  \[
  (\Phi_1, \Phi_2)_{KG} = i \int_S \left[ \Phi_2^* \nabla_\mu \Phi_1 - \Phi_1 \nabla_\mu \Phi_2^* \right] dS^\mu
  \]

Involves time derivative of $\Phi$
Quantum scalar field $\Phi$

**Step 2 - canonical quantization**

- Canonical conjugate momentum

$$\Pi_{\Phi} = \frac{\delta L_{\Phi}}{\delta \dot{\Phi}} \quad \dot{\Phi} = \partial_t \Phi$$

- Impose equal-time commutation relations on $t = \text{constant}$

$$[\hat{\Phi}(t, x), \hat{\Pi}_{\Phi}(t, x')] = i\delta(x, x')$$

$$[\hat{\Phi}(t, x), \hat{\Phi}(t, x')] = 0 = [\hat{\Pi}_{\Phi}(t, x), \hat{\Pi}_{\Phi}(t, x')]$$

$$\int_S \delta(x, x') dS = 1$$
Quantum scalar field Φ

Step 3 - definition of quantum states

(a) Choice of time $t$
Pick a suitable time co-ordinate $t$

(b) Choice of positive frequency modes $φ_j$
Solutions of the Klein-Gordon equation such that

$$φ_j \propto e^{-i\omega t} \quad \omega > 0$$

(c) Orthonormal basis of field modes
- Positive frequency modes $φ^+_j$ have positive norm
- Negative frequency modes $φ^-_j$ have negative norm
- Complete set of positive and negative frequency modes $\{φ^+_j, φ^-_j\}$
Quantum scalar field $\Phi$

**Step 3 - definition of quantum states**

- Expand classical field in terms of orthonormal basis
  $$\Phi = \sum_j a_j \phi_j^+ + a_j^\dagger \phi_j^-$$

- Promote expansion coefficients to operators $\hat{a}_j, \hat{a}_j^\dagger$ with
  $$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk} \quad [\hat{a}_j, \hat{a}_k] = 0 \quad [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0$$

- $\hat{a}_j$ - particle annihilation operators
  $\hat{a}_j^\dagger$ - particle creation operators

- Define the *vacuum* state $|0\rangle$:
  $$\hat{a}_j |0\rangle = 0$$
Choice of time

- Choice of ‘time’ co-ordinate important
- Definition of positive frequency
- Implications for definition of vacuum
- Not completely unrestricted - positive frequency modes must have positive norm
Quantum fermion field $\Psi$

**Step 1 - classical Lagrangian**

- Classical Lagrangian density for a fermion field of mass $M$
  \[
  \mathcal{L}_\Psi = \overline{\Psi} \left[ i \gamma^\mu \nabla_\mu - M \right] \Psi
  \]

- Dirac equation
  \[
  \left[ i \gamma^\mu \nabla_\mu - M \right] \Psi = 0
  \]

- Dirac inner product
  \[
  (\Psi_1, \Psi_2)_D = \int \overline{\Psi}_1 \gamma^\mu \Psi_2 \, dS_\mu
  \]

*All* modes have positive norm, regardless of the choice of positive frequency.
Quantum fermion field $\Psi$

**Step 2 - canonical quantization**

- Canonical conjugate momentum

$$\Pi_\Psi = \frac{\delta L_\Psi}{\delta \dot{\Psi}} \quad \dot{\Psi} = \partial_t \Psi$$

