The Philosophical Logic of Homotopy Type Theory

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Acknowledgments

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http://bristol.ac.uk/homotopy-type-theory

or search for *bristol homotopy type theory*

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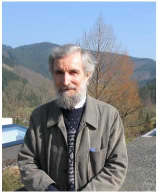

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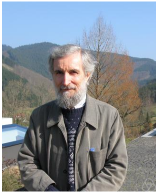

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  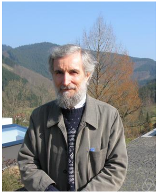

- Aside from its interest as an area of mathematics connecting algebraic topology with logic and computer science, it has also been proposed as a new foundation for mathematical practice.
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- Voevodsky: "Simpson claimed to have constructed a counterexample, but he was not able to show where the mistake was in our paper. Because of this, it was not clear whether we made a mistake somewhere in our paper or he made a mistake somewhere in his counterexample."

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- "**This story got me scared.**"

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- "There were several groups developing such systems, but none of them was in any way appropriate for the kind of mathematics for which I needed a system."
- "The primary challenge that needed to be addressed was that *the foundations of mathematics were unprepared for the requirements of the task*."

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- In this talk I will explore some of the philosophically interesting features of HoTT/UF.

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- For the purposes of this talk I will consider only the theory of the HoTT book.

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- There are two kinds of identity and they do not reflect each other.
- Is identity in HoTT really identity or is it really some other relation such as indiscernibility?

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- What is the difference between equivalence and isomorphism, and does mathematical structuralism motivate Univalence?

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- For example, there is a type $\mathbb{N}$, the natural numbers, whose tokens are individual natural numbers e.g. $5 : \mathbb{N}$, $24 : \mathbb{N}$, etc.
- Another example: given some $\mathtt{A}$ and $\mathtt{B}$ we can define the type $\mathtt{Iso}(\mathtt{A}, \mathtt{B})$, which is the type of isomorphisms between $\mathtt{A}$ and $\mathtt{B}$.

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- The rules of type formation and token construction then correspond to the basic operations of logic such as conjunction, disjunction, and implication.

The Language of HoTT

Starting from the **Curry-Howard correspondence**, HoTT has type constructions corresponding to logical connectives, to Existential and Universal quantification, and to equality/identity statements.

Logic	Type theory	Notation
Implication	Function type	$A \to B$
Conjunction	Product	$A \times B$
Disjunction	Coproduct	$A + B$
Existential quantification	Dependent pair type	$\sum_{x:A} P(x)$
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The resulting logic is constructive.

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- Similarly, given a function of type $(X \to 0) \to 0$ we cannot in general produce a token of X. Thus we do not have Double Negation Elimination.
- This is the sense in which the logic of HoTT is **constructive**.

Constructivity of HoTT

- To prove a disjunction $(\mathtt{A} + \mathtt{B})$ we must be able to prove at least one of the disjuncts.
- A function of type $\mathtt{A} \to \mathtt{B}$ is an algorithm or procedure that, when given a token of $\mathtt{A}$, produces a token of type $\mathtt{B}$.
- While the Law of Excluded Middle and Double Negation Elimination are not laws of the logic of HoTT, we are always free to posit any particular instance of LEM or DNE as a premise: we can use classical principles, we just have to be explicit about these uses.
- Constructivism need not be motivated by any anti-classical or Brouwerian sentiments.

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- The theory is **proof-relevant** – it matters which certificate to a proposition we have. e.g. when A and B are isomorphic it matters *which* isomorphism between them has been constructed.

Products and ordered pairs

- Given any two types $\mathtt{A}$ and $\mathtt{B}$ we can form the **product** type $\mathtt{A} \times \mathtt{B}$. Tokens of this are ordered pairs $(\mathtt{a}, \mathtt{b}) : \mathtt{A} \times \mathtt{B}$, where $\mathtt{a} : \mathtt{A}$ and $\mathtt{b} : \mathtt{B}$.

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- This is one of the basic constructions in HoTT, not derivative as in ZFC set theory (where ordered pairs are sets of a certain form).
- Note that the product $\mathtt{A} \times \mathtt{B}$ is distinct from the product $\mathtt{B} \times \mathtt{A}$ so philosophers would call the theory ‘hyperintensional’. However, these products are equivalent and under univalence they are identified.

