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# Markov distributional equilibrium dynamics in games with complementarities and no aggregate risk*

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**Abstract**

We present a new approach for studying equilibrium dynamics in a class of stochastic games with a continuum of players with private types and strategic complementarities. We introduce a suitable equilibrium concept, called Markov Stationary Distributional Equilibrium (MSDE), prove its existence, and provide constructive methods for characterizing and comparing equilibrium distributional transitional dynamics. To analyze equilibrium transitions for the distributions of private types, we develop an appropriate dynamic (exact) law of large numbers. Finally, we show that our models can be approximated as idealized limits of games with a large (but finite) number of players. We provide numerous applications of the results including: dynamic models of growth with status concerns, social distance, and paternalistic bequest with endogenous preference for consumption.

**Keywords:** large games, distributional equilibria, supermodular games, comparative dynamics, non-aggregative games, law of large numbers, social interactions
**JEL classification:** C62, C72, C73

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# 1 Introduction

This paper presents a new constructive method for studying the equilibrium dynamics in a class of games with complementarities and a continuum of players, where each player's type is private and evolves stochastically over time. The types may be interpreted as agents's endowment, their social rank, payoff relevant private information, parameterization of behavioral traits, etc., depending on the economic problem at hand. We study the evolution of the equilibrium joint distributions of types and actions in the population. Importantly, our approach allows for a robust characterization of both equilibrium distributional transitional dynamics as well as equilibrium comparative statics. Finally, we show how such large games can be used to approximate equilibria in dynamic games with a large (but finite) number of players.

Large dynamic games with private information find numerous applications to diverse fields in economics, including: growth with heterogeneous agents and endogenous social structure (as in Cole et al., 1992), inequality with endogenous preferences formation (as in Genicot and Ray, 2017), industry dynamics with heterogeneous firms (as in Weintraub et al., 2008), dynamic network formation (as in Mele, 2017; Xu, 2018), economics of identity and social dissonance (as in Akerlof and Kranton, 2000; Bisin et al., 2011), models of endogenous formation of social norms (as in Acemoglu and Jackson, 2017), macroeconomic models with public or private sunspots (as in Angeletos and Lian, 2016), or Bewley-Huggett-Aiyagari models of wealth distribution in the presence of incomplete markets (see Cao, 2020).[1] The principal questions regarding each of these models include how to compute, calibrate, and estimate dynamic equilibria. This concern is related to both stochastic steady-states and equilibrium distributional dynamics.

The theoretical literature concerning equilibrium dynamics in games is very limited, even in games with finitely many players.[2] One obvious reasons for this is that characteriz-

[1] Acemoglu and Jensen (2015, 2018) discuss the relation between large dynamic economies and large anonymous games. Notably, the former may be view as the latter (e.g., Bewley-Huggett-Aiyagari models).

[2] From a theoretical perspective, little is known about the nature of convergence of equilibrium transitional dynamics to stochastic steady-states. This question is complicated by the presence of equilibrium multiplicities and stability issues related to equilibrium transitional paths. The lack of theoretical foundation makes counterfactuals from these models difficult to implement and interpret.

ing sequential and Markovian equilibria dynamics in such models can become analytically intractable very quickly as the number of players grows and the state space becomes large and complex. Additionally, due to heterogeneity of private types, characterizing the updating of players' beliefs both on and off equilibrium paths is non-trivial. Even providing sufficient conditions for existence of sequential equilibria is very challenging, let alone describing (and computing) how types and actions evolve over time.

Due to such complications, the literature has focused on alternative notions of equilibria that simplify dynamic interactions. In particular, there have been two dominant methodological approaches to this question. The first approach exploits inherit *aggregative structure* in the game, where the players' interactions are limited to some statistic or aggregate that summarizes the population distribution, as well as imposes some notion of equilibrium (stochastic) steady state. A second approach (often used in conjunction with aggregation and stationarity) is to simplify interaction in the equilibrium by imposing some behavioral features in its definition. Such approaches include notions of *oblivious equilibria* (as in Lasry and Lions, 2007, Achdou et al., 2014, Bertucci et al., 2018, Light and Weintraub, 2019, Achdou et al., 2020), *mean-field equilibria* (as in Weintraub et al., 2008, Adlakha et al., 2015, and Ifrach and Weintraub, 2016), or *imagined-continuum equilibria* (as in Kalai and Shmaya, 2018), among others.[3] In this paper, we argue that such simplifications need not play a crucial role if one wants to analyze the equilibrium dynamics in a class of games with strategic complementarities we consider.

**Our results** This paper tackles the above theoretical and numerical problems within a unified methodological framework of large stochastic anonymous games with strategic complementarities[4] and no aggregate risk.[5] To obtain our results, we exploit the nature of games with infinitely many agents, where individuals have negligible impact on actions of others, thus, sufficiently limiting their interactions. This approach enables us to define

[3] See also the solution concept proposed in Krusell and Smith (1998) for Bewley models with aggregate risk, where agents know only the moments of the random measure determining the distribution of idiosyncratic shocks and assets. See also Doncel et al. (2016); Kwok (2019); Lacker (2018); Nutz (2018).

[4] See Topkis (1978), Vives (1990), Veinott (1992) and Milgrom and Shannon (1994) for some early contributions and motivations for studying games of strategic complemenentarities.

[5] Following Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992); Karatzas et al. (1994).

and provide sufficient conditions for the existence of a *Markov stationary distributional equilibrium* (henceforth MSDE). Our solution concept consists of a probability measure over types and actions in the population of players, and a law of motion that specifies the evolution of such distributions. Critically, a MSDE is defined over a minimal set of state variables[6] and thus resembles the extensively used notion of recursive competitive equilibria in macroeconomics.[7] We discuss this further in the motivating example below.

Our equilibrium concept is inherently dynamic and enables us to characterize and compare equilibrium transition paths. Notably, the results hold without the need of restricting our analysis to any aggregative structure. In fact, in our economic applications it is essential to study the *entire distribution* of types and actions of players.[8]

The particular structure of games with strategic complementarities is indispensable for our results. First of all, it allows us to prove existence of extremal MSDE (with respect to an appropriate order). To do this, a key tool is to provide a tractable formulation of the evolution of (distributional) equilibrium beliefs. Using these tools, we develop a new order-theoretic approach to characterizing the order structure of (Markovian) distributional equilibrium. Moreover, by analyzing a measure space of agents, we avoid the well-known technical issues that can emerge in extensive-form supermodular games with a finite number of players and private information.[9] Our approach delivers a new collection of *computable* equilibrium comparative statics/dynamics results. Thus, we complement and extend the recent *stationary equilibrium* comparative statics results from a class of mean field games and oblivious equilibria to distributional games and dynamic equilibrium (see Acemoglu and Jensen, 2015; Light and Weintraub, 2019).

To analyze the transition of private types/signals between periods, as well as issues related to characterizing the dynamics of players' beliefs, we develop a new version of

[6] By the *minimal state space*, we mean a domain that includes only current individual type and probability measures summarizing current population distribution of types.

[7] It bears mentioning, that there are no general results on the existence of minimal state space Markovian (or recursive equilibrium) in large dynamic economies. Cao (2020) provides the *generalized* Markov equilibrium existence result for a class of Krusell-Smith economics (which include Bewley-Huggett-Aiyagari models as a special case) that is not minimal state space.

[8] Equilibrium distributions are also important in econometric evaluation of heterogeneous models with macroeconomic data. See, e.g., Parra-Alvarez et al. (2017) and Auclert et al. (2019).

[9] See Echenique (2004), Vives (2009), and Mensch (2020) for discussions of these complications.

the *dynamic exact law of large numbers* (henceforth D-ELLN). Our D-ELLN builds on the important work of Sun (2006), Podczeck (2010) and He et al. (2017), among others, and allows us to (i) simplify our analysis by allowing for independent draws of types for a continuum of players; (ii) simplify the dynamics of the aggregate law of motion of distributions over types-actions in the population; and (iii) simplify the problem of an individual agent, who forms their beliefs using the law of large numbers, rather than updating their beliefs on (the product of) other players' types. We consider these results to be of independent interest themselves, as they can be applied in other dynamic settings that have micro-level idiosyncratic risk but no aggregate risk.

Eventually, we address the question of approximation as well as some theoretical/behavioral justification of MSDE. Specifically, we define the precise notion in which our large game can be interpreted as an idealized limit of a related stochastic game with a finite number of players. This is particularly useful in applications, as in some settings, large dynamic economies are used as a tool to characterize properties of finite models.

We organize the rest of the paper as follows. In the remainder of this section we present a motivating example to discuss precisely the new issues that emerge relative to the class of games studied in the literature. Section 2 is devoted to the presentation of the main model and our analysis of equilibrium. Our monotone comparative dynamics results are then presented in Section 3, and our approximation result in Section 4. Multiple applications of our results are discussed in Section 5. In Section 6, we explore the broad literature related to our work and provide a deeper connection of this paper to the existing literature. Proofs and auxiliary results omitted in the paper, together with some preliminaries on the law of large numbers and lattice theory, can be found in the online appendix.[10]

**A motivating example** Consider a growth model in which individuals are concerned with their relative social status. The society consists of a continuum of players. Each time period $n \in \{1, 2, \dots, \infty\}$, a typical player is endowed with some (private) wealth/capital $t \in T = [0, 1]$ that constitutes their type. This wealth can be transformed into consumption $c \in [0, 1]$ or investment $a \in A = [0, 1]$ using a simple one-to-one technology,

[10]The online appendix is enclosed at the end of this document.

thus, introducing the constraint $t = c + a$. By investing $a \in [0, t]$, the agent influences their wealth $t'$ in the following period via a stochastic technology $q$. Whenever $a$ units of wealth is being invested, the cumulative probability of attaining the capital $t'$ is $q(t'|a)$. We assume that higher investments make higher wealth more likely, i.e., distribution $q(\cdot|a)$ increases stochastically in $a$. Moreover, the realization of the future capital $t'$ is independent across players.