- Impose equal-time *anti-commutation* relations on $t = \text{constant}$

$$\{ \hat{\Psi}(t, x), \hat{\Pi}_\Psi(t, x') \} = i\delta(x, x')$$
$$\{ \hat{\Psi}(t, x), \hat{\Psi}(t, x') \} = 0 = \{ \hat{\Pi}_\Psi(t, x), \hat{\Pi}_\Psi(t, x') \}$$

$$\int_S \delta(x, x')\; dS = 1$$
Quantum fermion field $\Psi$

Step 3 - definition of quantum states

(a) *Choice of time $t$*
Pick a suitable time co-ordinate $t$

(b) *Choice of positive frequency modes $\psi_j$*
Solutions of the Dirac equation such that

$$\psi_j \propto e^{-i\omega t} \quad \omega > 0$$

(c) *Orthonormal basis of field modes*
- Both positive frequency modes $\psi_j^+$ and negative frequency modes $\psi_j^-$ have positive norm
- Complete set of positive and negative frequency modes $\{\psi_j^+, \psi_j^-\}$
Quantum fermion field $\Psi$

**Step 3 - definition of quantum states**

- Expand classical field in terms of orthonormal basis

$$\Psi = \sum_j b_j \psi_j^+ + c_j^\dagger \psi_j^-$$

- Promote expansion coefficients to operators $\hat{b}_j, \hat{c}_j$ with

$$\{b_j, b_k^\dagger\} = \delta_{jk} = \{c_j, c_k^\dagger\}$$

- $\hat{b}_j, \hat{c}_j$ - particle annihilation operators
  $\hat{b}_j^\dagger, \hat{c}_j^\dagger$ - particle creation operators

- Define the *vacuum* state $\ket{0}$:

$$\hat{b}_j \ket{0} = \hat{c}_j \ket{0} = 0$$
Observables for physical interpretation of states

Massive scalar field $\Phi$

Stress-energy tensor

$$\hat{T}_{\mu\nu} = (1 - 2\xi) \nabla_\mu \Phi \nabla_\nu \Phi + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu} \nabla_\lambda \Phi \nabla^\lambda \Phi - 2\xi \nabla_\mu \nabla_\nu \Phi$$

$$+ 2\xi g_{\mu\nu} \Phi \Box \Phi + \xi \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \Phi^2 - \frac{1}{2} g_{\mu\nu} M^2 \Phi^2$$

Massive fermion field $\Psi$

Stress-energy tensor

$$\hat{T}_{\mu\nu} = \frac{i}{8} \left\{ \left[ \hat{\Psi}, \gamma_\mu \nabla_\nu \hat{\Psi} \right] + \left[ \hat{\Psi}, \gamma_\nu \nabla_\mu \hat{\Psi} \right] - \left[ \nabla_\mu \hat{\Psi}, \gamma_\nu \hat{\Psi} \right] - \left[ \nabla_\nu \hat{\Psi}, \gamma_\mu \hat{\Psi} \right] \right\}$$
An example in Minkowski space

- Minkowski space in cylindrical co-ordinates
  \[ ds^2 = -dt^2 + d\rho^2 + \rho^2 \, d\varphi^2 + dz^2 \]

- Scalar field modes
  \[ \phi_j = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\omega t + im\varphi + ikz} J_m(q\rho) \]

- Norm of the modes is positive if \( \omega > 0 \)
- Frequency of the modes is positive if \( \omega > 0 \)
Rindler space

Rindler co-ordinates

\[ t = e^{g\zeta} \sinh (g\tau) \quad z = e^{g\zeta} \cosh (g\tau) \]

Rindler scalar field modes

\[ \phi_j = \frac{1}{2|\tilde{\omega}|} e^{-i\tilde{\omega}\tau} \tilde{\phi} (x, y, \zeta) \]

Rindler vacuum

- \( \tilde{\omega} > 0 \) - positive frequency
- \( \tilde{\omega} > 0 \) - positive norm
- Rindler vacuum not equivalent to Minkowski vacuum

[ Figure taken from Fulling and Matsas Scholarpedia 9 31789 (2014) ]
Rotating quantum states on Minkowski space-time
Minkowski space in rotating co-ordinates

Minkowski space in cylindrical co-ordinates

\[ ds^2 = -dt^2 + d\rho^2 + \rho^2\, d\phi^2 + dz^2 \]

