Implied Volatility of Leveraged ETF Options: Consistency and Scaling

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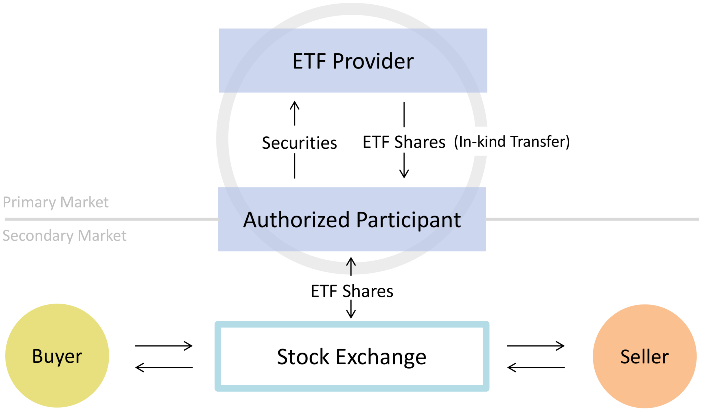
Finance and Stochastics (FAST) Seminar
University of Sussex
March 16, 2015

ETF Market


ETF Provider
Securities
ETF Shares (In-kind Transfer)
Primary Market
Secondary Market
Authorized Participant
ETF Shares
Buyer
Stock Exchange
Seller

LETFs and Their Options

- Leveraged Exchange Traded Funds (LETFs) promise a fixed multiple of the returns of an index/underlying asset.
- Most typical leverage ratios are: (long) $2, 3$, and (short) $-2, -3$.

- LETFs have also led to increased trading of options written on LETFs.
- In 2012, total daily notional on LETF options is $40-50 bil, as compared to $90 bil for S&P 500 index options.
- ETFs with the highest options volume: SPY (S&P 500), IWM (Russell 2000), QQQ (Nasdaq 100) and GLD (gold).

Leveraged ETF Returns

- By design, an LETF seeks to provide a constant multiple β of the *daily* returns R_j of a reference index or asset:

$$L_n = L_0 \cdot \prod_{j=1}^{n} (1 + \beta\, R_j).$$

- By differentiation, we get

$$\frac{d}{d\beta}\left(\log\left(\frac{L_n}{L_0}\right)\right) = \sum_{j=1}^{n} \frac{R_j}{1 + \beta\, R_j}.$$

- With a positive leverage ratio $\beta > 0$, if $R_j > 0$ for all j, then $\log\left(\frac{L_n}{L_0}\right)$, or equivalently the value L_n, is increasing in β.
- That is, when the reference asset is increasing in value, a larger, positive leverage ratio is preferred.

Example: non-directional movements

Day	ETF	%-change	$+2$x LETF	%-change	-2x LETF	%-change
0	100		100		100	
1	98	-2%	96	-4%	104	4%
2	99.96	2%	99.84	4%	99.84	-4%
3	97.96	-2%	95.85	-4%	103.83	4%
4	99.92	2%	99.68	4%	99.68	-4%
5	97.92	-2%	95.69	-4%	103.67	4%
6	99.88	2%	99.52	4%	99.52	-4%

- Even though the ETF records a tiny loss of 0.12% after 6 days, the $+2$x LETF ends up with a loss of 0.48%, which is greater (in abs. value) than 2 times the return (-0.12%) of the ETF.
- At the terminal date, *both* the long and short LETFs have recorded net losses of 0.48%.
- These can occur for over a longer/shorter period.

Empirical LETF Prices

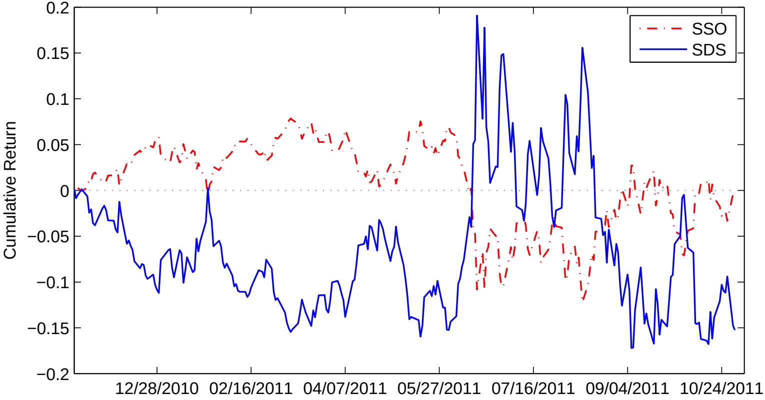


Figure: SSO $(+2)$ and SDS (-2) cumulative returns from Dec 2010 to Nov 2011. Observe that both SSO and SDS can give negative returns simultaneously over several periods in time.

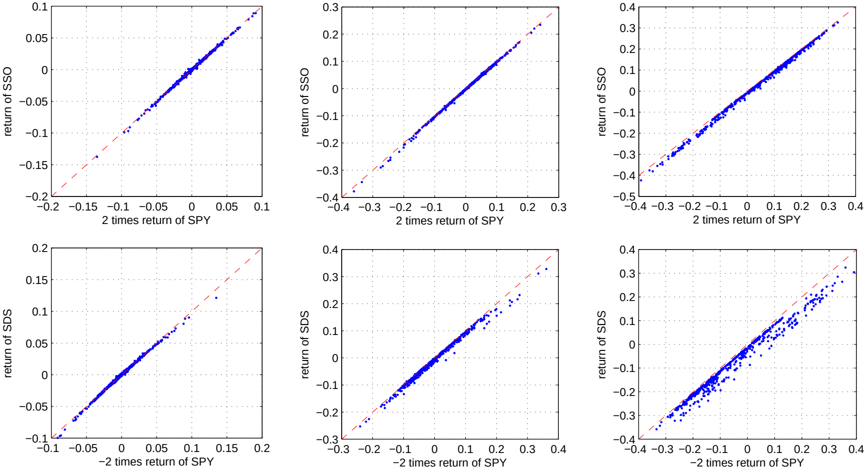


Figure: 1-day (left), 2-week (mid) and 2-month (right) returns of SPY against SSO (top) and SDS (bottom), in logarithmic scale. We considered 1-day, 2-week and 2-month rolling periods from Sept 29, 2010 to Sept 30, 2012.