Status of each agent is determined by both their current consumption $c$ and wealth $t$. In each period, every individual interacts randomly with one other member of the society. If an agent with capital $t$ consuming $c$ encounters an individual of wealth $\tilde{t}$ consuming $\tilde{c}$, the former receives utility $U(c, \tilde{c}, t, \tilde{t}) = m(t-\tilde{t})+w(c-\tilde{c})$, where $m$ and $w$ are continuous, strictly increasing, and concave functions. Thus, meeting individuals with lower wealth and consumption is preferable due to, e.g., the feeling of superiority.

We assume that (given their current wealth $t$), the individual has to determine their consumption $c$ and investment $a$ at the beginning of each period, i.e., before the interaction with other members takes place. In order to do so, they need to evaluate their belief about the distribution $\mu$ over capital-investments pairs $(\tilde{t}, \tilde{a})$ across the society, where $\tilde{a} = \tilde{t} - \tilde{c}$ determines their expected payoff in that particular period given by:

$$r(t, a, \mu) \;=\; \int_{A\times T} \Big[m(t-\tilde{t}) + w(t - a - \tilde{t} + \tilde{a})\Big]\mu(d\tilde{a} \times d\tilde{t}).$$

In order to specify sequential payoffs of the agent, suppose all other players play a symmetric stationary strategy $\sigma : T \to A$ that maps the current capital/wealth $t$ to the level of investment $a = \sigma(t)$. Given the distribution $\tau_n$ of types at time $n$, the joint distribution of types *and* actions is denoted by $\mu_n(S) = \tau_n\big(\{t \in T : (t, \sigma(t)) \in S\}\big)$.

Given the sequences of wealth and wealth-investment distributions $\{\tau_n\}$, $\{\mu_n\}$, the sequential payoff of a player endowed with an initial capital $t_0$ is

$$\max_{\{a_n\}} \left\{ (1-\beta) E_{t_1,\{\tau_n\}} \left[ \sum_{n=1}^{\infty} \beta^{n-1} \int_{A\times T} \Big[m(t_n - \tilde{t}) + w(t_n - a_n - \tilde{t} + \tilde{a})\Big]\mu_n(d\tilde{a} \times d\tilde{t}) \right] \right\},$$

where $\beta \in (0, 1)$ is a discount factor and the expectation $E_{t_1,\{\tau_n\}}$ is taken with respect to realization of the sequences of private types $\{t_n\}$ of that individual, induced by $q$.

We are interested in studying the socioeconomic dynamic *distributional* equilibrium in this game. More generally, we want to investigate how the distributions of types and actions in the population evolve and interact when (a) distributions of types and actions are determined by strategies of individuals and the stochastic transition $q$ defining the evolution of private types, and (b) individuals form beliefs over future distributions of types and actions that are consistent with the law of motion governing the distribution of private types (e.g., capital levels), given the joint strategy of all players.

As we show in the sequel, our approach to studying equilibrium in this environment benefits from the following observation: although the players' problem is sequential with each private capital type drawn randomly each period, it can be reformulated as a standard *Markov decision problem* (henceforth MDP) once the sequence of distributions of types $\{\tau_n\}$ and types-actions $\{\mu_n\}$ in the population are taken as fixed. This can be done precisely because infinitesimal individuals do not affect those distributions directly. In such a case, the measure $\mu_n$ (or $\tau_n$) serves as an additional state variable at time $n$.

A recursive formulation of the player's problem can be obtained by allowing the players to share a *macro belief* $\Phi$, i.e., a transition function of capital-investment distributions between periods, where $\mu_{n+1} = \Phi(\mu_n)$. Together with an initial distribution $\mu_1$, this allows players to conjecture a candidate equilibrium path of the game, enabling us to reformulate their sequential problem as a recursive one with the value function $v^*$ satisfying

$$v^*(t, \mu; \Phi) \;=\; \max_{a \in [0,t]} \left\{ (1-\beta) r(t, a, \mu) + \beta \int_T v^*\big(t', \Phi(\mu); \Phi\big) q(dt'|a) \right\}.$$

Our notion of MSDE consists of a measure $\mu^*$ over types-actions in the population and a macro belief transition $\Phi^*$ such that, conditional on the above, (almost) every player solves their MDP and the resulting distribution of types-actions coincides with $\mu^*$. Hence, the perceived dynamics of distributions (which are deterministic under the D-ELLN) is required to be consistent with the actual transition $q$ and it's initial distribution. Moreover, under a D-ELLN, one can associate probabilities $q$ with empirical population distributions on $T$. Since any equilibrium pair $(\mu^*, \Phi^*)$ generates a sequence of "equilibrium" measures $\{\mu_n^*\}$, where $\mu_1^* = \mu^*$ and $\mu_{n+1}^* = \Phi^*(\mu_n^*)$, our concept is inherently dynamic, and allows naturally to evaluate and compare distributional equilibrium *transition* paths.

Importantly, with our equilibrium concept we can study interactions between individual players and the *entire distribution* of types-actions in the population. This is resembled in the motivating example, which is inherently non-aggregative. Indeed, evaluating payoff at time $n$ requires the entire distribution of capital (types) and investments (actions) in the population at time $n$, and beliefs about the future distributions. It is not sufficient to substitute the measure $\mu_n$ with its summary statistics or an aggregate. This is a common feature of large dynamic economies with social interactions.

The motivating example is also a game with *dynamic* strategic complementarities. That is, in this game, it is optimal for every individual to increase their own wealth and consumption as the distribution of wealth and consumption in the population "increases" stochastically. More importantly, such complementarities are present *within* and *across* periods. Concentrating on the latter, we study situations where higher anticipated distributions of types tomorrow create dynamic complementarities for each player to increase their own type in the next period.[11] Whether a game exhibits such complementarities depends critically on two reinforcing conditions: (i) increasing differences between private type (status) and anticipated population distribution the next period; and (ii) agents forming monotone beliefs, i.e., expecting higher population distribution tomorrow when faced with higher distribution today.[12] Our example possess both features. This is in stark contrast to analyzing complementarities on stochastic *steady-state* equilibria only.

Finally, it is important to determine the comparative structure of equilibrium transitional paths in such models (in addition to comparisons of stationary equilibria). In particular, how changes in parameters of the game (e.g., discount factor, preference or technology parameters, the initial distribution $\tau_1$) affect the paths of equilibrium distributions $\{\mu_n^*\}$ (as implied by equilibrium $\mu^*$ and $\Phi^*$). Importantly, since $\mu_n^*$ is defined over the space of types *and* actions, one needs to provide an equilibrium comparative statics result for appropriate spaces of *multidimensional* distributions.

[11] More formally, there is a dynamic single-crossing condition satisfied between the current investment and the future anticipated type-action distribution in the population.

[12] Our work is hence related to recent work on characterizing single crossing in distribution (e.g., Quah and Strulovici, 2012 and Kartik et al., 2019).

# 2 Large stochastic games with complementarities

Consider a stochastic game in discrete time of an infinite horizon. Let $(\Lambda, \mathcal{L}, \lambda)$ be a probability space of players, which we assume to be super-atomless. The latter is critical since it allows us to apply the dynamic law of large numbers. One example of such a space is the product measure on $[0,1]^I$, where each factor is endowed with Lebesgue measure and $I$ is uncountable. See Section A.1 of the online appendix for details.

In each period $n \in \{1, 2, \dots, \infty\}$, a player is endowed with a private type $t \in T \subseteq \mathbb{R}^p$, where $T$ is compact and $\mathcal{T}$ denotes its Borel $\sigma$-algebra. Given a distribution $\tau$ of types of all (other) players, the player chooses an action $a$ in $\tilde{A}(t, \tau) \subseteq A$, where $A \subseteq \mathbb{R}^k$ is a compact space of all conceivable action endowed with the Borel $\sigma$-algebra $\mathcal{A}$. Endow $T$ and $A$ with the natural product partial order $\geq$.

Let $\mathcal{M}$ be a set of probability measures on $\mathcal{T} \otimes \mathcal{A}$ and $\mathcal{M}_T$ be the set of probability measures on $\mathcal{T}$. Endow both spaces with its induced first order stochastic dominance order.[13] The player's payoff in a particular period is determined by a bounded function $r : T \times A \times \mathcal{M} \to \mathbb{R}$ taking values $r(t, a, \mu)$, for a private type $t$, an action $a$, and the probability measure $\mu$ over types and actions of all players.

In this paper, we investigate dynamic games in which private types of players are determined stochastically in each period. The transition probability is represented by a function $q : T \times A \times \mathcal{M} \to \mathcal{M}_T$ that assigns a probability measure $q(\cdot|t, a, \mu)$ over the individual payer's types in the following period, given their current type $t$, action $a$, and measure $\mu$ of types-actions in the population.

## 2.1 Decision problems for the players

In order to define properly the sequential decision problems for each player, it is fundamental to specify how the individual is forming beliefs about types of all players in the game, based on the current distribution of types and strategies of other players. We begin

[13] For any two probability measures $\mu$ and $\nu$ over $Y$, we say that $\mu$ dominates $\nu$ in the first order stochastic sense, if $\int f(y)\mu(dy) \geq \int f(y)\nu(dy)$, for any measurable, bounded function $f : Y \to \mathbb{R}$ that increases with respect to the corresponding ordering $\geq_Y$.

with some basic assumptions on the primitives of the game:

**Assumption 1.** *Assume the following.*

*(i) For all $\tau \in \mathcal{M}_T$, correspondence $t \to \tilde{A}(t, \tau)$ is measurable and compact-valued.*

*(ii) For all $\mu \in \mathcal{M}$, function $(a, t) \to q(\cdot|t, a, \mu)$ is Borel-measurable.*

The super-atomless probability space of players together with Assumption 1 guarantee that the (endogenous) transition of private signals satisfies the *no aggregate uncertainty* condition in each period and evolves deterministically. Specifically, for the current distribution $\mu$ of types and actions of all players, the measure of players with privates types in some measurable set $S$ in the following period is determined by

$$\phi(\mu)(S) \; := \int\limits_{T \times A} q(S|t, a, \mu)\mu(dt \times da). \tag{1}$$

We now state a critical theorem that is applied repeatedly in the paper. It posits that, under Assumption 1, the exact dynamic law of large numbers holds. See Section A.1 in the online appendix for a formal definition of a (rich) Fubini extension.