Rotating co-ordinates

\[ t \rightarrow \tilde{t}, \quad \phi \rightarrow \tilde{\phi} = \phi - \Omega t \]

Rotating metric

\[ ds^2 = -\left(1 - \rho^2\Omega^2\right)\, d\tilde{t}^2 + 2\rho^2\Omega\, d\tilde{t}\, d\tilde{\phi} + d\rho^2 + \rho^2\, d\tilde{\phi}^2 + dz^2 \]

Speed of light surface (SOL) when \( \rho = \Omega^{-1} \)
Scalar field modes

\[
\phi_j = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\omega t + im\phi + ikz} J_m(q\rho) = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\tilde{\omega} t + im\tilde{\phi} + ikz} J_m(q\rho)
\]

\[
\tilde{\omega} = \omega - m\Omega
\]
Scalar field modes

\[
\phi_j = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\omega t + im\varphi + ikz} J_m(q\rho) = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\tilde{\omega} \tilde{t} + im\tilde{\varphi} + ikz} J_m(q\rho)
\]

\[
\tilde{\omega} = \omega - m\Omega
\]

Norm of these modes

\[
(\phi_j, \phi_{j'})_{KG} = \frac{\omega}{|\omega|} \delta (j, j')
\]
Scalar field modes

\[ \phi_j = \frac{1}{\sqrt{8\pi^2|\omega|}} e^{-i\omega t + im\varphi + ikz} J_m(q\rho) = \frac{1}{\sqrt{8\pi^2|\omega|}} e^{-i\tilde{\omega} \tilde{t} + im\tilde{\varphi} + i\tilde{k}z} J_m(q\rho) \]

\[ \tilde{\omega} = \omega - m\Omega \]

Norm of these modes

\[ (\phi_j, \phi_{j'})_{KG} = \frac{\omega}{|\omega|} \delta (j, j') \]

Frequency of these modes

Corotating Hamiltonian \( H = i\partial_{\tilde{t}} \neq i\partial_t \)

Frequency in rotating co-ordinates \( \tilde{\omega} \)
Defining states

Rotating vacuum state

- Positive frequency modes $\phi_j^+$ must have positive norm
- We must therefore choose positive frequency modes $\phi_j$ with $\omega > 0$
- Negative frequency modes are $\phi_j^*$
- Expansion of the field

$$\Phi = \sum_m \int_M \int d\omega \int dk \left[ a_j \phi_j + a_j^+ \phi_j^* \right]$$

- Vacuum state $|0\rangle$ is then identical to the non-rotating Minkowski vacuum

[ Letaw and Pfautsch PRD 22 1345 (1980) ]
Defining states

Rotating thermal state

- Frequency in rotating co-ordinates $\tilde{\omega} = \omega - m\Omega$
- Energy in rotating thermal expectation values at inverse temperature $\beta = T^{-1}$

$$\langle a_j^{\dagger} a_{j'} \rangle_\beta = \frac{\delta (j, j')}{\exp (\beta \tilde{\omega}) - 1}$$

- Modes with $\omega > 0$ but $\tilde{\omega} < 0$
  - Limit $\beta \to \infty$ is non-zero
  - Divergent when $\tilde{\omega} \sim 0$

- Rotating thermal states are ill-defined everywhere

[ Vilenkin *PRD* 21 2260 (1980) ]

[ Duffy and Ottewill *PRD* 67 044002 (2003) ]
Defining states with a boundary present

1. \( \Phi = 0 \) at reflecting boundary at \( \rho = R \)
2. Field modes
   
   \[
   \phi_j = \frac{1}{\sqrt{8\pi^2|\omega|}} e^{-i\tilde{\omega}t + im\tilde{\phi} + ikz} J_m(\eta_m,n\rho/R)
   \]
3. Frequency
   
   \[
   \omega = \pm \sqrt{k^2 + \eta_{m,n}^2/R^2} \quad \tilde{\omega} = \omega - m\Omega
   \]
   