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- Externally identical elements may be substituted for one another in any context, with no restrictions.
- Internal identity is not so simple, but this is from where much of the interest and power of HoTT comes.
- Importantly, internal identity does not imply external identity. This is another sense in which the theory is *intensional*. Thus in HoTT internally identifying two tokens of a type does not amount to just collapsing them together.

Identity Types

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- For two tokens that are distinct the corresponding identity type will be uninhabited, but two identical tokens may have *multiple* identifications.
- Every token of every type is trivially identical to itself, but *non-trivial* identifications are also allowed.
- Nothing we can express within the language of HoTT can distinguish tokens that are identified. In particular, all functions respect identity (i.e. $\mathtt{x} = \mathtt{y}$ implies $\mathtt{f}(\mathtt{x}) = \mathtt{f}(\mathtt{y})$).

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- If we have two tokens α, β of this type then we can form the higher identity type

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and so on . . .

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$$\mathtt{BPI} : \prod_{\mathtt{A}:\mathtt{TYPE}} \prod_{\mathtt{a}:\mathtt{A}} \prod_{\mathtt{K}:\mathtt{E}_\mathtt{a}\to\mathtt{TYPE}} \left[\mathtt{K}(\mathtt{a}, \mathtt{refl}_\mathtt{a}) \to \prod_{(\mathtt{b},\mathtt{p}):\mathtt{E}_\mathtt{a}} \mathtt{K}(\mathtt{b}, \mathtt{p}) \right]$$

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- To prove theorems involving identifications we use the powerful technique of *path induction*.
- Fix a type $\mathtt{A}$ and consider all identifications between tokens of $\mathtt{A}$. If we want to prove that some property holds of all such identifications, path induction says it is sufficient to prove that the property holds of the trivial self-identifications $\mathtt{refl_x}$ for each $\mathtt{x} : \mathtt{A}$.

$$\mathtt{BPI} : \prod_{\mathtt{A:TYPE}} \prod_{\mathtt{a:A}} \prod_{\mathtt{K:E_a \to TYPE}} \left[\mathtt{K}(\mathtt{a}, \mathtt{refl_a}) \to \prod_{(\mathtt{b,p}):\mathtt{E_a}} \mathtt{K}(\mathtt{b}, \mathtt{p}) \right]$$

- This often makes such proofs very simple or even trivial. The justification of path induction in the official presentation of HoTT appeals to the homotopy interpretation but it can be otherwise justified.

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- However, in an *extensional* type theory, from $\mathtt{p} : \mathtt{Id}_{\mathtt{A}}(\mathtt{a}, \mathtt{b})$ we could derive $\mathtt{a} \equiv \mathtt{b}$. Then the above $\mathtt{Q}$ would be well-typed, and the proof that all identifications are trivial goes through.

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- While internal identity is reflexive (and thus every token has a trivial self-identification) tokens may also have *non-trivial self-identifications*.
- The *Univalence Axiom* (which we'll discuss later) says that two types may be identified in virtue of being isomorphic.

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- If we read $\mathtt{Id}_{\mathtt{A}}(\mathtt{a}, \mathtt{b})$ as indiscernibility instead of identity then the above features are no longer unexpected, and fits better with standard views.

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- While any entity is trivially indiscernible from itself, if it has any non-trivial automorphisms then these provide additional ways in which the self-indiscernibility may be witnessed.
- On this view of $\mathtt{Id}_{\mathtt{A}}(\mathtt{a}, \mathtt{b})$, the Univalence Axiom says that types that are equivalent to each other (in a certain sense) are indiscernible from each other

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- In other words, if two entities have all their (non-identity-involving) properties in common, then they should be counted as 'identical'.
- To study this further we should formalise the definition of *indiscernibility* and the Principle of the Identity of Indiscernibles (PII).

Discernibility and Indiscernibility

- The natural definition of discernibility to use is *absolute discernibility*, which we formalise as:

$$\mathtt{Dis_A(a, b)} :\equiv \sum_{\mathtt{P:A}\to\mathcal{U}} \mathtt{P(a)} \times \neg\mathtt{P(b)}$$

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- Rather, we define indiscernibility as

$$\mathtt{InDis}_{\mathtt{A}}(\mathtt{a}, \mathtt{b}) :\equiv \prod_{\mathtt{P}:\mathtt{A}\to\mathcal{U}} \mathtt{P}(\mathtt{a}) \leftrightarrow \mathtt{P}(\mathtt{b})$$

"for every property P, token a satisfies the property iff b does".