**Theorem 1.** *Under Assumption 1, there is a sampling probability space $(\Omega, \mathcal{F}, P)$ and a rich Fubini extension $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that, for any sequence $\sigma = (\sigma_n)_{n \in \mathbb{N}}$ of functions $\sigma_n : T \times \mathcal{M}_T \to A$, any initial state $t \in T$, and any initial distribution $\tau \in \mathcal{M}_T$,there is sequence of $(\mathcal{L} \boxtimes \mathcal{F})$-measurable functions $X_n : \Lambda \times \Omega \to T$ satisfying:*

*(i) For all $n \in \mathbb{N}$, the random variables $\big((X_n)_\alpha\big)_{\alpha \in \Lambda}$ are (conditional on the history) essentially pairwise independent.*[14]

*(ii) For all $n \in \mathbb{N}$ and $P$-almost every $\omega \in \Omega$, we have*

$$\tau_n \; := \; \lambda(X_n)_\omega^{-1} \; = \; (\lambda \boxtimes P) X_n^{-1};$$ [15]

*as well as $\mu_n(\sigma) := \lambda\big(X_n, \sigma_n(X_n, \tau_n)\big)_\omega^{-1} = (\lambda \boxtimes P)\big(X_n, \sigma_n(X_n, \tau_n)\big)^{-1}$.*

[14] Recall that $(X_n)_\alpha$ denotes the section of $X_n$, for a fixed $\alpha \in \Lambda$.

[15] To clarify our notation, recall that we denote $\lambda(X_n)_\omega^{-1} = \lambda\big(\{\alpha \in \Lambda : (X_n)_\omega(\alpha) \in U\}\big)$, for any $U \in \mathcal{T}$. We define the remaining measures analogously.

*(iii) The distribution of the random variable* $(X_{n+1})_\alpha$*, conditional on* $\big((X_j)_\alpha\big)_{j\le n}$*, is given by* $q\big(\cdot\,|(X_n)_\alpha, \sigma_n\big((X_n)_\alpha, \mu_n(\sigma)\big), \mu_n(\sigma)\big)$*.*

The proof of this theorem is in the online appendix. As pointed out in the introduction, this theorem could be of independent interest for any large dynamic game/economy with micro-level idiosyncratic risk that induces no aggregate risk.

We now define the decision problem of a player in a candidate Markov stationary distributional equilibrium. Let $H_\infty$ be a set of all histories $\big\{(t_n, a_n, \tau_n)\big\}_{n\in\mathbb{N}}$, where $a_n \in \tilde{A}(t_n, \tau_n)$. Let $H_n$ be the set of histories up to time $n$, that is $H_n := \big\{(t_j, a_j, \tau_j)_{j=1}^n : a_j \in \tilde{A}(t_j, \tau_j)\big\}$. A strategy is a sequence of functions $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n : H_{n-1} \times T \times \mathcal{M}_T \to A$ is Borel-measurable in $(t_1, t_2, \ldots, t_n) \in T^n$, and $\sigma_n(h_{n-1}, t_n, \tau_n) \in \tilde{A}(t_n, \tau_n)$, where we have $H_0 = \emptyset$ and initial values of $t_1, \tau_1$ are given.

A strategy profile is called *Markov* if in each period $n$, the strategy profile depends only on the partition of histories consisting of the current state $(t, \tau)$. A strategy profile is *stationary* if it is time-invariant. By Theorem 1, given any initial private state $t$, a public distributional state $\tau$, a Markov strategy profile $\sigma'$ of other players, a Markov strategy $\sigma$ induces the unique private measure $P_{t,\tau}^{\sigma,\sigma'}$ on histories of the game.[16] This implies the sequential objective function for each player is:

$$R\big(t, (\sigma, \sigma'), \tau\big) \;:=\; (1-\beta)\mathbb{E}_{t,\tau}^{\sigma,\sigma'}\left[r(t, \sigma_1(t,\tau), \mu_1^{\sigma'}) + \sum_{n=2}^{\infty} \beta^{n-1} r\big(t_n, \sigma_n(t_n, \tau_n), \mu_n^{\sigma'}\big)\right], \quad (2)$$

where $\beta \in (0,1)$ is a discount factor and $\mathbb{E}_{t,\tau}^{\sigma,\sigma'}$ is the expectation induced by $P_{t,\tau}^{\sigma,\sigma'}$ and $\mu_n^{\sigma'} := \tau_n(id_T, \sigma'(\cdot, \tau_n))^{-1}$. We impose the following additional assumptions.

**Assumption 2** (Payoffs)**.** *The function* $r$ *(i) is continuous in* $(t, a)$*; (ii) is monotone sup- and inf-preserving in* $\mu$*; (iii) is increasing in* $t$*; (iv) is supermodular in* $a$*; and (v) has increasing differences in* $\big(a, (t, \mu)\big)$*.*[17]

**Assumption 3** (Transition probability)**.** *The transition kernel* $q(\cdot|t, a, \mu)$ *(i) is continuous in* $(t, a)$*; (ii) is monotone sup- and inf-preserving in* $\mu$*; (iii) is stochastically increasing in*

[16] For $\lambda-$almost every $\alpha \in \Lambda$, $(X_n^\alpha)_{n\in\mathbb{N}}$ has the same distribution, which exists by Ionescu-Tulcea Theorem (Dynkin and Yushkevich, 1979; see also Theorem 15.26 in Aliprantis and Border, 2006).

[17] See Section A in this paper for a formal definition of monotone sup- and inf-preserving functions. Supermodularity and increasing differences are defined in Section A.2 of the online appendix.

*$(t,a,\mu)$; (iv) is stochastically supermodular in $a$;*[18] *and (v) has stochastically increasing differences in $\big(a,(t,\mu)\big)$ and $(t,\mu)$.*[19]

**Assumption 4** (Feasible actions)**.** *The feasible action correspondence $\tilde{A}: T\times\mu_T \to A$ (i) is upper hemi-continuous; (ii) its values are compact sublattices; (iii) increases with $t$ in the sense of set inclusion;*[20] *and (iv) satisfies strict complementarities.*[21]

Most of these assumptions are standard in dynamic games with complementarities (see Curtat, 1996 or Balbus et al., 2014) with the exception of some monotonicity requirements on the payoff and transition functions. As shown later in the paper, these are indispensable to preserve strategic complementarities across periods in the extensive formulation of the game under Markovian strategies. Importantly, our framework encompasses the linear social interaction models studied in the econometric literature by Blume et al. (2015); Kline and Tamer (2020); Kwok (2019).[22] Additionally, an interesting example of a transition function $q$ satisfying Assumption 3 is

$$q(\cdot|t,a,\mu) \ := \ g(t,a,\mu)\rho(\cdot)+\big(1-g(t,a,\mu)\big)\nu(\cdot),$$

where $g(t,a,\mu)$ is supermodular in $a$; has increasing differences in $\big(a,(t,\mu)\big)$ and $(t,\mu)$; and is increasing in $(a,t,\mu)$; while $\rho$, $\nu$ are probability distributions over $\mathcal{T}$ such that $\rho$ first order stochastically dominates $\nu$. This class of transitions was introduced in Curtat (1996) and Amir (2002), and has been successfully applied in the related literature.[23]

[18] The transition function $q: X\to\mathcal{M}_T$ is *stochastically supermodular* if the function $x\to\int f(t')q(dt'|x)$ is supermodular, for any $\mathcal{T}$-measurable, bounded, and increasing function $f:T\to\mathbb{R}$.

[19] The transition function $q: X\times Y\to\mathcal{M}_T$ has *stochastically increasing differences* in $(x,y)$ if the function $g(x,y):=\int f(t')q(dt'|x,y)$ has increasing differences in $(x,y)$ for any $\mathcal{T}$-measurable, bounded, and increasing function $f:T\to\mathbb{R}$.

[20] That is, if $t\geq t$, then $\tilde{A}(t,\tau)\subseteq\tilde{A}(t',\tau)$.

[21] See Section A.2 of the online appendix for a definition of a sublattice and strict complementarities.

[22] There, we have $r(t,a,\mu)=\big[\beta_1 t+\beta_2\int_T f_1(t,t')t'\,\mu_T(dt')\big]a-\frac{1}{2}a^2-\frac{\beta_3}{2}\big[a-\beta_4\int_{T\times A}f_2(t')a'\,\mu(dt'\times da')\big]^2$, for some positive $\beta_i$'s and linear, positive, increasing functions $f_1, f_2$ that weight social interaction by measuring contextual and peer network effects respectively. Our computable monotone comparative statics/dynamics results developed in the following section may be very useful in developing and characterizing estimators to test equilibrium distributions in empirical models. See, e.g., Echenique and Komunjer (2009, 2013), DePaula (2013), and Uetake and Watanabe (2013), among others.

[23] For example, see Balbus et al. (2013) for a discussion on the nature of these assumptions.

**Remark 1.** Our assumption on a stochastic transition generally implies that transition can *not* be deterministic. Indeed, supermodularity and increasing differences of the integrand $\int f(t')q(dt'|t,a,\mu)$ must hold for any integrable and monotone function $f$, which is generally not satisfied by deterministic transitions. However, if $A \subseteq \mathbb{R}$ (an important special case in the applied literature), then the deterministic transition given by $q(S|t,a,\mu) = 1$ if $g(a) \in S$, and $q(S|t,a,\mu) = 0$ otherwise, for some continuous and increasing function $g : A \to T$, satisfies our assumption.