   4. If \( R < \Omega^{-1} \), by the properties of the zeros of the Bessel functions, \( \tilde{\omega} > 0 \) for all \( \omega > 0 \)
5. In this case rotating thermal states are well-defined

[ Duffy and Ottewill *PRD* 67 044002 (2003) ]
Fermion field modes

\[ \psi_j = \frac{1}{\sqrt{8\pi^2}} e^{-i\tilde{E}t + ikz} \left( \frac{\chi}{2\lambda E} \right) \]

\[ \chi = \frac{1}{\sqrt{2}} \left( \sqrt{1 + \frac{2\lambda k}{p}} e^{im\tilde{\varphi}} J_m(q\rho) \right)^{\frac{1}{2}i\lambda \sqrt{1 - \frac{2\lambda k}{p} e^{i(m+1)\tilde{\varphi}}} J_{m+1}(q\rho)} \]

\[ \tilde{E} = E - \Omega \left( m + \frac{1}{2} \right) \]

[Ambrus and EW PLB 734 296 (2014)]
Fermion field modes

\[ \psi_j = \frac{1}{\sqrt{8\pi^2}} e^{-i\tilde{E}t + ikz} \left( \frac{\chi}{2\lambda E} \right) \]

\[ \chi = \frac{1}{\sqrt{2}} \left( \sqrt{1 + \frac{2\lambda k}{\rho}} e^{im\tilde{\phi}} J_m(q\rho) \right) \left( 2i\lambda \sqrt{1 - \frac{2\lambda k}{\rho}} e^{i(m+1)\tilde{\phi}} J_{m+1}(q\rho) \right) \]

\[ \tilde{E} = E - \Omega \left( m + \frac{1}{2} \right) \]

Norm of these modes

\[ (\psi_j, \psi_{j'})_D = \delta (j, j') \]

[ Ambrus and EW PLB 734 296 (2014) ]
Rotating vacuum state

Vilenkin quantization [ Vilenkin PRD 21 2260 (1980) ]

- Positive frequency $E > 0$
- Positive frequency modes $\psi_j$
- Negative frequency modes $\tilde{\psi}_j = i\gamma^2 \psi_j^*$
- Expansion of the field

$$\Psi_V = \sum_m \int_M dE \int dk \left[ b_{j;V} \psi_j + c_{j;V}^{\dagger} \tilde{\psi}_j \right]$$

- Vacuum state $|0_V\rangle$ is then identical to the non-rotating Minkowski vacuum
Rotating vacuum state

Iyer quantization [ Iyer PRD 26 1900 (1982) ]

- Positive frequency $\tilde{E} > 0$
- Expansion of the field

$$\Psi_I = \sum_m \int_{\tilde{E} > 0, |E| > M}^\infty dE \int dk \left[ b_{j;I} \psi_j + c_{j;I}^\dagger \tilde{\psi}_j \right]$$

- Vacuum state $|0_I\rangle$ is then not the non-rotating Minkowski vacuum

$$b_{j;I} = \begin{cases} b_{j;V} & E > 0 \\ i^{2m+1} c_{j';V}^\dagger & E < 0 \end{cases}$$
Rotating thermal state

- Frequency in rotating co-ordinates: \( \tilde{E} = \omega - \Omega \left( m + \frac{1}{2} \right) \)
- Energy in rotating thermal expectation values at inverse temperature: \( \beta = T^{-1} \)

\[
\langle b_j^{\dagger} b_{j'} \rangle_\beta = \langle c_j^{\dagger} c_{j'} \rangle_\beta = \frac{\delta(j, j')}{\exp\left( \beta \tilde{E} \right) + 1}
\]

- Limit \( \beta \to \infty \) is non-zero for modes with \( \tilde{E} < 0 \)
- Rule out such modes by using Iyer quantization
- Fermi-Dirac density of states factor finite for all \( \tilde{E} \)
- Rotating thermal state can be defined on the unbounded space-time for fermions but not bosons

