Implied Volatility of Leveraged ETF Options: Consistency and Scaling

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ETF Market

ETF Provider

Authorized Participant

Primary Market

Secondary Market

Buyer

Stock Exchange

Seller

ETF Shares

ETF Shares (In-kind Transfer)

Securities
Leveraged Exchange Traded Funds (LETFs) promise a fixed multiple of the returns of an index/underlying asset.

Most typical leverage ratios are: (long) $2$, $3$, and (short) $-2$, $-3$.

LETFs have also led to increased trading of options written on LETFs.

In 2012, total daily notional on LETF options is $40-50$ bil, as compared to $90$ bil for S&P 500 index options.

ETFs with the highest options volume: SPY (S&P 500), IWM (Russell 2000), QQQ (Nasdaq 100) and GLD (gold).
Leveraged ETF Returns

- By design, an LETF seeks to provide a constant multiple $\beta$ of the daily returns $R_j$ of a reference index or asset:

$$L_n = L_0 \cdot \prod_{j=1}^{n} (1 + \beta R_j).$$

- By differentiation, we get

$$\frac{d}{d\beta} \left( \log \left( \frac{L_n}{L_0} \right) \right) = \sum_{j=1}^{n} \frac{R_j}{1 + \beta R_j}.$$

- With a positive leverage ratio $\beta > 0$, if $R_j > 0$ for all $j$, then $\log \left( \frac{L_n}{L_0} \right)$, or equivalently the value $L_n$, is increasing in $\beta$.

- That is, when the reference asset is increasing in value, a larger, positive leverage ratio is preferred.
Example: non-directional movements

<table>
<thead>
<tr>
<th>Day</th>
<th>ETF</th>
<th>%-change</th>
<th>+2x LETF</th>
<th>%-change</th>
<th>−2x LETF</th>
<th>%-change</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>100</td>
<td></td>
<td>100</td>
<td></td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>98</td>
<td>-2%</td>
<td>96</td>
<td>-4%</td>
<td>104</td>
<td>4%</td>
</tr>
<tr>
<td>2</td>
<td>99.96</td>
<td>2%</td>
<td>99.84</td>
<td>4%</td>
<td>99.84</td>
<td>-4%</td>
</tr>
<tr>
<td>3</td>
<td>97.96</td>
<td>-2%</td>
<td>95.85</td>
<td>-4%</td>
<td>103.83</td>
<td>4%</td>
</tr>
<tr>
<td>4</td>
<td>99.92</td>
<td>2%</td>
<td>99.68</td>
<td>4%</td>
<td>99.68</td>
<td>-4%</td>
</tr>
<tr>
<td>5</td>
<td>97.92</td>
<td>-2%</td>
<td>95.69</td>
<td>-4%</td>
<td>103.67</td>
<td>4%</td>
</tr>
<tr>
<td>6</td>
<td>99.88</td>
<td>2%</td>
<td>99.52</td>
<td>4%</td>
<td>99.52</td>
<td>-4%</td>
</tr>
</tbody>
</table>

Even though the ETF records a tiny loss of 0.12% after 6 days, the +2x LETF ends up with a loss of 0.48%, which is greater (in abs. value) than 2 times the return (−0.12%) of the ETF.

At the terminal date, both the long and short LETFs have recorded net losses of 0.48%.

These can occur for over a longer/shorter period.
Empirical LETF Prices

Figure: SSO (+2) and SDS (−2) cumulative returns from Dec 2010 to Nov 2011. Observe that both SSO and SDS can give negative returns simultaneously over several periods in time.
Figure: 1-day (left), 2-week (mid) and 2-month (right) returns of SPY against SSO (top) and SDS (bottom), in logarithmic scale. We considered 1-day, 2-week and 2-month rolling periods from Sept 29, 2010 to Sept 30, 2012.
Figure: 1-day (left), 2-week (mid) and 2-month (right) returns of SPY against UPRO (top) and SPXU (bottom), in logarithmic scale. We considered 1-day, 2-week and 2-month rolling periods from Sept 29, 2010 to Sept 30, 2012.
LETF Price Dynamics

- Suppose the reference asset price follows

\[
\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t,
\]

under the historical measure \( \mathbb{P} \).

- A long LETF \( L \) on \( X \) with leverage ratio \( \beta > 1 \) is constructed by
  - investing the amount \( \beta L_t \) (\( \beta \) times the fund value) in \( S \),
  - borrowing the amount \( (\beta - 1) L_t \) at the risk-free rate \( r \),
  - charging a small expense fee at rate \( c \).