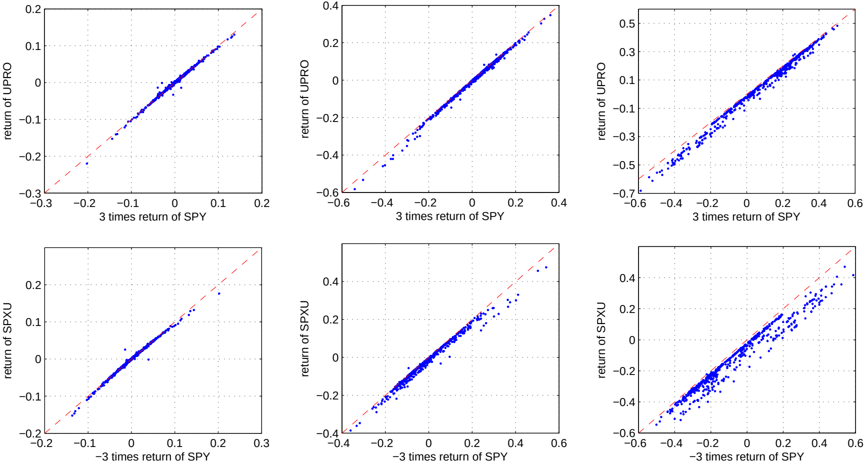


Figure: 1-day (left), 2-week (mid) and 2-month (right) returns of SPY against UPRO (top) and SPXU (bottom), in logarithmic scale. We considered 1-day, 2-week and 2-month rolling periods from Sept 29, 2010 to Sept 30, 2012.

LETF Price Dynamics

- Suppose the reference asset price follows
$$\frac{dS_t}{S_t} = \mu_t\,dt + \sigma_t\,dW_t,$$
under the historical measure $\mathbb{P}$.
- A long LETF L on X with leverage ratio $\beta > 1$ is constructed by
  - investing the amount βL_t (β times the fund value) in S,
  - borrowing the amount $(\beta - 1)L_t$ at the risk-free rate r,
  - charging a small expense fee at rate c.
- For a short ($\beta \leq -1$) LETF, $\$|\beta|L_t$ is shorted on S, and $\$(1 + |\beta|)L_t$ is kept in the money market account.
- The LETF price dynamics:
$$\begin{aligned}\frac{dL_t}{L_t} &= \beta\left(\frac{dS_t}{S_t}\right) - ((\beta - 1)r + c)\,dt \\ &= [\beta\mu_t - ((\beta - 1)r + c)]\,dt + \beta\sigma_t\,dW_t\,.\end{aligned}$$

LETF Returns and Volatility Decay

- The LETF value L can be written in terms of S:

$$\frac{L_T}{L_0} = \left(\frac{S_T}{S_0}\right)^{\beta} e^{-(r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\int_0^T \sigma_t^2\,dt},$$

or equivalently in log-returns:

$$\log\left(\frac{L_T}{L_0}\right) = \beta \log\left(\frac{S_T}{S_0}\right) - (r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\int_0^T \sigma_t^2\,dt.$$

- Hence, the realized volatility will lead to attrition in fund value.
- This phenomenon is called the volatility decay.

A Static LETF Portfolio

- A well known industry strategy consists of shorting two LETFs on the same reference.
- We short $\omega \in (0,1)$ of $\beta_+ > 0$ LETF and $1-\omega$ of $\beta_- < 0$ LETF.
- At time T, the (normalized) return of the portfolio is

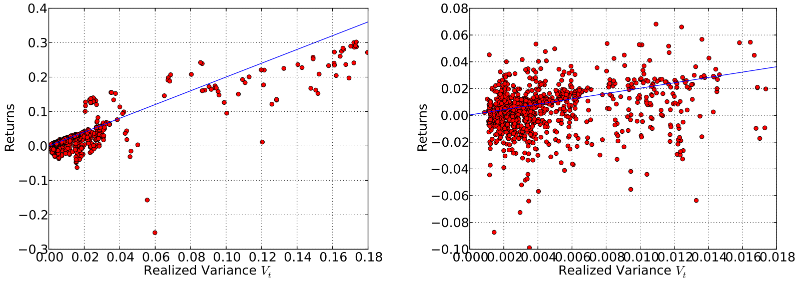
$$R_T = 1 - \omega \frac{L_T^+}{L_0^+} - (1-\omega)\frac{L_T^-}{L_0^-}.$$

- If we apply $\frac{L_T^\pm}{L_o^\pm} \approx \ln \frac{L_T^\pm}{L_0^\pm} + 1$, and set $\omega^* = \frac{-\beta_-}{\beta_+ - \beta_-}$, we get

$$R_T = \frac{-\beta_-\beta_+}{2} V_T - \frac{\beta_-}{\beta_+ - \beta_-}(f_+ - f_-)T + (f_- - r)T.$$

- Portfolio is long volatility since $\frac{-\beta_-\beta_+}{2} > 0$, and it's also Δ-neutral.

Empirical Performance



(a) DJUSEN-DIG-DUG

(b) GOLDLNPM-UGL-GLL

Figure: Empirical return vs realized variance for a double short strategy over 30-day holding periods. Here, $\beta = \pm 2$ for each LETF pair. The blue line gives a predicted (not regressed) return as a linear function of the realized variance. For empirical backtesting, we set $\omega^* = 0.5$, and $T = 30/252$.

Time Series of Empirical Performance

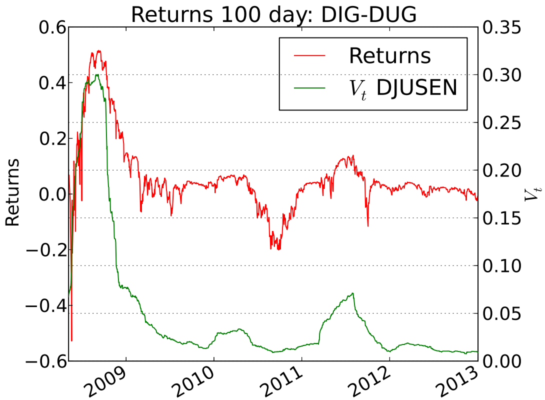


Figure: Time series of returns for a double short strategy over 100-day (rolling) holding periods. Notice how during the periods of greatest volatility our double short trading strategy had the greatest return.