[ Ambrus and EW PLB 734 296 (2014) ]
Rotating thermal state

\[ \ln[T_{tt}] \]

- \( \mu = 0 \)
- \( \mu = 2, \beta \Omega = 0.8 \)
- \( \mu = 2, \beta \Omega = 1. \)
- \( \mu = 2, \beta \Omega = 1.25 \)
- \( \mu = 2, \beta \Omega = 2. \)

[ Ambrus and EW PLB 734 296 (2014) ]
Rotating quantum states on anti-de Sitter space-time
Anti-de Sitter (adS) space-time

Metric

\[ ds^2 = a^2 \sec^2 \rho \left[ -d\tau^2 + d\rho^2 + \sin^2 \rho \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right] \]

a - radius of curvature of adS

[ Figure taken from Avis, Isham and Storey, *PRD* 18 3565 (1978) ]
Rotating co-ordinates

$$\tau \rightarrow \tilde{\tau}, \quad \varphi \rightarrow \tilde{\varphi} = \varphi - \Omega \tau$$

$$ds^2 = a^2 \sec^2 \rho \left[ -\varepsilon d\tilde{\tau}^2 + d\rho^2 + 2\Omega \sin^2 \rho \sin^2 \theta d\tilde{\tau} d\tilde{\varphi} + \sin^2 \rho \left( d\theta^2 + \sin^2 \theta d\tilde{\varphi}^2 \right) \right]$$

where

$$\varepsilon = (1 - \Omega^2 \sin^2 \rho \sin^2 \theta)$$

Speed-of-light surface (SOL) where $$\varepsilon = 0$$

Scalar field modes

\[ \phi_{n\ell m} = N_{n\ell} e^{-i\omega \tau + i m \varphi} R_{n\ell}(\rho) \Theta_{\ell m}(\theta) = N_{n\ell} e^{-i\tilde{\omega} \tilde{\tau} + i m \tilde{\varphi}} R_{n\ell}(\rho) \Theta_{\ell m}(\theta) \]
Scalar field modes

\[ \phi_{n\ell m} = N_{n\ell} e^{-i\omega \tau + im \phi} R_{n\ell}(\rho) \Theta_{\ell m}(\theta) = N_{n\ell} e^{-i\tilde{\omega} \tilde{\tau} + im \tilde{\phi}} R_{n\ell}(\rho) \Theta_{\ell m}(\theta) \]

Quantum numbers

\[ \tilde{\omega} = \omega - m \Omega \quad \omega = 2n + \ell + \kappa \]

\[ \kappa = \frac{3}{2} + \sqrt{M^2 a^2 + \xi \mathcal{R} a^2 + \frac{9}{4}} > 0 \]

\[ n \geq 0 \quad \ell \geq |m| \geq 0 \]
Scalar field modes

\[ \phi_{n\ell m} = N_{n\ell} e^{-i\omega \tau + i m \phi} R_{n\ell}(\rho) \Theta_{\ell m}(\theta) = N_{n\ell} e^{-i\tilde{\omega} \tau + i m \tilde{\phi}} R_{n\ell}(\rho) \Theta_{\ell m}(\theta) \]

Quantum numbers

\[ \tilde{\omega} = \omega - m \Omega \quad \omega = 2n + \ell + \kappa \]

\[ \kappa = \frac{3}{2} + \sqrt{M^2 a^2 + \xi \mathcal{R} a^2 + \frac{9}{4}} > 0 \]

\[ n \geq 0 \quad \ell \geq |m| \geq 0 \]

Norm of these modes

\[ (\phi_{n\ell m}, \phi_{n'\ell' m'})_{KG} = \frac{\omega}{|\omega|} \delta_{nn'} \delta_{\ell \ell'} \delta_{mm'} \]
Defining states