- For a short \( (\beta \leq -1) \) LETF, \( |\beta| L_t \) is shorted on \( S \), and \$\( (1 + |\beta|) L_t \) is kept in the money market account.

- The LETF price dynamics:

\[
\frac{dL_t}{L_t} = \beta \left( \frac{dS_t}{S_t} \right) - ((\beta - 1)r + c) \, dt
\]

\[
= [\beta \mu_t - ((\beta - 1)r + c)] \, dt + \beta \sigma_t \, dW_t.
\]
The LETF value $L$ can be written in terms of $S$:

$$\frac{L_T}{L_0} = \left(\frac{S_T}{S_0}\right)^\beta e^{-\left(r(\beta - 1) + c\right)T - \frac{\beta(\beta - 1)}{2} \int_0^T \sigma_t^2 \, dt},$$

or equivalently in log-returns:

$$\log \left(\frac{L_T}{L_0}\right) = \beta \log \left(\frac{S_T}{S_0}\right) - \left(r(\beta - 1) + c\right)T - \frac{\beta(\beta - 1)}{2} \int_0^T \sigma_t^2 \, dt.$$

Hence, the realized volatility will lead to attrition in fund value.

This phenomenon is called the volatility decay.
A Static LETF Portfolio

- A well known industry strategy consists of shorting two LETFs on the same reference.
- We short \( \omega \in (0, 1) \) of \( \beta_+ > 0 \) LETF and \( 1 - \omega \) of \( \beta_- < 0 \) LETF.
- At time \( T \), the (normalized) return of the portfolio is

\[
R_T = 1 - \omega \frac{L^+_T}{L^+_0} - (1 - \omega) \frac{L^-_T}{L^-_0}.
\]

- If we apply \( \frac{L^+_T}{L^+_0} \approx \ln \frac{L^+_T}{L^+_0} + 1 \), and set \( \omega^* = \frac{-\beta_-}{\beta_+ - \beta_-} \), we get

\[
R_T = \frac{-\beta_- \beta_+}{2} V_T - \frac{\beta_-}{\beta_+ - \beta_-} (f_+ - f_-) T + (f_- - r) T.
\]

- Portfolio is long volatility since \( \frac{-\beta_- \beta_+}{2} > 0 \), and it’s also \( \Delta \)-neutral.
Empirical Performance

(a) DJUSEN-DIG-DUG

(b) GOLDNPM-UGL-GLL

Figure: Empirical return vs realized variance for a double short strategy over 30-day holding periods. Here, $\beta = \pm 2$ for each LETF pair. The blue line gives a predicted (not regressed) return as a linear function of the realized variance. For empirical backtesting, we set $\omega^* = 0.5$, and $T = 30 / 252$. 
Time Series of Empirical Performance

![Diagram showing time series of returns for a double short strategy over 100-day (rolling) holding periods. Notice how during the periods of greatest volatility our double short trading strategy had the greatest return.]

**Figure:** Time series of returns for a double short strategy over 100-day (rolling) holding periods. Notice how during the periods of greatest volatility our double short trading strategy had the greatest return.
Figure: Average returns from a double short trading strategy by commodity LETF pairs over no. of days. For all pairs, $\beta = \pm 2$. 
Pricing LETF Options
If \( \sigma \) is constant, then \( L \) is lognormal, and the no-arb price a European call option on \( L \) is

\[
C^{(\beta)}_{BS}(t, L; K, T) = e^{-r(T-t)} \mathbb{E}^Q\{(L_T - K)^+ | L_t = L\} = C_{BS}(t, L; K, T, r, c, \beta|\sigma),
\]

where \( C_{BS}(t, L; K, T, r, c, \sigma) \) is the Black-Scholes formula for a call.

Given the market price \( C^{obs} \) of a call on \( L \), the implied volatility is given by

\[
I^{(\beta)}(K, T) = (C^{(\beta)}_{BS})^{-1}(C^{obs}) = \frac{1}{|\beta|}C^{-1}_{BS}(C^{obs}).
\]

We normalize by the \( |\beta|^{-1} \) factor in our definition of implied volatility so that they remain on the same scale.
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Proposition

The slope of the implied volatility curve admits the (model-free) bound:

\[-\frac{e^{-(r-c)(T-t)}}{|\beta|L\sqrt{T-t}} \frac{(1 - N(d_2^{(\beta)}))}{N'(d_1^{(\beta)})} \leq \frac{\partial I^{(\beta)}(K)}{\partial K} \leq \frac{e^{-(r-c)(T-t)}}{|\beta|L\sqrt{T-t}} \frac{N(d_2^{(\beta)})}{N'(d_1^{(\beta)})},\]

where \(d_2^{(\beta)} = d_1^{(\beta)} - |\beta|I^{(\beta)}(K)\sqrt{T-t}\), with \(\sigma = I^{\beta}(K)\).