**Remark 2.** Whenever the action space $A$ is one-dimensional and the transition function $q$ depends only on action $a$, our results remain true even if the payoff function $r$ and the correspondence $\tilde{A}$ are *not* increasing in $t$ (in the appropriate sense). This follows directly from our constructive argument in Section 2.3 and will be clear in a moment.

An important feature of our framework is that the original problem in (2) admits a recursive representation. Specifically, suppose that function $\Phi : \mathcal{M} \to \mathcal{M}$ determines the next period distribution $\Phi(\mu)$ over types and actions of all players based on the current distribution $\mu$. Given our observation in (1), the marginal of $\Phi(\mu)$ over $T$ must be $\phi(\mu)(S) = \int_{T\times A} q(S|t,a,\mu)\mu(dt \times da)$, for any measurable set $S$. Moreover, we restrict our attention to functions $\Phi$ that are monotone inf-preserving. Formally, let

$$\mathcal{D} := \Big\{\Phi : \mathcal{M} \to \mathcal{M} : \Phi \text{ is increasing and monotone} \\ \text{inf-preserving and } \operatorname{marg}_T(\Phi(\mu)) = \phi(\mu), \text{ for any } \mu \in \mathcal{M}\Big\}, \quad (3)$$

endowed with the componentwise order.

**Remark 3.** Dually, we can consider $\mathcal{D}' := \Big\{\Phi : \mathcal{M} \to \mathcal{M} : \Phi$ is increasing and monotone sup-preserving and $\operatorname{marg}_T\big(\Phi(\mu)\big) = \phi(\mu)$, for any $\mu \in \mathcal{M}\Big\}$. For expositional reasons, we focus on $\mathcal{D}$ but all our constructions and results have their counterpart in $\mathcal{D}'$.

Denote $\mu_T := \operatorname{marg}_T(\mu)$. In the remainder of this section, we show that for any initial distribution $\mu$ and any function $\Phi$, the value corresponding to the problem (2) satisfies

$$v^*(t,\mu;\Phi) = \max_{a\in\tilde{A}(t,\mu_T)} \left\{(1-\beta)r(t,a,\mu) + \beta\int_T v^*\big(t',\Phi(\mu);\Phi\big)q(dt'|t,a,\mu)\right\}.^{24} \quad (4)$$

[24] Equivalently, one may use $t,\tau$ as state variables and construct $\mu$ by composing $\tau$ and a strategy $\sigma : T \to A$. In such a case, the strategy $\sigma$ would have to be another parameter of the value function.

Given the initial distribution $\mu$ and a perceived law of motion $\Phi$, the player's problem is a MDP with uncertainty about the private signal $t$ only. Thus, under D-ELLN, the sequence of aggregate distributions $\{\mu_n\}_{n\in\mathbb{N}}$ is *deterministic*. Using standard arguments, we can show that the the best response correspondence of each player can be written as Markov on the natural state space of $t$ and $\mu$. However, our definition of equilibrium requires consistency between such policy correspondence and the perceived law of motion $\Phi$. Since $\Phi$ also specifies *beliefs* of players on continuation paths of the game, we write $v^*(t,\mu;\Phi)$ to stress that the value function and the corresponding policy depend on the beliefs.[25] We discuss the importance of this construction in the next section.

## 2.2 Markov stationary distributional equilibria

We are ready to specify the notion of equilibrium in the game.[26]

**Definition 1** (Markov Stationary Distributional Equilibrium)**.** A pair $(\mu^*,\Phi^*)\in\mathcal{M}\times\mathcal{D}$ is a Markov Stationary Distribution Equilibrium (MSDE) whenever:

(i) there is a function $v^*$ such that, for any $\mu\in\mathcal{M}$, and $\lambda$-almost every $t\in T$,

$$v^*(t,\mu;\Phi^*) \;=\; \max_{a\in\tilde{A}(t,\mu_T)}\left\{(1-\beta)r(t,a,\mu)+\beta\int_T v^*\big(t',\Phi^*(\mu);\Phi^*\big)q(dt'|t,a,\mu)\right\};$$

(ii) there is a measurable selection $\sigma_{\mu,\Phi^*}$ of correspondence $\Sigma_{\mu,\Phi^*}:T\rightrightarrows A$, where

$$\Sigma_{\mu,\Phi^*}(t) \;:=\; \arg\max_{a\in\tilde{A}(t,\mu_T)}\left\{(1-\beta)r(t,a,\mu)+\beta\int_T v^*\big(t',\Phi^*(\mu);\Phi^*\big)q(dt'|t,a,\mu)\right\},$$

and $\mu^*=\mu^*_T\big(id_T,\sigma_{\mu^*,\Phi^*}\big)^{-1}$ and $\Phi^*(\mu)=\phi(\mu)\big(id_T,\sigma_{\Phi^*(\mu),\Phi^*}\big)^{-1}$, for any $\mu\in\mathcal{M}$.[27]

An MDSE consists of an initial distribution $\mu^*$ and a Markov transition function $\Phi^*$. It also involves an equilibrium policy $\sigma_{\mu,\Phi^*}:T\to A$ (or equivalently $\sigma^*:T\times\mathcal{M}_T\to A$). Our equilibrium is stationary in the sense that strategies and beliefs of players are time-invariant. Nevertheless, we allow for a *dynamic* interaction of players in with future periods distributions (generated by the law of motion $\Phi^*$) and summarized by the value

[25] Compare with Markov equilibrium in Kalai and Shmaya (2018) for large but finite repeated games.
[26] Dually, we can define MSDE in $\mathcal{M}\times\mathcal{D}'$.
[27] That is, $\mu^*(S)=\mu^*_T\big(\{t\in T:(t,\sigma_{\mu^*,\Phi^*}(t))\in S\}\big)$, $\Phi^*(\mu)(S)=\phi(\mu)\big(\{t\in T:(t,\sigma_{\Phi(\mu^*),\Phi^*}(t))\in S\}\big)$.

$v^*\big(\cdot, \Phi^*(\mu^*); \Phi^*\big)$. Condition (i) is a standard Bellman equation that characterizes players best reply correspondences, while (ii) imposes a two-fold consistency. On one hand, we have $\mu^* = \mu_T^*(id_T, \sigma_{\mu^*,\Phi^*})^{-1}$, hence, the distribution of actions must be generated by the equilibrium strategy $\sigma_{\mu^*,\Phi^*}$, given the initial distribution of types and the equilibrium law of motion. In addition, we require that $\Phi^*(\mu) = \phi(\mu)(id_T, \sigma_{\Phi^*(\mu),\Phi^*})^{-1}$. Thus the perceived (macro belief) and the actual law of motion (i.e., generated by the best-response selection $\sigma$) for aggregate distributions coincide.[28] The Markov transition $\Phi^*$ specifies common beliefs which players each use to determine future paths of candidate equilibrium distributions. In macroeconomic literature on recursive equilibrium, such beliefs are often called *rational*. Since we require $\Phi^*(\mu) = \phi(\mu)(id_T, \sigma_{\Phi^*(\mu),\Phi^*})^{-1}$, for any $\mu \in \mathcal{M}$, these are beliefs on and off equilibrium paths.

**Theorem 2.** *Under Assumptions 1–4, there exists the greatest MSDE of the game in $\mathcal{M} \times \mathcal{D}$ and the least in $\mathcal{M} \times \mathcal{D}'$.*

The above theorem requires some comment. First, apart from providing sufficient conditions to guarantee the existence of an MSDE, Theorem 2 implies the greatest MSDE that determines the upper bound for all equilibria in the space $\mathcal{M} \times \mathcal{D}$. Similarly, there exists the least MSDE that is also a lower bound for all equilibria in $\mathcal{M} \times \mathcal{D}'$. Moreover, whenever the set of maximizers corresponding to the optimization problem on the right hand-side in (4) is unique, then the set of MSDE is *chain complete*, i.e., closed under monotone sequences of equilibria in $\mathcal{M} \times \{\mathcal{D} \cap \mathcal{D}'\}$.

**Remark 4.** Any MSDE induces a *sequential distributional equilibrium* as defined by Jovanovic and Rosenthal (1988), i.e., $\{\mu_n^*\}_{n\in\mathbb{N}}$, where $\mu_1^* = \mu^*$ and $\mu_n^* = \Phi^*(\mu_{n-1}^*)$.

Given this, a natural question arises as to whether there is an invariant distribution induced by MSDE. Hence, the following proposition. We omit the proof.

**Proposition 1** (Invariant distributions). *Under assumptions 1–4, there exists the greatest invariant distribution $\bar{\nu}$ induced by the greatest MSDE $(\bar{\mu}^*, \overline{\Phi}^*)$, i.e., $\bar{\nu} = \overline{\Phi}^*(\bar{\nu})$ and the least invariant distribution $\underline{\nu}$ induced by the least MSDE $(\underline{\mu}^*, \underline{\Phi}^*)$.*

[28] Since we work with no aggregate uncertainty, we do not require for $\Phi^*$ to be measurable.