Rotating vacuum state

- Positive frequency modes $\phi_{n\ell m}^+$ must have positive norm
- We must therefore choose positive frequency modes $\phi_{n\ell m}$ with $\omega > 0$
- Negative frequency modes are $\phi_{n\ell m}^*$
- Expansion of the field

$$\Phi = \sum_n \sum_\ell \sum_m \left[ a_{n\ell m} \phi_{n\ell m} + a_{n\ell m}^* \phi_{n\ell m}^* \right]$$

- Vacuum state $|0\rangle$ is then identical to the non-rotating anti-de Sitter vacuum

Defining states

Rotating thermal state

- Frequency in rotating co-ordinates $\tilde{\omega} = \omega - m\Omega$
- Energy in rotating thermal expectation values at inverse temperature $\beta = T^{-1}$

$$\langle a_j^{\dagger}a_{j'} \rangle_\beta = \frac{\delta (j,j')}{\exp (\beta \tilde{\omega}) - 1}$$

- From the properties of the quantum numbers

$$\omega = \kappa + 2n + \ell \geq \kappa + 2n + |m| > |m|$$

$$\Rightarrow \quad \tilde{\omega} \geq |m| - m\Omega > 0 \quad \text{if } \Omega < 1$$

- If $\Omega < 1$ rotating thermal states will be well-defined
- If $\Omega < 1$ there is no speed-of-light surface
Fermion field modes

\[ \psi_{E \ell m} = e^{-i\tilde{E}\tau} \chi_{E \ell m}(\rho, \theta, \tilde{\phi}) \]

[Cotaescu *PRD* 60 124006 (1999)]
[Ambrus and EW arXiv:1405.2215 [gr-qc]]
Fermion field modes

\[ \psi_{E \ell m} = e^{-i\tilde{E}\tilde{\tau}} \chi_{E \ell m}(\rho, \theta, \tilde{\varphi}) \]

Quantum numbers

\[ \tilde{E} = E - m\Omega \quad E = Ma + 2n + \ell + 2 \]
\[ n \geq 0 \quad \ell \geq |m| \geq 0 \]

[ Cotaescu PRD 60 124006 (1999)]
Fermion field modes

$$\psi_{E\ell m} = e^{-i\tilde{E}\tilde{\tau}} \chi_{E\ell m}(\rho, \theta, \tilde{\phi})$$

Quantum numbers

$$\tilde{E} = E - m\Omega$$

$$E = Ma + 2n + \ell + 2$$

$$n \geq 0$$

$$\ell \geq |m| \geq 0$$

Norm of these modes

$$(\psi_{E\ell m}, \psi_{E'\ell' m'})_D = \delta_{\ell \ell'} \delta_{mm'} \delta (E, E')$$

[Cotaescu PRD 60 124006 (1999)]

[Ambrus and EW arXiv:1405.2215 [gr-qc]]
Defining states

Rotating vacuum state

- All fermion modes have positive norm
- Non-rotating vacuum state arises from choosing $E > 0$ to be positive frequency
- Rotating vacuum state arises from choosing $\tilde{E} > 0$ to be positive frequency
- From the properties of the quantum numbers

$$E = Ma + 2n + \ell + 2 \geq Ma + 2n + |m| + 2 > |m|$$

$$\Rightarrow \quad \tilde{E} > |m| - m\Omega \geq 0 \quad \text{if } \Omega \leq 1$$

- If $\Omega \leq 1$, the rotating vacuum coincides with the non-rotating vacuum
- If $\Omega > 1$ the rotating and non-rotating vacua are different
Rotating thermal state

- Frequency in rotating co-ordinates $\tilde{E} = E - m\Omega$
- Energy in rotating thermal expectation values at inverse temperature $\beta = T^{-1}$

$$\langle b_j^{\dagger} b_{j'} \rangle_\beta = \langle c_j^{\dagger} c_{j'} \rangle_\beta = \frac{\delta (j, j')}{\exp \left( \beta \tilde{E} \right) + 1}$$