- It follows from the fact that observed and model call prices must be decreasing in \(K\):
  \[
  \frac{\partial C^{obs}}{\partial K} = \frac{\partial C_{BS}}{\partial K}(t, L; K, T, r, c, |\beta|I^{(\beta)}(K)) \quad = \quad \frac{\partial C_{BS}}{\partial K}(|\beta|I^{(\beta)}(K)) + |\beta|\frac{\partial C_{BS}}{\partial \sigma}(|\beta|I^{(\beta)}(K)) \frac{\partial I^{(\beta)}(K)}{\partial K} \leq 0.
  \]

- Rearranging terms gives the upper bound.
- Repeat this using put prices yields the lower bound.
Empirical Implied Volatilities – SPX, SPY (+1)

Figure: SPX (blue cross) and SPY (red circles) implied volatilities on Sept 1, 2010 for different maturities (from 17 to 472 days) plotted against log-moneyness:

\[ LM = \log \left( \frac{\text{strike}}{(L)\text{ETF price}} \right) \]
Empirical Implied Volatilities – SPY (+1), SSO (+2)

Figure: *SPY* (blue cross) and *SSO* (red circles) implied volatilities against log-moneyness.
Empirical Implied Volatilities – SPY (+1), SSO (+2)

Figure: SPY (blue cross) and SSO (red circles) implied volatilities against log-moneyness.
Empirical Implied Volatilities – SPY (+1), SDS (−2)

Figure: SPY (blue cross) and SDS (red circles) implied volatilities against log-moneyness (LM) for increasing maturities.
Empirical Implied Volatilities – SPY (+1), SDS (−2)

Figure: SPY (blue cross) and SDS (red circles) implied volatilities against log-moneyness (LM) for increasing maturities.
Observations

The most salient features of the empirical implied volatilities:

- IV skew for SSO (+2) is flatter than that for SPY,
- IV skew is downward sloping for long ETFs (e.g. SPY, SSO, UPRO),
- IV is skew is upward sloping for short ETFs (e.g. SDS, SPXU).

Intuitively,

- a put on a long-LETf and a call on a short-LETf are both bearish,
- IVs should be higher for smaller (larger) LM for long (short) LETf.

Traditionally, IV is used to compare option contracts across strikes & maturities. What about IVs across leverage ratios?

Which pair of LETf options should have comparable IV?
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- Which pair of LETF options should have comparable IV?
Assume zero interest rate, dividend rate, fee.

Under the risk-neutral measure, we model the underlying index $S$ by the SDEs:

$$S_t = e^{X_t},$$
$$dX_t = -\frac{1}{2}\sigma^2(t, X_t, Y_t)dt + \sigma(t, X_t, Y_t)dW^x_t,$$
$$dY_t = c(t, X_t, Y_t)dt + g(t, X_t, Y_t)dW^y_t,$$
$$d\langle W^x, W^y \rangle_t = \rho(t, X_t, Y_t)dt.$$  

Then, the LETF price $L$ follows

$$L_t = e^{Z_t},$$
$$dZ_t = -\frac{1}{2}\beta^2\sigma^2(t, X_t, Y_t) dt + \beta\sigma(t, X_t, Y_t) dW^x_t.$$
Local-Stochastic Framework

- **Stochastic Volatility**: When $\sigma$ and $\rho$ are functions of $(t, y)$ only, such as Heston, then options written on $Z$ can be priced using $(Y, Z)$ only.

- **Local Volatility**: if both $\sigma$ and $\rho$ are dependent on $(t, x)$ only, such as CEV, then the ETF follows a local vol. model *but not* the LETF. Options on $Z$ must be analyzed in conjunction with $X$.

- **Local-Stochastic Volatility**: If $\sigma$ and/or $\rho$ depend on $(x, y)$, such as SABR, then to analyze options on $Z$, one must consider the triple $(X, Y, Z)$. 
LETIF Option Price

- With a terminal payoff \( \varphi(Z_T) \), the LETIF option price is given by the risk-neutral expectation

\[
u(t, x, y, z) = \mathbb{E}^Q[\varphi(Z_T)|X_t = x, Y_t = y, Z_t = z].\]