The greatest and least invariant distributions can be obtained through simple iterations on the mappings $\overline{\Phi}^*$ and $\underline{\Phi}^*$, respectively. Also note that for any MSDE $(\mu^*, \Phi^*)$, and the pair $\big(\Phi^*(\mu^*),\ \Phi^*\big)$ is also an MSDE. Thus, the pair $(\nu, \Phi^*)$ is also an MSDE, for any invariant distribution $\nu$ generated by $\Phi^*$.[29]

Although we prove Theorem 2 in the following section, we make an important observation at this point. Importantly, our approach to MSDE is constructive. That is, we can introduce an *explicit* iterative algorithm that can be used to approximate the *greatest* equilibrium by successive approximation. To present our construction, we need to introduce some additional notation. For any $\mu \in \mathcal{M}$, $\Phi \in \mathcal{D}$, and function $v$, let

$$\Gamma(t, \mu, \Phi; v) \ := \ \arg \max_{a \in \tilde{A}(t,\mu_T)} \left\{ (1-\beta) r(t, a, \mu) + \beta \int_T v\big(t', \Phi(\mu), \Phi\big) q(dt'|t, a, \mu) \right\}, \quad (5)$$

which is the set of maximizers of the player's MDP. Define the greatest element of the set by $\overline{\gamma}(t, \mu, \Phi; v)$, whenever it exists. Let $\star$ be a binary operation between $\tau \in \mathcal{M}_T$ and the set of measurable functions $h : T \to A$ returning probability measure on $T \times A$:[30]

$$\tau \star h \ := \ \tau(id_T, h)^{-1}. \quad (6)$$

Define operator $\overline{\Psi}$ mapping $\mathcal{M} \times \mathcal{D}$ to itself, where $\overline{\Psi}(\mu, \Phi) = (\mu', \Phi')$ and

$$\mu' \ := \ \mu_T \star \overline{\gamma}(\cdot, \mu, \Phi; v^*) \text{ and } \ \Phi'(\nu) \ := \ \phi(\nu) \star \overline{\gamma}\big(\cdot, \Phi(\nu), \Phi; v^*\big), \quad (7)$$

for all $\nu \in \mathcal{M}$, where $v^* : T \times \mathcal{M} \times \mathcal{D} \to \mathbb{R}$ is a function solving (4).

**Proposition 2** (Bounds approximation)**.** *Let $\bar{\mu}$ and $\overline{\Phi}$ be the greatest elements of $\mathcal{M}$ and $\mathcal{D}$, respectively. Under Assumptions 1–4, $\lim_{n\to\infty} \overline{\Psi}^n(\bar{\mu}, \overline{\Phi})$ is the greatest MSDE.*

Again, a similar construction allows to approximate the least MSDE.

## 2.3 Construction of equilibria

We devote this subsection to the proof of Theorem 2. We discuss the main intuition of the argument and state the auxiliary results which may be of independent interest. Here,

[29] However, it must be that $\bar{\nu}$ is dominated by $\bar{\mu}^*$.

[30] That is, $(\tau \star h)(S) = \tau\big(\{t \in T : (t, h(t)) \in S\}\big)$, for any measurable set $S$.

we concentrate on the greatest MSDE and space $\mathcal{M} \times \mathcal{D}$ only. The argument for the least MSDE is analogous. Let Assumptions 1–4 be satisfied throughout. We begin by showing that the problem of each player in (2) admits a recursive representation. In particular, for any Markov transition function $\Phi \in \mathcal{D}$, there is a unique function $v$ satisfying (4).

Consider the space $\mathcal{V}$ of functions $v : T \times \mathcal{M} \times \mathcal{D} \mapsto \mathbb{R}$ such that: (i) functions $v$ are uniformly bounded by a value $\bar{r} > 0$, (ii) $v(\cdot, \mu, \Phi)$ is increasing and continuous, for any $(\mu, \Phi) \in \mathcal{M} \times \mathcal{D}$, (iii) $v(t, \cdot, \cdot)$ is monotone inf-preserving, for any $t \in T$, (iv) $v$ has increasing differences in $\big(t, (\mu, \Phi)\big)$. Endow $\mathcal{V}$ with natural sup-norm topology $|| \cdot ||_\infty$.

**Lemma 1.** *$\mathcal{V}$ is complete metric space.*

Given that $\mathcal{V}$ is a subset of all bounded functions, it is a subset of a Banach space. Hence, it suffices to show the set is closed. Noting the fact that continuity, monotonicity, and increasing differences are preserved in the sup-norm convergence, the main difficulty is to show that any limit of monotone inf-preserving functions preserves this property. The proof of this claim is shown in the online appendix.

The next lemma provides an important feature of the Markov transition functions $\Phi$. The proof is immediate from Lemma B.1 in the online appendix and we omit it.

**Lemma 2.** *Let $\{\mu_k\}_{k\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{M}$ that weakly converges to $\mu$ in $\mathcal{M}$. Let $\{\Phi_k\}_{k\in\mathbb{N}}$ be an decreasing sequence in $\mathcal{D}$ that pointwise weakly converges to some $\Phi$ in $\mathcal{D}$. Then $\big\{(\Phi_k(\mu_k)\big\}_{k\in\mathbb{N}}$ weakly converges to $\Phi(\mu)$.*

Define an operator $B : \mathcal{V} \to \mathcal{V}$ as

$$(Bv)(t, \mu, \Phi) \; := \; \max_{a\in\tilde{A}(t,\mu_T)} \left\{ (1-\beta)r(t, a, \mu) + \beta \int_T v\big(t', \Phi(\mu), \Phi\big) q(dt'|t, a, \mu) \right\}. \tag{8}$$

Some basic properties of the operator $B$ are provided in the following lemma.

**Lemma 3.** *For any $v \in \mathcal{V}$, function $(Bv)$ is continuous and increasing in $t$, jointly monotone inf-preserving in $(\mu, \Phi)$, and has increasing differences in $\big(t, (\mu, \Phi)\big)$.*

To keep our notation compact, denote the function within the brackets in (8) by

$$F(t, a, \mu; v, \Phi) \; := \; (1-\beta)r(t, a, \mu) + \beta \int_T v\big(t', \Phi(\mu), \Phi\big) q(dt'|t, a, \mu).$$

Given Assumptions 2–4, $F(t, a, \mu; v, \Phi)$ is increasing in $t$, jointly continuous in $(t, a)$ and has increasing differences in $\big(a, (t, \mu, \Phi)\big)$ and $\big(t, (\mu, \Phi)\big)$. We claim it is also monotone inf-preserving and monotone sup-preserving in $(\mu, \Phi)$. We will show the former property, where the latter property follows by a similar argument. So see monotone inf-preserving, suppose that $\big\{(\mu_n, \Phi_n)\big\}_{k\in\mathbb{N}}$ is a decreasing sequence that converges to $(\mu, \Phi)$. By Lemma 2, we have $\Phi_n(\mu_n) \to \Phi(\mu)$. By Assumption 2 and the choice of the set $\mathcal{V}$, it must be that both $r(t, a, \mu_k) \to r(t, a, \mu)$ and $v\big(t, \Phi_k(\mu_k), \mu_k\big) \to v\big(t, \Phi(\mu), \mu\big)$. Moreover, we have $\int_T v\big(t', \Phi_k(\mu_k), \mu_k\big) q(dt'|t, a, \mu_k) \to \int_T v\big(t', \Phi_k(\mu_k), \mu_k\big) q(dt'|t, a, \mu_k)$, which follows from Lemma B.2 in the online appendix. We are ready to prove Lemma 3.

*Proof of Lemma 3.* Continuity of $(Bv)$ follows from Berge's Maximum Theorem (see Theorem 17.31 in Aliprantis and Border, 2006). Monotonicity of $(Bv)$ in $t$ is implied by monotonicity of $F$ and the fact that $\tilde{A}$ increases in $t$ in the sense of set inclusion. To show that it is monotonically inf-preserving in $(\mu, \Phi)$, take any decreasing sequence $\big\{(\mu_k, \Phi_k)\big\}_{k\in\mathbb{N}}$ that converges to some $(\mu, \Phi)$. We know that $F(t, a_k, \mu_k; v, \Phi_k) \to F(t, a, \mu; v, \Phi)$ whenever $a_k \to a$. By Lemma B.3 in the online appendix, this suffices for $(Bv)(t, \mu_k, \Phi_k) \to (Bv)(t, \mu, \Phi)$. Finally, the fact that $(Bv)$ has increasing differences in $\big(t, (\mu, \Phi)\big)$ can be shown as in the proof of Lemma 1 in Hopenhayn and Prescott (1992). □

The conditions guaranteeing that the value function has increasing differences in both arguments (i.e., in $t, \mu$) along with the transition $\Phi^*$ being monotone allows us to avoid the problems in characterizing dynamic complementarities in actions between periods and beliefs that have been reported in the literature (e.g., Mensch, 2020). As a result, we dispense with some continuity assumptions that are typically critical for existence of equilibria in these games. This is due to no aggregate uncertainty and the fact that a player has no influence on aggregate distribution and macro beliefs.[31] The next result follows immediately and concerns the solution to equation (4).

**Proposition 3.** *Operator $B : \mathcal{V} \to \mathcal{V}$ has a unique fixed point in $\mathcal{V}$.*

[31] See also Kalai and Shmaya (2018).

Indeed, Lemma 3 guarantees that $B$ is well-defined operator that maps a complete metric space into itself. Since it is also a contraction, it has a unique fixed point $v^*$. Finally, showing that the value coincides with the value of the original problem (2) can be done using standard arguments. See, e.g., Theorem 9.2 in Stokey et al. (1989).

We now proceed with the second half of the argument in which we prove existence of the greatest MSDE. First, recall the definition of the correspondence $\Gamma$ from (5), with its greatest selection $\overline{\gamma} : T \times \mathcal{M} \times \mathcal{D} \to A$. Consider the following lemma.