Average Returns by LETF Pairs

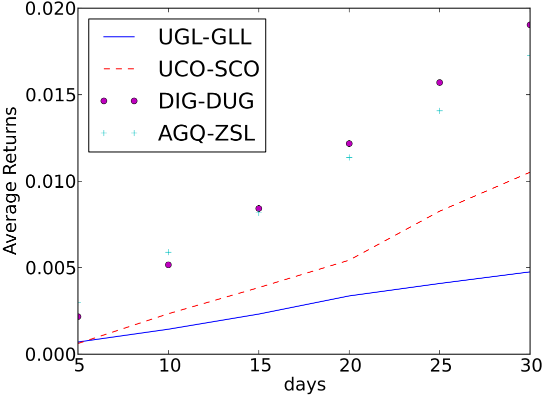


Figure: Average returns from a double short trading strategy by commodity LETF pairs over no. of days. For all pairs, $\beta = \pm 2$.

Pricing LETF Options

LETF Option Price

- If σ is constant, then L is lognormal, and the no-arb price a European call option on L is

$$C_{BS}^{(\beta)}(t, L; K, T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\{(L_T - K)^+ \mid L_t = L\}$$
$$= C_{BS}(t, L; K, T, r, c, |\beta|\sigma),$$

where $C_{BS}(t, L; K, T, r, c, \sigma)$ is the Black-Scholes formula for a call.

- Given the market price C^{obs} of a call on L, the implied volatility is given by

$$I^{(\beta)}(K, T) = (C_{BS}^{(\beta)})^{-1}(C^{obs}) = \frac{1}{|\beta|}C_{BS}^{-1}(C^{obs}).$$

- We normalize by the $|\beta|^{-1}$ factor in our definition of implied volatility so that they remain on the same scale.

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Implied Volatility (IV) of LETF Options

Proposition

The slope of the implied volatility curve admits the (model-free) bound:

$$-\frac{e^{-(r-c)(T-t)}}{|\beta|L\sqrt{T-t}}\,\frac{(1-N(d_2^{(\beta)}))}{N'(d_1^{(\beta)})} \leq \frac{\partial I^{(\beta)}(K)}{\partial K} \leq \frac{e^{-(r-c)(T-t)}}{|\beta|L\sqrt{T-t}}\,\frac{N(d_2^{(\beta)})}{N'(d_1^{(\beta)})},$$

where $d_2^{(\beta)} = d_1^{(\beta)} - |\beta| I^{(\beta)}(K)\sqrt{T-t}$, *with* $\sigma = I^{\beta}(K)$.

- It follows from the fact that observed and model call prices must be decreasing in K:

$$\frac{\partial C^{obs}}{\partial K} = \frac{\partial C_{BS}}{\partial K}(t, L; K, T, r, c, |\beta| I^{(\beta)}(K))$$
$$= \frac{\partial C_{BS}}{\partial K}(|\beta| I^{(\beta)}(K)) + |\beta| \frac{\partial C_{BS}}{\partial \sigma}(|\beta| I^{(\beta)}(K)) \frac{\partial I^{(\beta)}(K)}{\partial K} \leq 0.$$

- Rearranging terms gives the upper bound.
- Repeat this using put prices yields the lower bound.

Empirical Implied Volatilities – SPX, SPY (+1)

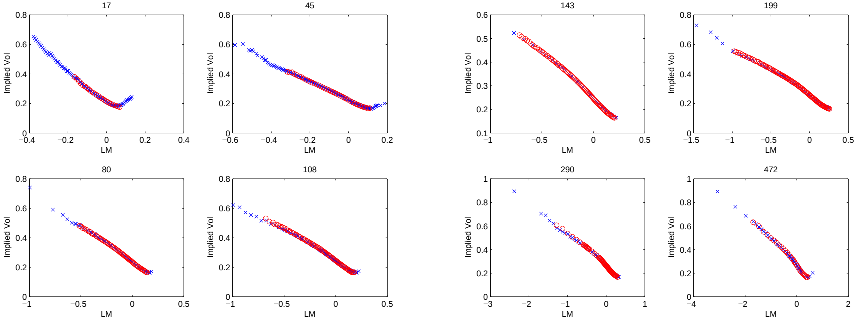

Figure: *SPX (blue cross) and SPY (red circles) implied volatilities on Sept 1, 2010 for different maturities (from* 17 *to* 472 *days) plotted against log-moneyness:*

$$LM = \log\left(\frac{\textit{strike}}{\textit{(L)ETF price}}\right).$$

Empirical Implied Volatilities – SPY $(+1)$, SSO $(+2)$

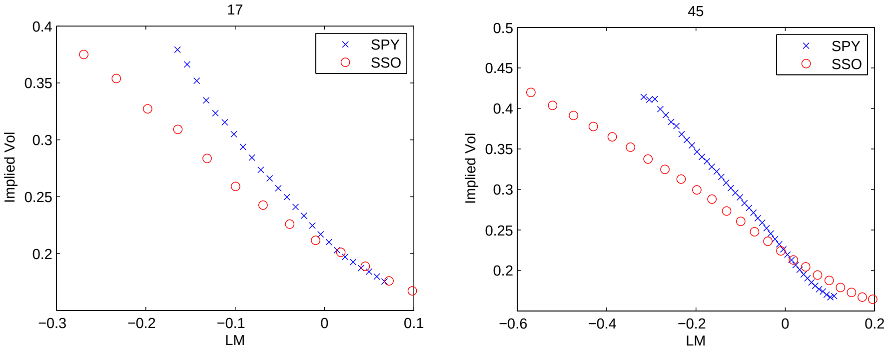

Figure: *SPY (blue cross) and SSO (red circles) implied volatilities against log-moneyness.*

Empirical Implied Volatilities – SPY $(+1)$, SSO $(+2)$

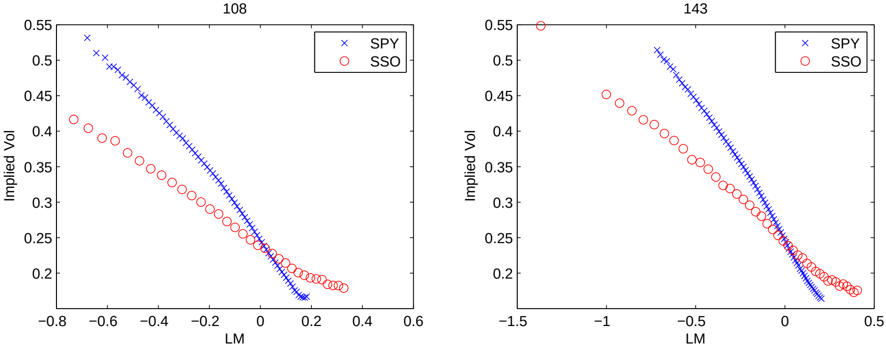

Figure: *SPY (blue cross) and SSO (red circles) implied volatilities against log-moneyness.*