- The price function \( u \) satisfies the Kolmogorov backward equation

\[
(\partial_t + \mathcal{A}(t)) u = 0, \quad u(T, x, y, z) = \varphi(z),
\]

where the operator \( \mathcal{A}(t) \) is given by

\[
\mathcal{A}(t) = a(t, x, y) \left((\partial_x^2 - \partial_x) + \beta^2 \left(\partial_z^2 - \partial_z\right) + 2\beta \partial_x \partial_z\right) \\
+ b(t, x, y) \partial_y^2 + c(t, x, y) \partial_y + f(t, x, y) (\partial_x \partial_y + \beta \partial_y \partial_z),
\]

with coefficients

\[
a(t, x, y) = \frac{1}{2} \sigma^2(t, x, y), \quad b(t, x, y) = \frac{1}{2} g^2(t, x, y), \\
f(t, x, y) = g(t, x, y) \sigma(t, x, y) \rho(t, x, y).
\]
Expand the coefficients \((a, b, c, f)\) of the operator \(A(t)\) as a Taylor series. For \(\chi \in \{a, b, c, f\}\), we write

\[
\chi(t, x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \chi_{n-k,k}(t) (x - \bar{x})^{n-k} (y - \bar{y})^{k},
\]

\[
\chi_{n-k,k}(t) = \frac{\partial_x^{n-k} \partial_y^k \chi(t, \bar{x}, \bar{y})}{(n-k)!k!}.
\]
By direct substitution, the operator $A(t)$ can now be written as

$$
A(t) = A_0(t) + B_1(t), \quad B_1(t) = \sum_{n=1}^{\infty} A_n(t),
$$

where

$$
A_n(t) = \sum_{k=0}^{n} (x - \bar{x})^{n-k}(y - \bar{y})^k A_{n-k,k}(t),
$$

$$
A_{n-k,k}(t) = a_{n-k,k}(t) \left( \left( \partial_x^2 - \partial_x \right) + \beta^2 \left( \partial_z^2 - \partial_z \right) + 2\beta \partial_x \partial_z \right)
+ b_{n-k,k}(t) \partial_y^2 + c_{n-k,k}(t) \partial_y + f_{n-k,k}(t) \left( \partial_x \partial_y + \beta \partial_y \partial_z \right),
$$

The price function now satisfies the PDE

$$
(\partial_t + A_0(t))u(t) = -B_1(t)u(t), \quad u(T) = \varphi.
$$
Asymptotic Expansion of Option Price

We define $\bar{u}_N$ our $N$th order approximation of $u$ by

$$\bar{u}_N = \sum_{n=0}^{N} u_n.$$ 

The 0th order term $u_0$ solves $(\partial_t + A_0)u_0 = 0$, and is given by

$$u_0(t) = \int_{\mathbb{R}} d\zeta \frac{1}{\sqrt{2\pi s^2(t, T)}} \exp \left( \frac{-(\zeta - m(t, T))^2}{2s^2(t, T)} \right) \varphi(\zeta),$$

$$m(t, T) = z - \beta^2 \int_t^T dt_1 a_{0,0}(t_1), \quad s^2(t, T) = 2\beta^2 \int_t^T dt_1 a_{0,0}(t_1).$$

In turn, $u_1$ would solve $(\partial_t + A_0 + A_1)u_1 = -A_1 u_0$. 
Asymptotic Expansion of Option Price

And the higher order terms are given by

\[ u_n(t) = \mathcal{L}_n(t, T) u_0(t), \]

where

\[ \mathcal{L}_n(t, T) = \sum_{k=1}^{n} \int_t^T dt_1 \int_{t_1}^T dt_2 \cdots \int_{t_{k-1}}^T dt_k \sum_{i \in I_{n,k}} G_{i_1}(t, t_1) G_{i_2}(t, t_2) \cdots G_{i_k}(t, t_k), \]

with

\[ G_n(t, t_i) := \sum_{k=0}^{n} (M_x(t, t_i) - \bar{x})^{n-k} (M_y(t, t_i) - \bar{y})^k A_{n-k,k}(t_i), \]

\[ M_x(t, t_i) := x + \int_t^{t_i} ds \left( a_{0,0}(s) (2 \partial_x + 2 \beta \partial_z - 1) + f_{0,0}(s) \partial_y \right), \]

\[ M_y(t, t_i) := y + \int_t^{t_i} ds \left( f_{0,0}(s) (\partial_x + \beta \partial_z) + 2 b_{0,0}(s) \partial_y + c_{0,0}(s) \right), \]

\[ I_{n,k} = \{ i = (i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 + i_2 + \cdots + i_k = n \}. \]
Implied Volatility Expansion

- The *Black-Scholes Call price* $u^{\text{BS}} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$u^{\text{BS}}(\sigma) := e^z N(d_+(\sigma)) - e^k N(d_-(\sigma)), \quad d_{\pm}(\sigma) := \frac{1}{\sigma \sqrt{\tau}} \left( z - k \pm \frac{\sigma^2 \tau}{2} \right),$$

where $\tau = T - t$, and $N$ is the standard normal CDF.