**Lemma 4.** *For any $v \in \mathcal{V}$, the greatest selection $\overline{\gamma}(t, \mu, \Phi; v)$ is a well-defined function, measurable in $t$, increasing in $(t, \mu, \Phi)$, and monotone inf-preserving.*

*Proof.* Take any $v \in \mathcal{V}$. Clearly, we have $\Gamma(t, \mu; v, \Phi) = \arg\max_{a \in \tilde{A}(t,\mu_T)} F(t, a, \mu; v, \Phi)$. It is straightforward to verify that $F$ is supermodular and continuous in $a$. Since set $\tilde{A}(t, \mu_T)$ is a complete sublattice of $A$, by Corollary 4.1 in Topkis (1978), set $\Gamma(t, \mu; v, \Phi)$ is a complete sublattice of $A$. Therefore, it admits both the greatest and least element. The proof of measurability of $\overline{\gamma}$ is in the online appendix. Monotonicity follows from increasing differences of $F$ and Theorem 6.2 in Topkis (1978). To show that $\overline{\gamma}$ is monotone inf-preserving, let $\{(\mu_k, \Phi_k)\}_{k\in\mathbb{N}}$ be decreasing sequence converging to $(\mu, \Phi)$. By the previous argument, sequence $\{\overline{\gamma}(t, \mu_k, \Phi_k; v)\}_{k\in\mathbb{N}}$ is decreasing. Suppose it converges to some $\gamma$, and thus $\overline{\gamma}(t, \mu_k, \Phi_k; v) \geq \gamma$, for all $k \in \mathbb{N}$. Since $F$ continuous and monotone inf-preserving, Lemma B.3. in the online appendix guarantees that $\gamma \in \Gamma(t, \mu; v, \Phi)$. Thus, it must be $\gamma \leq \overline{\gamma}(t, \mu, \Phi; v)$, and so $\gamma \leq \overline{\gamma}(t, \mu, \Phi; v) \leq \overline{\gamma}(t, \mu_k, \Phi_k; v)$. □

Next, recall the definition of operator $\star$ from (6).

**Lemma 5.** *Take any measures $\tau, \tau' \in \mathcal{M}_T$ such that $\tau'$ first order stochastically dominates $\tau$, and increasing functions $h, h' : T \times A \to A$ such that $h'$ dominates $h$ pointwise. Then, the measure $(\tau' \star h')$ first order stochastically dominates the measure $(\tau \star h)$.*

The proof of the above lemma is straightforward, hence, we omit it.

**Lemma 6.** *Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{M}_T$ converging to some $\tau$, and let $\{h_k\}_{k\in\mathbb{N}}$ be a (pointwise) decreasing sequence converging to some $h$, where $h_k : T \times A \to A$ are increasing and monotone inf-preserving functions. Then $(\tau_k \star h_k) \to (\tau \star h)$ weakly.*

*Proof.* This follows from Lemma B.1 in the online appendix.[32] We only need to show that any of $\tau \star h$ is inf-preserving in $h$. Let $\tau \in \mathcal{M}_T$ be arbitrary and let $h_k$ be a decreasing sequence of Borel functions from $T$ to $A$. Let $h = \lim_{k\to\infty} h_k$. Then, for any measurable, continuous, and bounded function $f : T \times A \to \mathbb{R}$, we obtain

$$\begin{aligned} \lim_{k\to\infty} \int_{T\times A} f(t,a)(\tau \star h_k)(dt \times da) \ &= \ \lim_{k\to\infty} \int_T f\big(t, h_k(t)\big)\tau(dt) \\ &= \ \int_T f\big(t, h(t)\big)\tau(dt) \ = \ \int_{T\times A} f(t,a)(\tau \star h)(dt \times da). \end{aligned}$$

Hence $(\tau \star h_k) \to (\tau \star h)$ weakly. This completes the proof. □

To prove Theorem 2, take the unique function $v^*$ that solves the equation (4). Define operator $\overline{\Psi}$ as in (7). Given monotonicity of $\overline{\gamma}(t, \mu, \Phi; v)$ and Lemma 5, we conclude that it is increasing. Moreover, by Lemmas 4 and 6, it is also monotonically inf-preserving.

**Lemma 7.** *The set $\mathcal{D}$ is a lower chain complete poset.*

*Proof.* Let $\{\Phi_j\}_{j\in J}$ be a chain of elements in $\mathcal{D}$. Let $\Phi := \bigwedge_{j\in J} \Phi_j$. It suffices to show that $\Phi$ is monotone inf-preserving. Let $\{\mu_k\}_{k\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{M}$ that converges to $\mu$. For any $k$, $j$, and increasing, measurable function $f : T \times A \to \mathbb{R}$,

$$\int_{T\times A} f(t,a)(\Phi\mu)(dt \times da) \ \le \ \int_{T\times A} f(t,a)(\Phi\mu_k)(dt \times da) \ \le \ \int_{T\times A} f(t,a)(\Phi_j\mu_k)(dt \times da).$$

As $k \to \infty$, we obtain

$$\begin{aligned} \int_{T\times A} f(t,a)(\Phi\mu)(dt \times da) \ &\le \ \liminf_{k\to\infty} \int_{T\times A} f(t,a)(\Phi\mu_k)(dt \times da) \\ &\le \ \limsup_{k\to\infty} \int_{T\times A} f(t,a)(\Phi\mu_k)(dt \times da) \ = \ \int_{T\times A} f(t,a)(\Phi_j\mu)(dt \times da). \end{aligned}$$

We conclude by taking the infimum with respect to $j$ on the right hand-side. □

We proceed with the proof of Theorem 2.

*Proof of Theorem 2.* It suffices to show that there is the greatest fixed point of $\overline{\Psi}$ defined in (7). Note that $\overline{\Psi}$ is monotone in $(\mu, \Phi)$. Indeed, by Lemma 4, $\overline{\gamma}(t, \mu, \Phi; v^*)$ is jointly

[32] Here the role of $\Xi$ plays $\mathcal{M}_T$, and the role of $f_k$ plays $h \mapsto (\tau_k \star h)$.

increasing in $(t, \mu, \Phi)$. By Lemma 6, this implies monotonicity of $\mu'$ in (7). By the same argument $\Phi'$ is increasing in $\mu$ and $\Phi$. Moreover, by Lemmas 4 and 6, we conclude that $\overline{\Psi}$ is a monotone inf-preserving self-map on $\mathcal{M} \times \mathcal{D}$. By applying Proposition A.1 at the end of this paper, we conclude that there exists the greatest MSDE. □

# 3 Monotone equilibrium comparative dynamics

We next discuss the nature of monotone equilibrium comparative dynamics in the class of games studied in Section 2. To do this, we parameterize primitives of our game with $\theta$ in a poset $\Theta$, and seek conditions under which MSDE are ordered in the deep parameters of the game. Given our definition of equilibrium, this means that a selection $\theta \to \mu^*(\theta)$ and the equilibrium law of motion $\theta \to \Phi^*(\theta)$ are increasing. Hence, the use of the term monotone comparative *dynamics* rather than monotone comparative statics.

We first define a positive shock.[33]

**Assumption 5** (Positive shock). *Let $\Theta$ be a poset, and assume the following: (i) Payoff function $r(t, a, \mu; \theta)$ has increasing differences in $(a, \theta)$ and $(t, \theta)$. (ii) Transition kernel $q(\cdot|t, a, \mu; \theta)$ is increasing in $\theta$ and has increasing differences in $(a, \theta)$ and $(t, \theta)$. (iii) Feasible action correspondence $\tilde{A}(t, \mu; \theta)$ has strict complementarities in $(t, \theta)$.*

**Theorem 3** (Monotone Comparative Dynamics). *Suppose that the parameterized mappings $r(\cdot, \theta)$, $q(\cdot; \theta)$, and $\tilde{A}(\cdot; \theta)$ satisfy Assumptions 1–4, for all $\theta \in \Theta$. Under Assumption 5, the greatest equilibrium $\left(\bar{\mu}^*(\theta), \overline{\Phi}^*(\theta)\right)$ of the parameterized game increases in $\theta$. Similarly does the least equilibrium $\left(\underline{\mu}^*(\theta), \underline{\Phi}^*(\theta)\right)$.*

*Proof.* We prove the case for the greatest equilibrium only. Let $\overline{\Psi}^\theta$ be the counterpart of the operator $\overline{\Psi}$ in the parameterized game with $\theta \in \Theta$. Similarly we denote $\phi^\theta$ and $\overline{\gamma}^\theta$. Given that $q(\cdot|t, a, \mu; \theta)$ is increasing in $\theta$, it suffices to show that $\theta \to \overline{\gamma}^\theta$ is increasing. Observe that, under our assumptions, the objective $(1 - \beta)r(t, a, \mu, \theta) +$

[33] Our notion of a positive shock is consistent with the terminology of Acemoglu and Jensen (2015). The difference here is we consider the situation of comparative equilibrium *transitional* dynamics. In a sense, our question here is more related to related issues for Bewley models studied in Huggett (1997).

$\beta \int_T v^*(t', \phi(\mu), \theta) q(dt'|t, a, \mu, \theta)$ has increasing differences in $(a, \theta)$ and $v^*(t, \mu, \theta)$ has increasing differences in $(t, \theta)$, for any $\mu \in \mathcal{M}$. By Theorem 6.2 in Topkis (1978), we conclude that $\overline{\gamma}$ is increasing in $\theta$. See also Hopenhayn and Prescott (1992). By Assumption 5 and definition we conclude that $\theta \to \phi^\theta$ is increasing. The same property is inherited by $\overline{\Psi}^\theta$ from its definition and Lemma 5. Moreover, similarly as in the proof of Theorem 2, we conclude that $\overline{\Psi}^\theta$ is an increasing operator, for any fixed $\theta$. To finish this proof we apply Proposition 4 at the end of this paper, recalling that a poset of distributions and poset of uniformly bounded functions are chain complete. $\square$

An immediate corollary to the above result is the following: Under Assumptions 1–4, the greatest equilibrium increases in the initial distribution of types $\tau_0$.[34] Indeed, if we let $\theta = \tau_0$ and $\Theta = \mathcal{M}_T$ is ordered in the stochastic sense, then Assumption 5 holds.