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  > *"it is not true that two substances may be exactly alike and differ only numerically, solo numero"*

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- Constructively the latter statement is *stronger* than the former.

Problems with Identity-as-Indiscernibility

- The behaviour of $\mathtt{InDis_A}(\mathtt{a}, \mathtt{b})$ is importantly different from that of $\mathtt{Id_A}(\mathtt{a}, \mathtt{b})$; specifically, it does not support *path induction*.

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- The role of *universes* in the definition of indiscernibility.

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  involves a quantification over all predicates defined on type $\mathtt{A}$.
- The type of the predicates that we quantify over is written as $\mathtt{A} \to \mathcal{U}$, where $\mathcal{U}$ is the *universe* – roughly, the type whose tokens are all the types under consideration.

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- If identity was replaced by indiscernibility then these definitions would be relative to universes.

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  (More details in our paper *Identity in HoTT part 2: The Conceptual and Philosophical Status of Identity in HoTT*, in *Philosophia Mathematica*.)
- Thus we cannot resolve the counterintuitive features of 'identity types' in HoTT by taking them to stand instead for indiscernibility.
- If we are to take HoTT seriously as a candidate foundation for mathematics then we must get used to the novel way it treats identity instead of reinterpreting away these unusual features.

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- Awodey and Voevodsky have given several considerations to motivate the need for HoTT as a new foundation.

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Mismatch between Foundation & Practice

- Most working mathematicians don't work in ZFC or category theory or any other official foundational system. They work informally, and if they appeal to the existence of these foundational systems at all it is as an 'in principle' guarantee that their work is secure.
- There can be a very large gap between what's done in practice and the 'in principle' formal underpinnings.

Abuse of Notation and other Sloppiness

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- Sometimes we do this by a kind of 'controlled sloppiness', using the same symbol for both structures and simply remembering (in the back of our mind) that there are actually two distinct entities.
- For example, Dedekind cuts are sets of rationals, whereas Cauchy sequences are sequences of rationals. When we talk about 'the reals' we freely switch back and forth between these two structures.

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- In particular, they fail to judge two distinct representations of a given structure (e.g. two different definitions of *ordered pair*, or $\mathbb{R}$) as equivalent – they compare the representations, not the thing represented.
- We need a formal language that relates better to mathematical practice. This is what HoTT promises (thanks in part to the magic of the Univalence Axiom, about which more later.)

Application to Physics

It has also been claimed by Mike Shulman and others that HoTT is a better foundation for mathematics because it helps us avoid conceptual problems in physics such as the hole argument.

What is a Foundation for Mathematics?

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- The most minimal idea of foundation is that of a unifying language and conceptual framework.
- Let's suppose that an affirmative answer to the question of whether HoTT is adequate as such a framework for mathematics. (See in particular the reconstructions of the natural numbers (Section 1.9), real numbers (Chapter 11), category theory (Chapter 9) and a model of ZFC set theory (Chapter 10).

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- Some mathematicians and philosophers think of a foundation as also answering semantic, metaphysical, epistemological, and/or methodological questions about mathematics.
- An important criterion for some is that a foundation be *autonomous*.

Five Components of a Foundation for Mathematics

There are five interrelated components to a foundation for mathematics, and each generates a series of questions that a given putative foundation for mathematics might be expected to answer.

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What role, if any, do axioms play?

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In particular, is the theory extensional or intensional?

Metaphysics

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Questions: Does the metaphysics posit any objects at all? Is the ontology (if any) to be understood as mind-dependent or mind-independent? What is the relationship between mathematical reality (if any) and physical reality?

Epistemology

An epistemology in the sense of an account of how we are able to know the truths (if any) of mathematics, given (2) and (3) (which may also include an account of the applicability of mathematics) and a justification of the axioms and rules of the framework.

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Given the role of proof in mathematical practice, what is the relationship between mathematical knowledge and proof? How are the rules justified given their interpretation (if any)? If there are axioms, what is their epistemological status – for example, are they taken to be known, or are they taken to be merely hypothetical statements that form the antecedent of conditionals?

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What is the relationship between mathematical knowledge and knowledge of physical reality?