- For fixed $(t, T, z, k)$, the *implied volatility* corresponding to a call price $u \in ((e^z - e^k)^+, e^z)$ is defined as the unique strictly positive real solution $I$ of the equation

$$u^{\text{BS}}(I) = u.$$

- We consider an expansion of the implied volatility

$$I = \sigma_0 + \sum_{n=1}^{\infty} \sigma_n$$
Implied Volatility Expansion

- Recall that the price expansion is of the form: \( u = u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} u_n. \)
- On the other hand, one expands \( u^{BS}(I) \) as a Taylor series about the point \( \sigma_0 \)

\[
u^{BS}(I) = u^{BS}(\sigma_0 + \eta) = u^{BS}(\sigma_0) + \eta \partial_{\sigma} u^{BS}(\sigma_0) + \frac{1}{2!} \eta^2 \partial^2_{\sigma} u^{BS}(\sigma_0) + \frac{1}{3!} \eta^3 \partial^3_{\sigma} u^{BS}(\sigma_0) + \ldots .\]

- In turn, one can solve iteratively for every term of \( (\sigma_n)_{n \geq 1} \). We have a general expression for the nth term.
- The first two terms are

\[
\sigma_1 = \frac{u_1}{\partial_{\sigma} u^{BS}(\sigma_0)}, \quad \sigma_2 = \frac{u_2 - \frac{1}{2} \sigma_1^2 \partial^2_{\sigma} u^{BS}(\sigma_0)}{\partial_{\sigma} u^{BS}(\sigma_0)},
\]

which can be simplified using the explicit expressions of \( u^{BS} \) and \( u^BS_{\sigma} \).
Implied Volatility Expansion

- In the time-homogeneous LSV setting, we write down up to the 1st order terms:

\[ \sigma_0 = |\beta| \sqrt{2a_{0,0}}, \quad \sigma_1 = \sigma_{1,0} + \sigma_{0,1}, \]

where

\[ \sigma_{1,0} = \left( \frac{\beta a_{1,0}}{2 \sigma_0} \right) \lambda + \tau \left( \frac{1}{4} (1 - \beta) \sigma_0 a_{1,0} \right), \]

\[ \sigma_{0,1} = \left( \frac{\beta^3 a_{0,1} f_{0,0}}{2 \sigma_0^3} \right) \lambda + \tau \left( \frac{\beta^2 a_{0,1} (2c_{0,0} + \beta f_{0,0})}{4 \sigma_0} \right), \]

\[ \lambda = k - z, \quad \tau = T - t. \]

- These expressions show explicitly the non-trivial dependence of IV on the leverage ratio \( \beta \).
Let $\sigma_Z(\tau, \lambda)$ (resp. $\sigma_X(\tau, \lambda)$) be the implied volatility of a call written on the LETF $Z$ (resp. $X$).

From the above IV expressions, we have

**LETF**: $\sigma_Z \approx |\beta| \sqrt{2a_{0,0}} + |\beta| \left( \frac{a_{1,0}}{2 \sqrt{2a_{0,0}}} + \frac{a_{0,1} f_{0,0}}{2(2a_{0,0})^{3/2}} \right) \frac{\lambda}{\beta} + O(\tau),$

**ETF**: $\sigma_X \approx \sqrt{2a_{0,0}} + \left( \frac{a_{1,0}}{2 \sqrt{2a_{0,0}}} + \frac{a_{0,1} f_{0,0}}{2(2a_{0,0})^{3/2}} \right) \lambda + O(\tau).$

**Implied volatility scaling**: the vertical axis of $\sigma_Z$ is scaled by a factor of $|\beta|$. Second, the horizontal axis is scaled by $1/\beta$. 
Empirical IV Scaling

Figure: Left: Empirical IV $\sigma_Z(\tau, \lambda)$ plotted as a function of log-moneyness $\lambda$ for SPY (red, $\beta = +1$), SSO (purple, $\beta = +2$), and SDS (blue, $\beta = -2$) on August 15, 2013 with $\tau = 155$ days to maturity. Note that the implied volatility of SDS is increasing in the LETF log-moneyness. Right: Using the same data, the scaled LETF implied volatilities $\sigma_Z^{(\beta)}(\tau, \lambda)$ nearly coincide.
Example: Heston Model