Our monotone comparative dynamics result improves upon and complements important results in the existing literature, e.g., Adlakha and Johari (2013), Acemoglu and Jensen (2010, 2015), Light and Weintraub (2019). These papers discuss equilibrium comparative statics of (a) the set of equilibrium invariant distributions and/or steady states, and (b) in games with aggregative structure. In contrast, we provide conditions under which MSDE equilibrium *transition paths* are increasing in the parameter. This extension is of utmost importance. The conditions in Acemoglu and Jensen (2015) or Light and Weintraub (2019) that determine comparative statics of invariant distributions are not sufficient for comparison of MSDE equilibrium transitional dynamics. In our case, as the equilibrium distribution $\mu^*(\theta)$ and the law of motion/belief $\Phi^*(\theta)$ increase in $\theta$, so does the distribution $\Phi^*\big(\mu^*(\theta)\big)(\theta)$ in the following period. This guarantees that the entire equilibrium path shifts with respect to the parameter $\theta$. This also suffices for the greatest invariant distribution $\bar{\nu}$ induced by the greatest equilibrium to be increasing in $\theta$.[35]

Additionally, our results apply to distributions over *multidimensional* space $\mathbb{R}^n$. In fact, the multidimensionality is inherent if one studies distributions over types *and* actions (like in our motivating example). Since spaces of measures over multidimensional spaces are not lattices, it is critical to employ the new tool from Proposition 4. See also the

[34] An analogous comparative equilibrium transitional dynamics results applies to the least MSDE.

[35] A similar argument works for the least equilibrium and the least invariant distribution.

interesting discussion in Section 3 of Light and Weintraub (2019).

## 4 Approximating dynamic Bayesian games with finitely many players

We now show how a large dynamic game of strategic complements can serve as an approximation (or "idealized limit") of its counterpart with a finite number of players $N$, for $N$ is sufficiently large. To relate the sets of equilibria in the two different classes of games, it is useful to introduce a behavioral equilibrium concept for a game with finitely may players. To do so, we first define an $N$-player, dynamic Bayesian game, where the sequence of priors from which player types are drawn in each period $n$ is given by $\tau_n$.[36] Then we impose the following behavioral assumption in the finite player game: each of the $N$ players believes that the law of large numbers holds and updates their beliefs accordingly. That is, they do not form beliefs about the possible private type profiles in the finite game; rather, they behave as if each period the types were drawn as in a game with infinitely many players.[37] Therefore, the belief regarding the distribution of types $\tau_n$ at time $n$ is determined as in (1), where $\tau_1 = \tau$, for some initial $\tau$, and

$$\tau_{n+1}(Z) = \int_{T \times A} q\big(Z|t, a, (\tau_n \star \sigma_n)\big)(\tau_n \star \sigma_n)(dt \times da),$$

for $n \geq 1$, where $\tau_n \star \sigma_n = \tau_n(id_T, \sigma_n)^{-1}$, for the symmetric strategy $\sigma_n : T \to A$ used by all players at time $n$. This sequence of priors is assumed to be common knowledge. In this section we formally compare the corresponding equilibrium in the finite game to MSDE in the game with continuum of players.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\tilde{T}_n : \Omega \to T^N$ be a random variable determining the types of players in period $n$. Define the mapping $\tilde{T}_n = (\tilde{T}_n^1, \ldots, \tilde{T}_n^N)$, where $\tilde{T}_n^l$ is the random variable determining the type of agent $l$, drawn i.i.d. from the theoretical distribution $\tau_n$. For any vector of types $\tilde{t} := (\tilde{t}^1, \ldots, \tilde{t}^N) \in T^N$ of players, i.e., vector of

[36] As a matter of notation, when it causes no confusion, we shall denote the set of players in the finite player game and its cardinality both by $N$.

[37] This is analogous of *imagined-continuum equilibrium* in Kalai and Shmaya (2018) for repeated games.

realizations of the random variable $\tilde{T}_n$, we construct the *empirical distribution*

$$\hat{\tau}_n^N(\tilde{t})(Z) \;=\; \frac{\#\big\{l \in \{1,2,\ldots,N\} : \tilde{t}^l \in Z\big\}}{N}.$$

We seek to compare symmetric equilibrium profiles of games with different number of players. To do this, suppose that all but the $j$'th player apply a sequence of (now fixed) Markov policies $(\sigma_n)_{n\in\mathbb{N}}$. That is, any player $l \neq j$, after observing $\tilde{t}^l \in T$ and knowing the theoretical distribution $\tau_n$ at time $n$, chooses the action $\sigma_n(\tilde{t}^l) \in \tilde{A}(\tilde{t}^l, \tau_n)$, where $\sigma_n$ is Borel measurable. Player $j$ at time $n$ selects strategy $S_n^j$ (a random variable). Let $\tilde{t}$ be a realization of $\tilde{T}_n$, $s^j$ a realization of $S_n^j$, and $s^l = \sigma_n(\tilde{t}^l)$ some realization for $l \neq j$. The empirical distribution on types-actions is given by:

$$\hat{\mu}_n^N(\tilde{t}, s^j)(D) \;:=\; \frac{\#\big\{l \in \{1,2,\ldots,N\} : (\tilde{t}^l, s^l) \in D\big\}}{N} \;=\; \frac{1}{N}\sum_{l\neq j} \mathbf{1}_D\big(\tilde{t}^l, \sigma_n(t^l)\big) + \frac{1}{N}\mathbf{1}_D(\tilde{t}^j, s^j).$$

The following preliminary lemma will allow us to formalize the appropriate notion of an idealized limit of this finite player dynamic game. It states that the empirical distribution over types-actions of our Bayesian game with finitely many players converges weakly to the theoretical one as the number of players increase.

**Lemma 8.** *For any $n \in \mathbb{N}$, let $\tilde{T}_n^{-j} = (\tilde{T}_n^l)_{l\neq j}$ be a collection of $T$-valued random types for $l \neq j$, drawn i.i.d. from $\tau_n$. Let $S_n^l := \sigma_n(\tilde{T}_n^l)$, for all $l \neq j$. For any $N$, let $(\xi^N, \eta^N)$ be an alternative random vector of type and policy for $j$, such that $(\xi^N, \eta^N) \in Gr\big(\tilde{A}_n(\cdot, \tau_n)\big)$ almost surely. Then $\hat{\mu}_n^N\big((\tilde{T}_{-j}, \xi^N), \eta^N\big)$ converges weakly to $(\tau_n \star \sigma_n)$, $\mathbb{P}$-almost surely.*

We now proceed with the formal definition of histories in the game and player's payoff. Let $\mathcal{F}_n$ be the sigma-algebra generated by the sequence of random variables of types (histories) $\tilde{T}_k$, for $k \leq n$. The selection $S_n^j$ for $j$ is called *admissible* if

$$\mathbb{P}\Big(S_n^j \in \tilde{A}_n(\tilde{T}_n^j, \tau_n) \,\big|\, \mathcal{F}_n\Big) \;=\; 1, \;\mathbb{P}\text{-almost surely.}$$

The selection for players other than $j$ is *admissible* by definition of $(\sigma_n)$. For any $l$, assume that $\tilde{T}_n^l$ is a Markov chain controlled by all players, and the transition probability satisfies

$$\mathbb{P}\Big(\tilde{T}_{n+1}^l \in Z \big| \mathcal{F}_n\Big) \;=\; q\Big(Z \big| \tilde{T}_n^l, S_n^l, \hat{\mu}_n^N(\tilde{T}_n, S_n^j)\Big), \text{ for any } Z \in \mathcal{T}, \;\mathbb{P}\text{-almost surely.}$$

Moreover, the random variables $\tilde{T}^1_{n+1}, \ldots, \tilde{T}^N_{n+1}$ are $\mathcal{F}_n$-conditionally independent.[38] The history is generated by types and actions of all players. The set of histories up to time $n$ is $H_n \subseteq \prod_{k=0}^n \left(\mathrm{Gr}(\tilde{A}_k)\right)^N$, with a generic element $h_n = (\tilde{t}^1_k, s^1_k, \ldots, \tilde{t}^N_k, s^N_k)_{k=0}^n$ and $s^l_k = \sigma_k(\tilde{t}^l_k)$, for all $l \neq j$. Moreover, for any $k$, $\tilde{t}^j_{k+1}$ is in the support of $q\big(\cdot\,|\tilde{t}^j_k, s^j_k, \hat{\mu}_n(\tilde{t}_k, s^j_k)\big)$.

Any initial type $\tilde{t}^j_0$ of player $j$, their (behavioral) policy $\pi$, policy of other players $(\sigma_n)$, initial distribution for all types $\tau_1$, and the transitions between types induce a unique private probability measure on histories and its expectation $\mathbb{E}^{\sigma,\pi}_{\tilde{t}^j_0}$.[39] If player $j$ unilaterally deviates from the Markov policy $(\sigma_n)$ to $\pi = (\pi_n)$, the strategy profile is $\big((\sigma_n)_{-j}, (\pi_n)\big)$, since $(\sigma_n)$ is symmetric for $l \neq j$. For an initial private state $\tilde{t}^j_0 = t$, player $j$ payoff is

$$\begin{aligned}\mathcal{R}^N(\sigma_{-j}, \pi)(t) &:= (1-\beta)\mathbb{E}^{\sigma,\pi}_t\left[\sum_{n=1}^{\infty} r^N_n(\tilde{t}^j_n, s^j_n)\beta^{n-1}\right] \\ &= (1-\beta)\mathbb{E}\left[\sum_{n=1}^{\infty} r^N_n(\tilde{T}^j_n, S^j_n)\beta^{n-1}\Big|\tilde{T}^j_1 = t\right],\end{aligned}$$

where $r^N_n$ is a reward function defined as follows

$$r^N_n(t,a) := \int_{T^{N-1}} r\Big(t, a, \hat{\mu}^N_n\big((t, \tilde{t}^{-j}), a\big)\Big)\tau^{N-1}_n(d\tilde{t}^{-j}),$$

where $\tau^{N-1}_n = \underbrace{\tau_n \otimes \tau_n \ldots \otimes \tau_n}_{N-1 \text{ times.}}$ Similarly, let

$$q^N_n(\cdot|t,a) := \int_{T^{N-1}} q\big(\cdot\,|t, a, \hat{\mu}^N_n((t, \tilde{t}^{-j}), a)\big)\tau^{N-1}_n(d\tilde{t}^{-j}).$$