Empirical Implied Volatilities – SPY $(+1)$, SDS (-2)

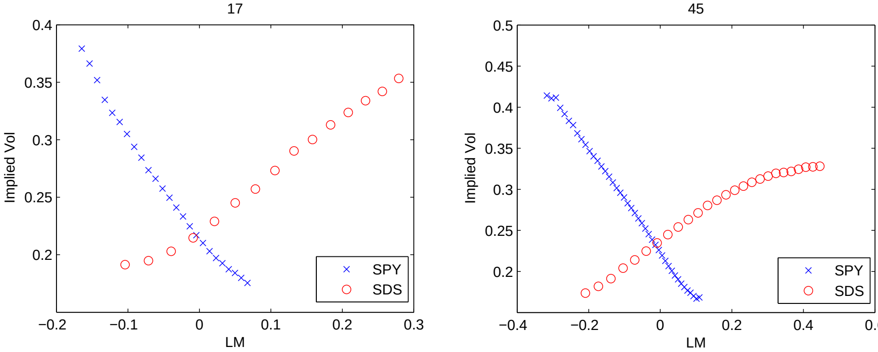

Figure: *SPY (blue cross) and SDS (red circles) implied volatilities against log-moneyness (LM) for increasing maturities.*

Empirical Implied Volatilities – SPY $(+1)$, SDS (-2)

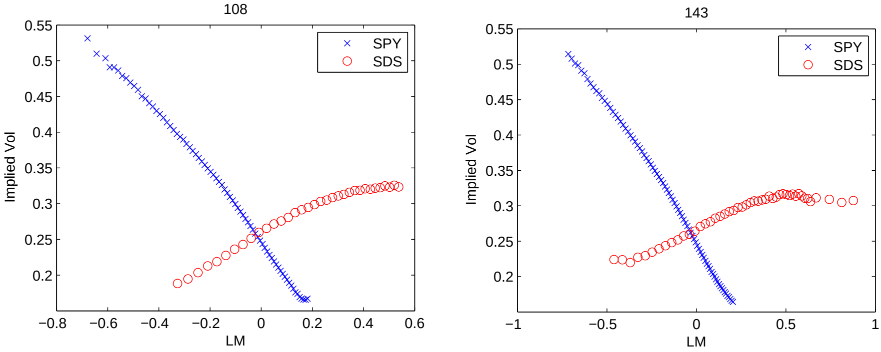

Figure: *SPY (blue cross) and SDS (red circles) implied volatilities against log-moneyness (LM) for increasing maturities.*

Observations

- The most salient features of the empirical implied volatilities:
  - IV skew for SSO $(+2)$ is flatter than that for SPY,
  - IV skew is downward sloping for long ETFs (e.g. SPY, SSO, UPRO),
  - IV is skew is upward sloping for short ETFs (e.g. SDS, SPXU).
- Intuitively,
  - a put on a long-LETF and a call on a short-LETF are both bearish,
  - IVs should be higher for smaller (larger) LM for long (short) LETF.
- Traditionally, IV is used to compare option contracts across strikes & maturities. What about IVs *across leverage ratios*?
- Which pair of LETF options should have comparable IV?

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Local-Stochastic Framework

- Assume zero interest rate, dividend rate, fee.
- Under the risk-neutral measure, we model the underlying index S by the SDEs:

$$
\begin{aligned}
S_t &= \mathrm{e}^{X_t}, \\
\mathrm{d}X_t &= -\frac{1}{2}\sigma^2(t, X_t, Y_t)\mathrm{d}t + \sigma(t, X_t, Y_t)\mathrm{d}W_t^x, \\
\mathrm{d}Y_t &= c(t, X_t, Y_t)\mathrm{d}t + g(t, X_t, Y_t)\mathrm{d}W_t^y, \\
\mathrm{d}\langle W^x, W^y \rangle_t &= \rho(t, X_t, Y_t)\mathrm{d}t.
\end{aligned}
$$

- Then, the LETF price L follows

$$
L_t = \mathrm{e}^{Z_t}, \qquad \mathrm{d}Z_t = -\frac{1}{2}\beta^2\sigma^2(t, X_t, Y_t)\,\mathrm{d}t + \beta\sigma(t, X_t, Y_t)\,\mathrm{d}W_t^x.
$$

Local-Stochastic Framework

- **Stochastic Volatility**: When σ and ρ are functions of (t, y) only, such as **Heston**, then options written on Z can be priced using (Y, Z) only.
- **Local Volatility**: if both σ and ρ are dependent on (t, x) only, such as **CEV**, then the ETF follows a local vol. model *but not* the LETF. Options on Z must be analyzed in conjunction with X.
- **Local-Stochastic Volatility**: If σ and/or ρ depend on (x, y), such as **SABR**, then to analyze options on Z, one must consider the triple (X, Y, Z).

LETF Option Price

- With a terminal payoff $\varphi(Z_T)$, the LETF option price is given by the risk-neutral expectation

$$u(t,x,y,z) = \mathbb{E}^{\mathbb{Q}}[\varphi(Z_T)|X_t = x, Y_t = y, Z_t = z].$$

- The price function u satisfies the Kolmogorov backward equation

$$(\partial_t + \mathcal{A}(t))\, u = 0, \qquad\qquad u(T,x,y,z) = \varphi(z),$$

where the operator $\mathcal{A}(t)$ is given by

$$\begin{aligned}\mathcal{A}(t) = a(t,x,y)\left(\left(\partial_x^2 - \partial_x\right) + \beta^2\left(\partial_z^2 - \partial_z\right) + 2\beta\,\partial_x\partial_z\right) \\ + b(t,x,y)\partial_y^2 + c(t,x,y)\partial_y + f(t,x,y)\left(\partial_x\partial_y + \beta\,\partial_y\partial_z\right),\end{aligned}$$

with coefficients

$$\begin{aligned}&a(t,x,y) = \tfrac{1}{2}\sigma^2(t,x,y), \quad b(t,x,y) = \tfrac{1}{2}g^2(t,x,y),\\ &f(t,x,y) = g(t,x,y)\sigma(t,x,y)\rho(t,x,y).\end{aligned}$$