- The Heston model (in log prices) is described by

\[
\begin{align*}
    dX_t &= -\frac{1}{2}e^{Y_t} dt + e^{\frac{1}{2}Y_t} dW^x_t, \\
    dY_t &= ((\kappa\theta - \frac{1}{2}\delta^2)e^{-Y_t} - \kappa) dt + \delta e^{-\frac{1}{2}Y_t} dW^y_t, \\
    dZ_t &= -\beta^2\frac{1}{2}e^{Y_t} dt + \beta e^{\frac{1}{2}Y_t} dW^x_t, \\
    d\langle W^x, W^y \rangle_t &= \rho dt.
\end{align*}
\]

- \(X_0 = x := \log S_0,\)

- \(Y_0 = y := \log V_0,\)

- \(Z_0 = z := \log L_0,\)

- The generator of \((X, Y, Z)\) is given by

\[
\begin{align*}
    \mathcal{A} &= \frac{1}{2}e^y \left( (\partial_x^2 - \partial_x) + \beta^2 (\partial_z^2 - \partial_z) + 2\beta \partial_x \partial_z \right) \\
    &\quad + \left( (\kappa\theta - \frac{1}{2}\delta^2)e^{-y} - \kappa \right) \partial_y + \frac{1}{2} \delta^2 e^{-y} \partial_y^2 + \rho \delta \left( \partial_x \partial_y + \beta \partial_x \partial_z \right),
\end{align*}
\]

- \(a(x, y) = \frac{1}{2}e^y,\)

- \(b(x, y) = \frac{1}{2}\delta^2 e^{-y},\)

- \(c(x, y) = \left( (\kappa\theta - \frac{1}{2}\delta^2)e^{-y} - \kappa \right),\)

- \(f(x, y) = \rho \delta.\)
Example: Heston Model

- The first two terms in the IV expansion are

\[
\sigma_0 = |\beta| \sqrt{e^y}, \\
\sigma_1 = \frac{\tau \left( -\beta^2 \left( \delta^2 - 2\theta \kappa \right) + (-2\kappa + \beta \delta \rho) \sigma_0^2 \right)}{8\sigma_0} + \frac{\beta \delta \rho}{4\sigma_0} (k - z).
\]

- Note that when \( X \) has Heston dynamics with parameters \((\kappa, \theta, \delta, \rho, y)\), then \( Z \) also admits Heston dynamics with parameters

\[(\kappa_Z, \theta_Z, \delta_Z, \rho_Z, y_Z) = (\kappa, \beta^2 \theta, |\beta| \delta, \text{sign}(\beta) \rho, y + \log \beta^2).\]

- This can also be inferred from our IV expressions. Indeed, the coefficients’ dependence on \( \beta \) is present only in the terms \( \beta^2 \theta, |\beta| \delta, \text{sign}(\beta) \rho, y + \log \beta^2 \).

For instance, we can write \( \sigma_0 = \sqrt{e^y + \log \beta^2} = \sqrt{e^{yz}} \), and the coeff. of \((k - z)\) in \( \sigma_1 \) is \( \beta \delta \rho / 4\sigma_0 = |\beta| \delta \text{sign}(\beta) \rho / 4\sigma_0 = \delta_Z \rho_Z / 4\sigma_0 \).
IV comparison

Figure: Exact (solid – computed by Fourier inversion) and approximate (dashed) scaled IV $\sigma_Z^{(\beta)}(\tau, \lambda)$ under Heston, plotted against $\lambda$. For comparison, we also plot the ETF’s exact Heston IV $\sigma_X(\tau, \lambda)$ (dotted). Parameters: $\kappa = 1.15$, $\theta = 0.04$, $\delta = 0.2$, $\rho = -0.4$, $y = \log \theta$, $\tau = 0.0625$. Left: $\beta = +2$. Right $\beta = -2$. 
Example: CEV Model

- The CEV model (in log prices) is described by

\[
\begin{align*}
\text{d}X_t &= -\frac{1}{2}\delta^2 e^{2(\gamma-1)X_t} \text{d}t + \delta e^{(\gamma-1)X_t} \text{d}W_t^x, \\
X_0 &= x := \log S_0. \\
\text{d}Z_t &= -\frac{1}{2}\beta^2 \delta^2 e^{2(\gamma-1)X_t} \text{d}t + \beta \delta e^{(\gamma-1)X_t} \text{d}W_t^x, \\
Z_0 &= z := \log L_0,
\end{align*}
\]

with \( \gamma \leq 1. \)