- Limit $\beta \to \infty$ is non-zero for modes with $\tilde{E} < 0$
- Rule out such modes by using rotating vacuum
- Fermi-Dirac density of states factor finite for all $\tilde{E}$
- Rotating thermal state can be defined on the unbounded space-time for all $\Omega$

Rotating thermal state

\[ k = 0., \beta = 0.8 \]

\[ T_{ii} = \begin{array}{ll}
\Omega = 0.9 \\
\Omega = 0.85 \\
\Omega = 0.8 \\
\Omega = 0.65 \\
\Omega = 0.3 \end{array} \]

Rotating thermal state

\[ k = 2, \Omega = 1 \]

\[ \beta = 0.8 \]

\[ \beta = 0.8 \]

\[ \beta = 1.2 \]

\[ \beta = 2.0 \]

\[ \log[T_{\hat{t}\hat{t}}] \]

Rotating thermal state

\[ \log[T_{\hat{t}\hat{t}}] \]

- \( k = 2, \quad \Omega = 1.5 \)
- \( k = 0 \)
- \( \beta = 0.8 \)
- \( \beta = 1.0 \)
- \( \beta = 1.2 \)
- \( \beta = 2.0 \)

Conclusions
Minkowski space

**Scalars**

- Positive frequency modes must have positive norm
- Rotating vacuum is the same as the Minkowski vacuum
- Rotating thermal states cannot be defined unless the system is enclosed in a boundary sufficiently close to the axis of rotation

**Fermions**

- All modes have positive norm
- Two possible rotating vacua: Vilenkin and Iyer
- Rotating thermal states can be defined on the unbounded space-time using the Iyer quantization
- Rotating thermal states diverge on the speed-of-light surface
Minkowski space

**Scalars**
- Positive frequency modes must have positive norm
- Rotating vacuum is the same as the Minkowski vacuum
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**Fermions**
- All modes have positive norm
- Two possible rotating vacua: Vilenkin and Iyer
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- Rotating thermal states diverge on the speed-of-light surface
Conclusions

Anti-de Sitter space

**Scalars**
- Rotating vacuum is identical to the non-rotating vacuum
- If $\Omega < 1$, positive norm modes have positive frequency as seen by the rotating observer
- Rotating thermal states well-defined in this case

**Fermions**
- If $\Omega \leq 1$, rotating vacuum is identical to the non-rotating vacuum
- If $\Omega > 1$, rotating vacuum is distinct from the non-rotating vacuum
- For all $\Omega$ rotating thermal states can be defined
Anti-de Sitter space

**Scalars**
- Rotating vacuum is identical to the non-rotating vacuum
- If $\Omega < 1$, positive norm modes have positive frequency as seen by the rotating observer
- Rotating thermal states well-defined in this case

**Fermions**
- If $\Omega \leq 1$, rotating vacuum is identical to the non-rotating vacuum
- If $\Omega > 1$, rotating vacuum is distinct from the non-rotating vacuum
- For all $\Omega$ rotating thermal states can be defined
Conclusions

Key points

- Fermions and bosons are different when it comes to defining quantum states
- Much more freedom in definition of fermionic quantum states
- Can define quantum states for fermions which have no analogue for bosons

Implications

Considerations apply to general curved space-times

No rotating thermal state exists on a Kerr black hole for bosonic fields

A rotating thermal state can be defined for fermions on a Kerr black hole

[Casals et al PRD 87 064027 (2013)]

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Sussex, November 2014
Conclusions

Key points

- Fermions and bosons are different when it comes to defining quantum states.
- Much more freedom in definition of fermionic quantum states.
- Can define quantum states for fermions which have no analogue for bosons.

Implications

- Considerations apply to general curved space-times.
- No rotating thermal state exists on a Kerr black hole for bosonic fields.
- A rotating thermal state can be defined for fermions on a Kerr black hole.

[ Casals et al *PRD* 87 064027 (2013) ]