Asymptotic Expansion of Option Price

- Expand the coefficients (a, b, c, f) of the operator $\mathcal{A}(t)$ as a Taylor series.
- For $\chi \in \{a, b, c, f\}$, we write

$$\chi(t, x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \chi_{n-k,k}(t)(x - \bar{x})^{n-k}(y - \bar{y})^{k},$$

$$\chi_{n-k,k}(t) = \frac{\partial_x^{n-k} \partial_y^{k} \chi(t, \bar{x}, \bar{y})}{(n-k)!k!}.$$

Asymptotic Expansion of Option Price

- By direct substitution, the operator $\mathcal{A}(t)$ can now be written as

$$\mathcal{A}(t) = \mathcal{A}_0(t) + \mathcal{B}_1(t), \qquad \mathcal{B}_1(t) = \sum_{n=1}^{\infty} \mathcal{A}_n(t),$$

where

$$\mathcal{A}_n(t) = \sum_{k=0}^{n} (x - \bar{x})^{n-k} (y - \bar{y})^k \mathcal{A}_{n-k,k}(t),$$

$$\mathcal{A}_{n-k,k}(t) = a_{n-k,k}(t) \left(\left(\partial_x^2 - \partial_x \right) + \beta^2 \left(\partial_z^2 - \partial_z \right) + 2\beta \, \partial_x \partial_z \right) + b_{n-k,k}(t) \partial_y^2 + c_{n-k,k}(t) \partial_y + f_{n-k,k}(t) \left(\partial_x \partial_y + \beta \, \partial_y \partial_z \right),$$

- The price function now satisfies the PDE

$$(\partial_t + \mathcal{A}_0(t)) u(t) = -\mathcal{B}_1(t) u(t), \qquad u(T) = \varphi.$$

Asymptotic Expansion of Option Price

We define $\bar{u}_N$ our Nth order approximation of u by

$$\bar{u}_N = \sum_{n=0}^{N} u_n.$$

The 0th order term u_0 solves $(\partial_t + \mathcal{A}_0)u_0 = 0$, and is given by

$$u_0(t) = \int_{\mathbb{R}} \mathrm{d}\zeta \, \frac{1}{\sqrt{2\pi s^2(t,T)}} \exp\left(\frac{-(\zeta - m(t,T))^2}{2s^2(t,T)} \right) \varphi(\zeta),$$

$$m(t,T) = z - \beta^2 \int_t^T \mathrm{d}t_1 \, a_{0,0}(t_1), \quad s^2(t,T) = 2\beta^2 \int_t^T \mathrm{d}t_1 \, a_{0,0}(t_1).$$

In turn, u_1 would solve $(\partial_t + \mathcal{A}_0 + \mathcal{A}_1)u_1 = -\mathcal{A}_1 u_0$.

Asymptotic Expansion of Option Price

And the higher order terms are given by

$$u_n(t) = \mathcal{L}_n(t,T)u_0(t),$$

where

$$\mathcal{L}_n(t,T) = \sum_{k=1}^{n} \int_t^T \mathrm{d}t_1 \int_{t_1}^T \mathrm{d}t_2 \cdots \int_{t_{k-1}}^T \mathrm{d}t_k \sum_{i \in I_{n,k}} \mathcal{G}_{i_1}(t,t_1)\mathcal{G}_{i_2}(t,t_2)\cdots\mathcal{G}_{i_k}(t,t_k),$$

with

$$\mathcal{G}_n(t,t_i) := \sum_{k=0}^{n} \left(\mathcal{M}_x(t,t_i) - \bar{x}\right)^{n-k} \left(\mathcal{M}_y(t,t_i) - \bar{y}\right)^k \mathcal{A}_{n-k,k}(t_i)$$

$$\mathcal{M}_x(t,t_i) := x + \int_t^{t_i} \mathrm{d}s \Big(a_{0,0}(s)\left(2\partial_x + 2\beta\partial_z - 1\right) + f_{0,0}(s)\partial_y \Big),$$

$$\mathcal{M}_y(t,t_i) := y + \int_t^{t_i} \mathrm{d}s \Big(f_{0,0}(s)\left(\partial_x + \beta\partial_z\right) + 2b_{0,0}(s)\partial_y + c_{0,0}(s) \Big),$$

$$I_{n,k} = \{ i = (i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 + i_2 + \cdots + i_k = n \}.$$

Implied Volatility Expansion

- The *Black-Scholes Call price* $u^{\mathrm{BS}} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$u^{\mathrm{BS}}(\sigma) := \mathrm{e}^z \mathcal{N}(d_+(\sigma)) - \mathrm{e}^k \mathcal{N}(d_-(\sigma)), \quad d_\pm(\sigma) := \frac{1}{\sigma\sqrt{\tau}}\left(z - k \pm \frac{\sigma^2 \tau}{2}\right),$$

  where $\tau = T - t$, and $\mathcal{N}$ is the standard normal CDF.
- For fixed (t, T, z, k), the *implied volatility* corresponding to a call price $u \in ((\mathrm{e}^z - \mathrm{e}^k)^+, \mathrm{e}^z)$ is defined as the unique strictly positive real solution I of the equation

$$u^{\mathrm{BS}}(I) = u.$$

- We consider an expansion of the implied volatility

$$I = \sigma_0 + \sum_{n=1}^{\infty} \sigma_n$$

Implied Volatility Expansion

- Recall that the price expansion is of the form: $u = u^{\mathrm{BS}}(\sigma_0) + \sum_{n=1}^{\infty} u_n$.
- On the other hand, one expands $u^{\mathrm{BS}}(I)$ as a Taylor series about the point σ_0

$$\begin{aligned} u^{\mathrm{BS}}(I) &= u^{\mathrm{BS}}(\sigma_0 + \eta) \\ &= u^{\mathrm{BS}}(\sigma_0) + \eta\,\partial_\sigma u^{\mathrm{BS}}(\sigma_0) + \frac{1}{2!}\eta^2 \partial_\sigma^2 u^{\mathrm{BS}}(\sigma_0) + \frac{1}{3!}\eta^3 \partial_\sigma^3 u^{\mathrm{BS}}(\sigma_0) + \ldots. \end{aligned}$$

- In turn, one can solve iteratively for every term of $(\sigma_n)_{n\geq 1}$. We have a general expression for the nth term.
- The first two terms are

$$\sigma_1 = \frac{u_1}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)}, \qquad \sigma_2 = \frac{u_2 - \frac{1}{2}\sigma_1^2 \partial_\sigma^2 u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)},$$

which can be simplified using the explicit expressions of u^{BS} and u_σ^{BS}.