- The generator of \((X, Z)\) is given by

\[
A = \frac{1}{2} \delta^2 e^{2(\gamma-1)x} \left( (\partial_x^2 - \partial_x) + \beta^2 (\partial_z^2 - \partial_z) + 2\beta \partial_x \partial_z \right).
\]

\[
a(x, y) = \frac{1}{2} \delta^2 e^{2(\gamma-1)x}, \quad b(x, y) = 0, \quad c(x, y) = 0, \quad f(x, y) = 0.
\]
The first 3 terms in the IV expansion are

\[
\sigma_0 = \beta \sqrt{e^{2x(\gamma-1)}\delta^2},
\]

\[
\sigma_1 = \frac{\tau(\beta - 1)(\gamma - 1)\sigma_0^3}{4\beta^2} + \frac{(\gamma - 1)\sigma_0}{2\beta}(k - z),
\]

\[
\sigma_2 = \frac{\tau(\gamma - 1)^2\sigma_0^3}{96\beta^4}
\left(4\beta^2 + t(13 + 2\beta(-13 + 6\beta))\sigma_0^2\right)
\]

\[
+ \frac{7\tau(\beta - 1)(\gamma - 1)^2\sigma_0^3}{24\beta^3}(k - z) + \frac{(\gamma - 1)^2\sigma_0}{12\beta^2}(k - z)^2,
\]

The factor \((\gamma - 1)\) appears in every term of these expressions.

If \(\gamma = 1\), \(\sigma_0 = \beta\delta\) and \(\sigma_1 = \sigma_2 = \sigma_3 = 0\). The higher order terms also vanish since \(a(x, y) = \frac{1}{2}\delta^2\) in this case. Hence, just as in the B-S case, the IV expansion becomes flat.
Figure: Exact (solid – computed by Fourier inversion) and approximate (dashed) scaled IV $\sigma_{Z}^{(\beta)}(\tau, \lambda)$ under CEV, plotted against $\lambda$. For comparison, we also plot the ETF’s exact CEV IV $\sigma_{X}(\tau, \lambda)$ (dotted). Parameters: $\delta = 0.2$, $\gamma = -0.75$, $x = 0$. Left: $\beta = +2$. Right: $\beta = -2$. 
Example: SABR Model

- The SABR model (in log prices) is described by

\[ dX_t = -\frac{1}{2} e^{2Y_t + (\gamma - 1)X_t} dt + e^{Y_t + (\gamma - 1)X_t} dW_t^x, \]

\[ dY_t = -\frac{1}{2} \delta^2 dt + \delta dW_t^y, \]

\[ dZ_t = -\frac{1}{2} \beta^2 e^{2Y_t + (\gamma - 1)X_t} dt + \beta e^{Y_t + (\gamma - 1)X_t} dW_t^x, \]

\[ d\langle W^x, W^y \rangle_t = \rho dt. \]

- The generator of \((X, Y, Z)\) is given by

\[ A = \frac{1}{2} e^{2y + (\gamma - 1)x} \left( (\partial_x^2 - \partial_x) + \beta^2 (\partial_z^2 - \partial_z) + 2\beta \partial_x \partial_y \right) \]

\[ - \frac{1}{2} \delta^2 \partial_y + \frac{1}{2} \delta^2 \partial_y^2 + \rho \delta e^{y + (\gamma - 1)x} (\partial_x \partial_y + \beta \partial_y \partial_z). \]

\[ a(x, y) = \frac{1}{2} e^{2y + (\gamma - 1)x}, \quad b = \frac{1}{2} \delta^2, \quad c = -\frac{1}{2} \delta^2, \quad f(x, y) = \rho \delta e^{y + (\gamma - 1)x}. \]
Example: SABR Model

The first 3 terms in the IV expansion are

\[ \sigma_0 = |\beta| \sqrt{e^{2y+2(\gamma-1)x}}, \]
\[ \sigma_1 = \sigma_{1,0} + \sigma_{0,1}, \]
\[ \sigma_2 = \sigma_{2,0} + \sigma_{1,1} + \sigma_{0,2}. \]

where

\[ \sigma_{1,0} = \frac{\tau(-1 + \beta)(\gamma - 1)\sigma_0^3}{4\beta^2} + \frac{(\gamma - 1)\sigma_0}{2\beta}(k - z), \]
\[ \sigma_{0,1} = -\frac{1}{4}\tau \delta \sigma_0 (\delta - \rho \text{sign}[\beta]\sigma_0) + \frac{1}{2}\delta \rho \text{sign}[\beta](k - z). \]
Example: SABR Model