Methodology

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Questions: How is the foundation to be used in practice? In particular, how is it to be applied in the physical sciences?

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- Linnebo & Pettigrew consider three kinds of *autonomy* a foundational system may have.
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- Briefly, a presentation of a system is autonomous iff all of its definitions, justification, and interpretation can be given *without appeal to existing mathematics*. A foundation is autonomous iff it can be given an autonomous presentation.
- The standard presentation of Homotopy Type Theory given in the HoTT Book is not autonomous, since it depends upon *homotopy theory*.

An autonomous justification of path induction

Path induction follows from the uniqueness principle for identity types and Leibniz's law.

Interpreting HoTT

- The first thing we would need to do in giving an autonomous account of HoTT as a foundation is to give an interpretation – what exactly do we take types and tokens to be?
- The authors of the HoTT Book are not primarily interested in giving an account of the philosophical underpinnings of the theory, of course, so they don't go into detail on this.
- Two interpretations that they use:
  - types and tokens as *mathematical objects*
  - types and tokens as *spaces* and *points* (the homotopy interpretation)
- (Of course, we could also be Formalist and just explicitly deny that the formal language is to be interpreted at all – but where's the fun in that?)

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- It doesn't naturally account for the zero type.

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- types are **general concepts**;
- tokens of a type are **specific concepts** *qua* instance of that general concept.

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- It accords with intensional type theory.
- Complex concepts are composed from simpler concepts.
- It accords with each token belonging to exactly one type.
- It gets the order of dependence right: we must have the general concept before we can have a specific concept *qua* instance of the general concept

Ontology

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- The foundation doesn't assert that any particular types exist (beyond the simple ones arising from the basic language). But if we posit the existence of, say, natural numbers, Hausdorff spaces, or non-principal ultrafilters, we can study them.

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- We are interested in *structural* properties – those that are preserved under structure-preserving maps – but we may be given a presentation of a structure that has other irrelevant properties.
- As in category theory, different kinds of mathematical structures have corresponding kinds of structure-preserving maps.
- The Univalence axiom permits us to set aside the specific properties of any particular presentation in order to study the properties of the structure being presented because it allows us to identify types when there is the right kind of map between them.

Univalence

- The Univalence Axiom is an identity criterion for types. It says that *equivalent types are identical*. (For now, read 'equivalent' to mean 'isomorphic' – but we'll return to this later.)
- If two types are identical then trivially there is an equivalence between them. We have a function that maps any identification between types to an equivalence between those types.

$$\texttt{id-to-eq} : \texttt{Id}(\texttt{A}, \texttt{B}) \to \texttt{Equiv}(\texttt{A}, \texttt{B})$$

- Univalence posits a function in the opposite direction, mapping any equivalence between types to an identification between them.

$$\texttt{eq-to-id} : \texttt{Equiv}(\texttt{A}, \texttt{B}) \to \texttt{Id}(\texttt{A}, \texttt{B})$$

- So whenever we have an equivalence between types Univalence says we can replace it with an identity.

- More precisely, Univalence says for any types $\mathtt{A}$ and $\mathtt{B}$,

$$(\mathtt{A} \simeq \mathtt{B}) \quad \simeq \quad (\mathtt{A} = \mathtt{B})$$

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- Note that this doesn't follow from the basic setup of HoTT – if we want Univalence we must add it as an axiom. Most people currently working with HoTT adopt Univalence without question.

Universes and Univalence

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Extending the concepts interpretation of types to universes.

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- What would normally be achieved by 'abuses of notation' can be officially sanctioned and carried out formally in a way that a proof checker can understand.
- This is one way in which Homotopy Type Theory closes the gap between foundations and practice.

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Univalence and Structuralism

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- An equivalence between types means that they're really just different presentations of some common structure.
- The Invariance Principle (IP): reasoning in the system should be invariant under (the appropriate notion of) equivalence, so that we are reasoning about the underlying structures themselves, not about some particular presentation.
- Thus any property satisfied by some type `A` should be satisfied by any type `B` that is equivalent to `A`. But if we consider the property of *being identical to* `A`, Univalence follows. So IP is equivalent to Univalence.

What is 'Equivalence'?

- Recall the statement of Univalence:

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- Thus if Univalence is to be consistent, $\simeq$ must be some other relation.