Implied Volatility Expansion

- In the time-homogeneous LSV setting, we write down up to the 1st order terms:

$$\sigma_0 = |\beta|\sqrt{2a_{0,0}}, \qquad \sigma_1 = \sigma_{1,0} + \sigma_{0,1},$$

where

$$\begin{aligned}
\sigma_{1,0} &= \left(\frac{\beta a_{1,0}}{2\sigma_0}\right)\lambda + \tau\left(\frac{1}{4}(-1+\beta)\sigma_0 a_{1,0}\right), \\
\sigma_{0,1} &= \left(\frac{\beta^3 a_{0,1} f_{0,0}}{2\sigma_0^3}\right)\lambda + \tau\left(\frac{\beta^2 a_{0,1}\left(2c_{0,0} + \beta f_{0,0}\right)}{4\sigma_0}\right), \\
\lambda &= k - z, \qquad \tau = T - t.
\end{aligned}$$

- These expressions show explicitly the non-trivial dependence of IV on the leverage ratio β.

Implied Volatility Scaling

- Let $\sigma_Z(\tau,\lambda)$ (resp. $\sigma_X(\tau,\lambda)$) be the implied volatility of a call written on the LETF Z (resp. X).
- From the above IV expressions, we have

$$\text{LETF}: \quad \sigma_Z \approx |\beta|\sqrt{2a_{0,0}} + |\beta|\left(\frac{a_{1,0}}{2\sqrt{2a_{0,0}}} + \frac{a_{0,1}f_{0,0}}{2(2a_{0,0})^{3/2}}\right)\frac{\lambda}{\beta} + \mathcal{O}(\tau),$$

$$\text{ETF}: \quad \sigma_X \approx \sqrt{2a_{0,0}} + \left(\frac{a_{1,0}}{2\sqrt{2a_{0,0}}} + \frac{a_{0,1}f_{0,0}}{2(2a_{0,0})^{3/2}}\right)\lambda + \mathcal{O}(\tau).$$

- Implied volatility scaling: the vertical axis of σ_Z is scaled by a factor of $|\beta|$. Second, the horizontal axis is scaled by $1/\beta$.

Empirical IV Scaling

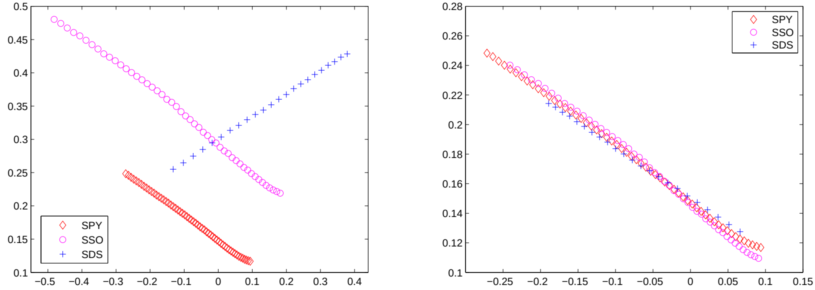

Figure: Left: Empirical IV $\sigma_Z(\tau, \lambda)$ plotted as a function of log-moneyness λ for SPY (red, $\beta = +1$), SSO (purple, $\beta = +2$), and SDS (blue, $\beta = -2$) on August 15, 2013 with $\tau = 155$ days to maturity. Note that the implied volatility of SDS is increasing in the LETF log-moneyness. Right: Using the same data, the scaled LETF implied volatilities $\sigma_Z^{(\beta)}(\tau, \lambda)$ nearly coincide.

Example: Heston Model

- The Heston model (in log prices) is described by

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\mathrm{e}^{Y_t}\mathrm{d}t + \mathrm{e}^{\frac{1}{2}Y_t}\mathrm{d}W_t^x, && X_0 = x := \log S_0,\\
\mathrm{d}Y_t &= \left((\kappa\theta - \tfrac{1}{2}\delta^2)\mathrm{e}^{-Y_t} - \kappa\right)\mathrm{d}t + \delta\,\mathrm{e}^{-\frac{1}{2}Y_t}\mathrm{d}W_t^y, && Y_0 = y := \log V_0,\\
\mathrm{d}Z_t &= -\beta^2\frac{1}{2}\mathrm{e}^{Y_t}\mathrm{d}t + \beta\mathrm{e}^{\frac{1}{2}Y_t}\mathrm{d}W_t^x, && Z_0 = z := \log L_0,\\
\mathrm{d}\langle W^x, W^y\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned}
$$

- The generator of (X, Y, Z) is given by

$$
\begin{aligned}
\mathcal{A} &= \frac{1}{2}\mathrm{e}^y\left((\partial_x^2 - \partial_x) + \beta^2(\partial_z^2 - \partial_z) + 2\beta\partial_x\partial_z\right)\\
&\quad + \left((\kappa\theta - \tfrac{1}{2}\delta^2)\mathrm{e}^{-y} - \kappa\right)\partial_y + \frac{1}{2}\delta^2\mathrm{e}^{-y}\partial_y^2 + \rho\,\delta\left(\partial_x\partial_y + \beta\partial_x\partial_z\right),\\
a(x,y) &= \frac{1}{2}\mathrm{e}^y, \qquad b(x,y) = \frac{1}{2}\delta^2\mathrm{e}^{-y},\\
c(x,y) &= \left((\kappa\theta - \tfrac{1}{2}\delta^2)\mathrm{e}^{-y} - \kappa\right), \qquad f(x,y) = \rho\,\delta.
\end{aligned}
$$

Example: Heston Model

- The first two terms in the IV expansion are

$$\sigma_0 = |\beta|\sqrt{\mathrm{e}^y},$$
$$\sigma_1 = \frac{\tau\left(-\beta^2\left(\delta^2 - 2\theta\kappa\right) + (-2\kappa + \beta\delta\rho)\sigma_0^2\right)}{8\sigma_0} + \frac{\beta\delta\rho}{4\sigma_0}(k - z).$$

- Note that when X has Heston dynamics with parameters $(\kappa,\ \theta,\ \delta,\ \rho,\ y)$, then Z also admits Heston dynamics with parameters

$$(\kappa_Z, \theta_Z, \delta_Z, \rho_Z, y_Z) = (\kappa, \beta^2\theta, |\beta|\delta, \mathsf{sign}(\beta)\rho, y + \log\beta^2).$$

- This can also be inferred from our IV expressions. Indeed, the coefficients' dependence on β is present *only* in the terms $\beta^2\theta, |\beta|\delta, \mathsf{sign}(\beta)\rho, y + \log\beta^2$. For instance, we can write $\sigma_0 = \sqrt{e^{y+\log\beta^2}} = \sqrt{e^{y_Z}}$,
and the coeff. of $(k - z)$ in σ_1 is $\beta\delta\rho/4\sigma_0 = |\beta|\delta\mathsf{sign}(\beta)\rho/4\sigma_0 = \delta_Z\rho_Z/4\sigma_0$.