\[
\sigma_{2,0} = \frac{\tau (\gamma - 1)^2 \sigma_0^3 \left(4\beta^2 + \tau (13 + 2\beta (-13 + 6\beta)) \sigma_0^2\right)}{96 \beta^4} + \frac{7\tau (-1 + \beta) (\gamma - 1)^2 \sigma_0^3}{24 \beta^3} (k - z) + \frac{(\gamma - 1)^2 \sigma_0}{12 \beta^2} (k - z)^2,
\]

\[
\sigma_{1,1} = \frac{\tau (\gamma - 1) \delta \sigma_0^2 (12 \rho|\beta| + \tau \sigma_0 (-9 (-1 + \beta) \delta + (-11 + 10 \beta) \rho \text{sign}[\beta] \sigma_0))}{48 \beta^2} - \frac{\tau (\gamma - 1) \delta \sigma_0 \left(3 \delta + \frac{5(1 - 2\beta) \rho \sigma_0}{|\beta|}\right)}{24 \beta} (k - z),
\]

\[
\sigma_{0,2} = \frac{1}{96} \tau \delta^2 \sigma_0 \left(32 + 5\tau \delta^2 - 12 \rho^2 + 2\tau \sigma_0 (-7 \delta \rho \text{sign}[\beta] + (-2 + 6 \rho^2) \sigma_0)\right) - \frac{1}{24} \tau \delta^2 \rho (\delta \text{sign}[\beta] - 3\rho \sigma_0) + \frac{\delta^2 (2 - 3\rho^2)}{12 \sigma_0} (k - z)^2,
\]
Figure: Exact (solid – computed by Fourier inversion) and approximate (dashed) scaled IV $\sigma^{(\beta)}_Z(\tau, \lambda)$ under SABR, plotted against $\lambda$. For comparison, we also plot the ETF’s exact SABR IV $\sigma_X(\tau, \lambda)$ (dotted). Parameters: $\delta = 0.5$, $\gamma = -0.5$, $\rho = 0.0$ $x = 0$, $y = -1.5$. Left: $\beta = +2$. Right: $\beta = -2$. 
Alternative IV Scaling

- In a general stochastic volatility model, the log LETF price is

\[
\log \left( \frac{L_T}{L_0} \right) = \beta \log \left( \frac{X_T}{X_0} \right) - (r(\beta - 1) + c)T - \frac{\beta(\beta - 1)}{2} \int_0^T \sigma_t^2 dt.
\]

- **Key idea:** Condition on that the *terminal* LM \( \log \left( \frac{X_T}{X_0} \right) \) equal to constant \( LM^{(1)} \).

- Then, the best estimate of the \( \beta \)-LETF’s LM is given by the cond’l expectation:

\[
LM^{(\beta)} := \beta LM^{(1)} - (r(\beta - 1) + c)T - \frac{\beta(\beta - 1)}{2} \mathbb{E}_Q \left\{ \int_0^T \sigma_t^2 dt \mid \log \left( \frac{X_T}{X_0} \right) = LM^{(1)} \right\}.
\]
Connecting Log-moneyness

- Assuming constant $\sigma$ as in the B-S model, we have the formula:

$$LM^{(\beta)} = \beta LM^{(1)} - (r(\beta - 1) + c)T - \frac{\beta(\beta - 1)}{2}\sigma^2T.$$  \hspace{1cm} (1)

- Hence, the $\beta$-LETF log-moneyness $LM^{(\beta)}$ is expressed as an affine function of the unleveraged ETF log-moneyness $LM^{(1)}$, reflecting the role of $\beta$.
- The moneyness scaling formula can be interpreted via Dual Delta matching.

**Proposition**

*Under the B-S model, an ETF call with log-moneyness $LM^{(1)}$ and a $\beta$-LETF call with log-moneyness $LM^{(\beta)}$ in (1) have the same Dual Delta.*

Recall: Dual Delta of an LETF call is $e^{-r(T-t)}N(d_2^{(\beta)})$, and $N(d_2^{(\beta)})$ represents the risk-neutral probability of the option ending up ITM.
Concluding Remarks

- We have discussed a local-stochastic volatility framework to understand the inter-connectedness of LETF options.
- Explicit price and IV expansions are provided for Heston, CEV, and SABR models.
- The method of moneyness scaling enhances the comparison of IVs with different leverage ratios.
- The connection allows us to use the richer unleveraged index/ETF option data to shed light on the less liquid LETF options market.
- Our procedure can be applied to identify IV discrepancies across LETF options markets.
Appendix
Predicting from SPY IVs to LETF IVs

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SSO Prediction

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SDS Prediction

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UPRO Prediction

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SPXU Prediction

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