Bi-invertibility

- A function $\mathtt{f} : \mathtt{A} \to \mathtt{B}$ is an **isomorphism** iff there is a function $\mathtt{g} : \mathtt{B} \to \mathtt{A}$ such that

$$\mathtt{g}(\mathtt{f}(\mathtt{a})) = \mathtt{a} \quad \text{for every } \mathtt{a} : \mathtt{A}$$
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- A function $\mathtt{f} : \mathtt{A} \to \mathtt{B}$ is **bi-invertible** iff there are functions $\mathtt{g}, \mathtt{h} : \mathtt{B} \to \mathtt{A}$ such that

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i.e. there is a left-inverse and right-inverse, but they're not required to be the same function.

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- What’s the difference?

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- If a function `f` has both a left-inverse `g` and a right-inverse `h` (i.e. it's bi-invertible) then we can prove that both `g` and `h` are full inverses of `f` – i.e. either of them is sufficient to prove that `f` is an isomorphism.

The difference between Iso and Equiv

- So the difference between $\texttt{Iso}(\texttt{A},\texttt{B})$ and $\texttt{Equiv}(\texttt{A},\texttt{B})$ is a proof-relevant distinction: it's about the specific evidence that's required in order to demonstrate that two types stand in that particular relation.

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- Equiv(A, B) requires a function f, and functions g and h, and certificates that they are respectively left- and right-inverse to f.
- But why should this difference be so important – important enough that Univalence is consistent while Isovalence is inconsistent?
- In one sense equivalence is weaker than isomorphism, but in another sense it is stronger: equivalence of types is a *mere proposition*.

Justifying Univalence

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- UA entails function extensionality
- Arguably the reason Univalence is useful is that it represents how mathematicians think by identifying types that would otherwise be kept distinct. For example, in HoTT as in set theory there are different ways to define ordered pairs, but in Univalent HoTT (unlike in set theory) alternative equivalent types of ordered pairs are identified.

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Examples of the meaning of UA

To see what it means to say that equivalent types are identical consider some examples:

(i) $\mathtt{A} \times \mathtt{B}$ and $\mathtt{B} \times \mathtt{A}$ are (externally) distinct types, but in almost any context in which we are interested in them there is no effective difference between them, so it makes sense to equate them. We can think of this as creating the type 'unordered pair'.

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(ii) The list-sorting algorithms MergeSort and InsertionSort. Clearly these are distinct algorithms, and in some contexts the differences between them (e.g. their running times on a given list) are important. But in another sense, regarded just as relations between inputs and outputs, they are identical since they produce the same output when given the same input.

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(III) All empty types, for example, *even divisors of 9* and *largest prime*, are equivalent to $\mathtt{0}$ and thus (under univalence) identical to it.

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- Purely intensional theories such as basic HoTT make distinctions that are too fine-grained for ordinary mathematical practice.
- Purely extensional theories such as set theory collapses some of these distinctions, but introduces unwanted distinctions between different ways of representing mathematical structures (such as ordered pairs).

- Univalence allows us to exploit the benefits of the intensional nature of HoTT – viz. non-trivial identities and higher identity structure – whilst recovering the extensional character of much mathematical thought.
- The language of HoTT without Univalence is so fine-grained as to distinguish types that differ only in how they are described. Univalence allows us to identify such types.
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- Purely extensional theories such as set theory collapses some of these distinctions, but introduces unwanted distinctions between different ways of representing mathematical structures (such as ordered pairs).
- Univalence strikes a balance between the two, introducing an element of extensionality into the intensional theory of HoTT.

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THANK YOU

- http://bristol.ac.uk/homotopy-type-theory
- A Primer on Homotopy Type Theory. www.bristol.ac.uk/homotopy-type-theory/, 2014.
- Does Homotopy Type Theory Provide a Foundation for Mathematics? *BJPS*, forthcoming.
- Identity in Homotopy Type Theory, Part I: The Justification of Path Induction. *Phil Math*, 23:386–406, 2015.
- Identity in Homotopy Type Theory: Part II, The Conceptual and Philosophical Status of Identity in HoTT. *Phil Math*.
- Universes and Univalence in Homotopy Type Theory. forthcoming, *The Review of Symbolic Logic*.
- Representation and Symmetry in Physics. Under review, *Phil Sci*.