IV comparison

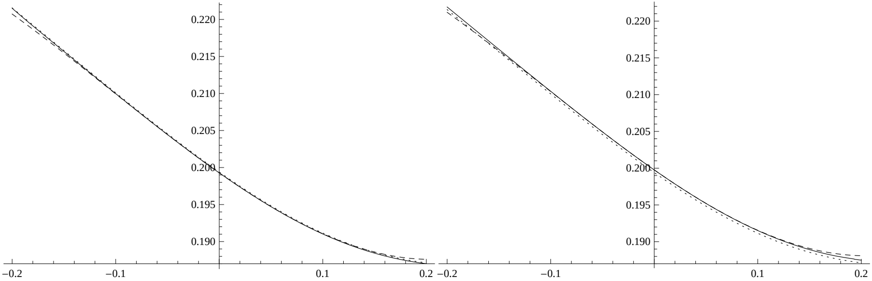

Figure: Exact (solid – computed by Fourier inversion) and approximate (dashed) scaled IV $\sigma_Z^{(\beta)}(\tau, \lambda)$ under Heston, plotted against λ. For comparison, we also plot the ETF's exact Heston IV $\sigma_X(\tau, \lambda)$ (dotted). Parameters: $\kappa = 1.15$, $\theta = 0.04$, $\delta = 0.2$, $\rho = -0.4$, $y = \log \theta$, $\tau = 0.0625$. Left: $\beta = +2$. Right $\beta = -2$.

Example: CEV Model

- The CEV model (in log prices) is described by

$$dX_t = -\frac{1}{2}\delta^2 e^{2(\gamma-1)X_t}dt + \delta\, e^{(\gamma-1)X_t}dW_t^x, \qquad X_0 = x := \log S_0.$$
$$dZ_t = -\frac{1}{2}\beta^2\delta^2 e^{2(\gamma-1)X_t}dt + \beta\delta\, e^{(\gamma-1)X_t}dW_t^x, \qquad Z_0 = z := \log L_0,$$

  with $\gamma \le 1$.
- The generator of (X, Z) is given by

$$\mathcal{A} = \frac{1}{2}\delta^2 e^{2(\gamma-1)x}\left((\partial_x^2 - \partial_x) + \beta^2(\partial_z^2 - \partial_z) + 2\beta\partial_x\partial_z\right).$$
$$a(x,y) = \frac{1}{2}\delta^2 e^{2(\gamma-1)x}, \;\; b(x,y) = 0, \;\; c(x,y) = 0, \;\; f(x,y) = 0.$$

Example: CEV Model

- The first 3 terms in the IV expansion are

$$
\begin{aligned}
\sigma_0 &= |\beta|\sqrt{e^{2x(\gamma-1)}\delta^2}, \\
\sigma_1 &= \frac{\tau(\beta-1)(\gamma-1)\sigma_0^3}{4\beta^2} + \frac{(\gamma-1)\sigma_0}{2\beta}(k-z), \\
\sigma_2 &= \frac{\tau(\gamma-1)^2\sigma_0^3\left(4\beta^2 + t(13+2\beta(-13+6\beta))\sigma_0^2\right)}{96\beta^4} \\
&\quad + \frac{7\tau(\beta-1)(\gamma-1)^2\sigma_0^3}{24\beta^3}(k-z) + \frac{(\gamma-1)^2\sigma_0}{12\beta^2}(k-z)^2,
\end{aligned}
$$

- The factor $(\gamma - 1)$ appears in every term of these expressions.
- If $\gamma = 1$, $\sigma_0 = |\beta|\delta$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$. The higher order terms also vanish since $a(x, y) = \frac{1}{2}\delta^2$ in this case. Hence, just as in the B-S case, the IV expansion becomes flat.

IV Comparison

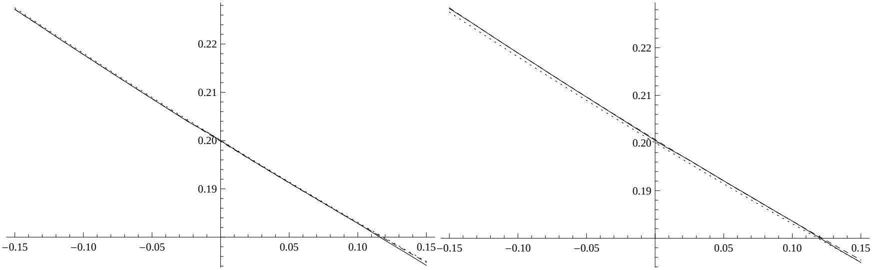

Figure: Exact (solid – computed by Fourier inversion) and approximate (dashed) scaled IV $\sigma_Z^{(\beta)}(\tau, \lambda)$ under CEV, plotted against λ. For comparison, we also plot the ETF's exact CEV IV $\sigma_X(\tau, \lambda)$ (dotted). Parameters: $\delta = 0.2$, $\gamma = -0.75$, $x = 0$. Left: $\beta = +2$. Right: $\beta = -2$.

Example: SABR Model

- The SABR model (in log prices) is described by

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\mathrm{e}^{2Y_t+2(\gamma-1)X_t}\mathrm{d}t + \mathrm{e}^{Y_t+(\gamma-1)X_t}\mathrm{d}W_t^x, \\
\mathrm{d}Y_t &= -\frac{1}{2}\delta^2\mathrm{d}t + \delta\,\mathrm{d}W_t^y, \\
\mathrm{d}Z_t &= -\frac{1}{2}\beta^2\mathrm{e}^{2Y_t+2(\gamma-1)X_t}\mathrm{d}t + \beta\mathrm{e}^{Y_t+(\gamma-1)X_t}\mathrm{d}W_t^x, \\
\mathrm{d}\langle W^x, W^y\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned}
$$

- The generator of (X, Y, Z) is given by

$$
\begin{aligned}
\mathcal{A} = \frac{1}{2}\mathrm{e}^{2y+2(\gamma-1)x}\left((\partial_x^2 - \partial_x) + \beta^2(\partial_z^2 - \partial_z) + 2\beta\partial_x\partial_y\right) \\
- \frac{1}{2}\delta^2\partial_y + \frac{1}{2}\delta^2\partial_y^2 + \rho\,\delta\,\mathrm{e}^{y+(\gamma-1)x}(\partial_x\partial_y + \beta\partial_y\partial_z).
\end{aligned}
$$

$$
a(x,y) = \frac{1}{2}\mathrm{e}^{2y+2(\gamma-1)x}, \;\; b = \frac{1}{2}\delta^2, \;\; c = -\frac{1}{2}\delta^2, \;\; f(x,y) = \rho\,\delta\,\mathrm{e}^{y+(\gamma-1)x}.
$$

Example: SABR Model

- The first 3 terms in the IV expansion are

$$\sigma_0 = |\beta|\sqrt{e^{2y+2(\gamma-1)x}},$$
$$\sigma_1 = \sigma_{1,0} + \sigma_{0,1},$$
$$\sigma_2 = \sigma_{2,0} + \sigma_{1,1} + \sigma_{0,2}.$$

where

$$\sigma_{1,0} = \frac{\tau(-1+\beta)(\gamma-1)\sigma_0^3}{4\beta^2} + \frac{(\gamma-1)\sigma_0}{2\beta}(k-z),$$
$$\sigma_{0,1} = -\frac{1}{4}\tau\delta\sigma_0\left(\delta - \rho\,\mathsf{sign}[\beta]\sigma_0\right) + \frac{1}{2}\delta\rho\,\mathsf{sign}[\beta](k-z).$$

Example: SABR Model

$$\sigma_{2,0} = \frac{\tau(\gamma-1)^2\sigma_0^3\left(4\beta^2+\tau(13+2\beta(-13+6\beta))\sigma_0^2\right)}{96\beta^4} + \frac{7\tau(-1+\beta)(\gamma-1)^2\sigma_0^3}{24\beta^3}(k-z) + \frac{(\gamma-1)^2\sigma_0}{12\beta^2}(k-z)^2,$$

$$\sigma_{1,1} = \frac{\tau(\gamma-1)\delta\sigma_0^2\left(12\rho|\beta| + \tau\sigma_0\left(-9(-1+\beta)\delta + (-11+10\beta)\rho\,\mathsf{sign}[\beta]\sigma_0\right)\right)}{48\beta^2} - \frac{\tau(\gamma-1)\delta\sigma_0\left(3\delta + \frac{5(1-2\beta)\rho\sigma_0}{|\beta|}\right)}{24\beta}(k-z),$$

$$\sigma_{0,2} = \frac{1}{96}\tau\delta^2\sigma_0\left(32+5\tau\delta^2-12\rho^2+2\tau\sigma_0\left(-7\delta\rho\,\mathsf{sign}[\beta]+\left(-2+6\rho^2\right)\sigma_0\right)\right) - \frac{1}{24}\tau\delta^2\rho\left(\delta\,\mathsf{sign}[\beta]-3\rho\sigma_0\right) + \frac{\delta^2\left(2-3\rho^2\right)}{12\sigma_0}(k-z)^2,$$

IV Comparison

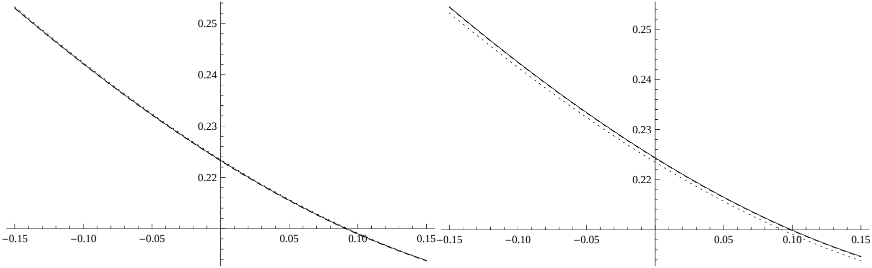

Figure: Exact (solid – computed by Fourier inversion) and approximate (dashed) scaled IV $\sigma_Z^{(\beta)}(\tau, \lambda)$ under SABR, plotted against λ. For comparison, we also plot the ETF's exact SABR IV $\sigma_X(\tau, \lambda)$ (dotted). Parameters: $\delta = 0.5$, $\gamma = -0.5$, $\rho = 0.0$ $x = 0$, $y = -1.5$. Left: $\beta = +2$. Right: $\beta = -2$.

Alternative IV Scaling

- In a general stochastic volatility model, the log LETF price is

$$\log\left(\frac{L_T}{L_0}\right) = \beta \log\left(\frac{X_T}{X_0}\right) - (r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\int_0^T \sigma_t^2 dt.$$

- **Key idea:** Condition on that the *terminal* LM $\log\left(\frac{X_T}{X_0}\right)$ equal to constant $LM^{(1)}$.
- Then, the best estimate of the β-LETF's LM is given by the cond'l expectation:

$$LM^{(\beta)} :=$$

$$\beta LM^{(1)} - (r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\mathbb{E}^{\mathbb{Q}}\left\{\int_0^T \sigma_t^2 dt \,\Big|\, \log\left(\frac{X_T}{X_0}\right) = LM^{(1)}\right\}.$$

Connecting Log-moneyness

- Assuming constant σ as in the B-S model, we have the formula:

$$LM^{(\beta)} = \beta\, LM^{(1)} - (r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\sigma^2 T. \qquad (1)$$

- Hence, the β-LETF log-moneyness $LM^{(\beta)}$ is expressed as an affine function of the unleveraged ETF log-moneyness $LM^{(1)}$, reflecting the role of β.
- The moneyness scaling formula can be interpreted via **Dual Delta matching**.

Proposition

Under the B-S model, an ETF call with log-moneyness $LM^{(1)}$ and a β-LETF call with log-moneyness $LM^{(\beta)}$ in (1) have the same Dual Delta.

Recall: Dual Delta of an LETF call is $e^{-r(T-t)}N(d_2^{(\beta)})$, and $N(d_2^{(\beta)})$ represents the risk-neutral probability of the option ending up ITM.

Concluding Remarks

- We have discussed a local-stochastic volatility framework to understand the inter-connectedness of LETF options.
- Explicit price and IV expansions are provided for Heston, CEV, and SABR models.
- The method of moneyness scaling enhances the comparison of IVs with different leverage ratios.
- The connection allows us to use the richer unleveraged index/ETF option data to shed light on the less liquid LETF options market.
- Our procedure can be applied to identify IV discrepancies across LETF options markets.

Appendix

Predicting from SPY IVs to LETF IVs