Given the evolution of $\tau_n$ specified earlier and the policy for all players $(\sigma_n)_{n\in\mathbb{N}}$, the problem for player $j$ is a Markov decision process with the value function

$$\tilde{v}^N_1(t) := \sup_{\pi\in\Sigma} \mathcal{R}(\sigma^{-j}, \pi)(t),$$

where $\Sigma$ is the set of all feasible policies, i.e., Borel measurable functions $\pi := (\pi_n)^\infty_n$ such that $\pi_n : H_n \times T \times \mathcal{M}_T \mapsto \mathcal{M}_A$ and $\pi_k\big(\tilde{A}_n(t^j_n, \tau_n)|h_n, t^j_n\big) = 1$ for all $n$, $t^j_n \in T$, $h_n \in H_n$, and all $\pi_n$ are Borel measurable function.[40]

[38] That is, $\tilde{T}_{n+1}$ has $\mathcal{F}_n$-conditional distribution $q^P\big(\cdot\,|\tilde{T}_n, S_n, \hat{\mu}^N_n(\tilde{T}_n, S^j_n)\big) := \bigotimes_{j=1}^N q\big(\cdot\,|\tilde{T}^j_n, \tilde{S}^j_n, \hat{\mu}^N_n(\tilde{T}_n, S^j_n)\big)$, that is $\mathbb{P}\big(\prod_{j=1}^N\{\tilde{T}^j_{n+1} \in Z_j\}|\mathcal{F}_n\big) = \prod_{j=1}^N q\big(Z_j|\tilde{T}^j_n, S^j_n, \hat{\mu}^N_n(\tilde{T}_n, S^j_n)\big)$, for any $Z_1, \ldots, Z_N$, all belonging to $\mathcal{T}$, and all $\omega \in \tilde{\Omega}$ (or modifying $\tilde{\Omega}$ on a null set if necessary).

[39] As previously, see Ionescu-Tulcea Theorem in Dynkin and Yushkevich (1979).

[40] We denote by $\mathcal{M}_A$ the set of probability distributions over $A$.

**Definition 2** (Approximation)**.** A profile $\hat{\sigma} = (\hat{\sigma}_n)$ is an $\epsilon$-*equilibrium* for an initial distribution $\tau_0$, if there is some $N_0 \in \mathbb{N}$ such that for any $N > N_0$, any player $j = 1, 2, \ldots, N$, any type $t \in T$, and any $\pi \in \Sigma$, we have

$$\epsilon + \mathcal{R}^N(\hat{\sigma})(t) \ \geq \ \mathcal{R}^N\big((\hat{\sigma})_{-j}, \pi\big)(t).$$

A symmetric action profile $\hat{\sigma}$ is an $\epsilon$-equilibrium if it constitutes an $\epsilon$-equilibrium for a sufficiently large $N$. Clearly, both $\epsilon$ and $N_0$ depend on the initial distribution $\tau_0$.

**Assumption 6.** *Suppose that (i) function $r$ is continuous, (ii) for any continuous function $f : T \to \mathbb{R}$, the function $(t, a, \mu) \to \int_T f(t''|t, a, \mu)$ is continuous, and (iii) for any $\tau$, the correspondence $t \to \tilde{A}(t, \tau)$ is continuous.*

It is important to note that in our specification of the dynamic Bayesian game with finitely many players, the agents do not control the *theoretical* distribution $\tau_n$ — rather, they only control the empirical distribution $\hat{\mu}_n$. Now, for some MSDE $(\mu^*, \Phi^*)$ in the counterpart large dynamic game, consider an associated equilibrium strategy profile $\sigma^*$ and its associated value function $v^*$. For $\mu_0^* = \mu^*$, consider the sequence of measures $(\mu_n^*)$ defined recursively by macro belief operator $\mu_{n+1}^* = \Phi^*(\mu_n^*)$. Similarly, take the associated distributions on types $\{\tau_n^*\}$, the policies $\{\sigma_n^*\}$, and values $v_n(t)$, where $\tau_n^* = \text{marg}_T(\mu_n^*)$, $\sigma_n^*(t) := \sigma^*(t, \mu_n^*) = \sigma_{\mu_n, \Phi^*}(t)$ and $v_n(t) := v^*(t, \mu_n^*; \Phi^*)$.

We then have the following main theorem of this section.

**Theorem 4.** *Under Assumption 6, for any MSDE $(\mu^*, \Phi^*)$ and $\epsilon > 0$, the sequence of implied policy functions $\{\sigma_n^*\}$ is an $\epsilon$-equilibrium for $\tau_1 = \tau^*$.*

We make a few remarks on this result relative to related results in the existing literature. Weintraub et al. (2008) and Adlakha et al. (2015), for example, study the asymptotic Markov properties of both oblivious (OE) and mean-field equilibrium (MFE). Specifically, they show in an OE-MFE for a dynamic game with finitely many players, the invariant distribution becomes "approximately optimal" as the number of players tends to infinity. Such approximation notion in their work requires both *uniqueness* and *continuity* of the best reply. Moreover, as the authors work with unbounded states spaces and unbounded payoffs, their result also requires the so-called *light tail condition*. Relative to this work, we dispense with uniqueness and continuity of the best reply, but impose stronger condition relative to the boundedness on the state space and payoffs (so no light tail condition is needed). Alternatively, Kalai and Shmaya (2018) show in their work that an imagined-continuum Bayesian equilibrium with a finite number of players is an $\epsilon$-equilibrium of the actual (repeated) game. Moreover, they also show that $\epsilon$ is arbitrarily small as $N$ tends to infinity. Aside from considering dynamic vs. repeated games, our results differs from theirs as we do not require the aggregative structure of the interactions in the dynamic game. Our results also complement earlier contributions by allowing for Markovian environment in a dynamic game, and without restricting the asymptotic analysis to invariant distributions, unique and continuous best replies, or aggregative games.[41]

# 5 Applications and examples

## 5.1 Motivating example revisited

Recall the motivating example from the Introduction. In each period, the type of a player was identified with their level of capital/wealth $t \in T = [0, 1]$. Their actions (investments) $a \in A = [0, 1]$ were chosen from the feasible correspondence $\tilde{A}(t, \tau) = [0, t]$. Given the distribution $\mu$ of types-actions of all players, the payoff in a single period was

$$r(t, a, \tau, \theta) \ := \int_{A \times T} \Big[\theta m(t - \tilde{t}) + w(t - a - \tilde{t} + \tilde{a})\Big] \mu(d\tilde{a} \times d\tilde{t}).$$

Here we introduce a positive parameter $\theta$ with respect to the initial example.

Given an investment $a$, the cumulative probability distribution of capital level $t'$ in the following period is $q(t'|a)$. Thus, conditional on the macro belief $\Phi$, the Bellman equation determining the player's value function in the infinite horizon game is

$$v(t, \tau; \Phi) \ = \ \max_{a \in \tilde{A}(t,\tau)} \left\{ (1 - \beta) r(t, a, \tau, \theta) + \beta \int v\big(t', \Phi(\tau); \Phi\big) q(dt'|a) \right\}.$$

[41] We also refer the reader to recent results of approximation of large static games by Carmona and Podczeck (2012, 2020) and related results in Qiao and Yu (2014); Qiao et al. (2016).

It is straightforward to verify this game satisfies Assumptions 1–4. Correspondence $\tilde{A}$ is measurable, continuous, compact valued, and increasing (both in the sense of set inclusion and strong set order). Given that functions $m$ and $w$ are continuous, increasing, and concave, function $r$ is continuous over $T \times A$, increasing over $T$, and has increasing differences in $\big(a, (t, \mu)\big)$ and $(t, \mu)$. The function is also (trivially) supermodular in $a$ and continuous in $\mu$. As long as the distribution $q$ is continuous in $a$, the requirements of Theorem 2 for existence of the greatest MSDE are satisfied.

As it was pointed out in the main body of the text, the equilibrium pair $(\mu^*, \Phi^*)$ generates the entire equilibrium path of distributions $\{\mu_n^*\}$, where $\mu_0^* = \mu^*$ and $\mu_{n+1} = \Phi^*(\mu_n)$, which allows us to investigate the dynamics of the model. Moreover, the sequence converges to an invariant distribution, allowing for the study of steady states.

Apart from existence and approximation of equilibria, Theorem 3 allows us to say more about its equilibrium comparative dynamics. In particular, the equilibrium $(\mu^*, \Phi^*)$ and the corresponding sequence $\{\mu_n^*\}$ increase as the initial distribution of types $\tau_0$ increases in the first order stochastic sense. That is, along the equilibrium path to a stationary equilibrium, players invest more and have higher capital levels (stochastically). In addition, the equilibrium changes monotonically with respect to the parameter $\theta$. One can easily verify that the return function $r$ has increasing differences in $(a, \theta)$ and $(t, \theta)$. Given that the correspondence $\tilde{A}$ and transition kernel $q$ are independent of $\theta$, this suffices for the equilibrium and its path to be increasing in $\theta$. Thus, the higher the weight of the wealth-driven status, the higher (stochastically) are investments in the population.

The above results would hold under a more elaborate transition kernel $q(\cdot|t, a, \mu)$, that would depend on the investment of a player, their type, and the distribution of wealth-investments in the population. However, this would require for Assumption 3 to hold.

## 5.2 Dynamics of social distance

We next analyze a dynamic model of *social distance*, described originally in Akerlof (1997).[42] Consider a measure space of agents. Let $T = [0, 1]$ be the set of all possi-

[42] The model is related to multiple strands of the social economics literature, including models of identity and economic choice as in Akerlof and Kranton (2000), or models with endogenous social reference

ble social positions in the population. Each period an individual is characterized by an *identity* $t \in T$ (type), which determines the social position to which the agent aspires. In every period an agent has to choose their own social position (action) $a \in A := [0, 1]$. The set of social positions feasible to agent with identity $t$ is $\tilde{A}(t, \tau) := \left[\underline{a}(t), \overline{a}(t)\right]$, where $\underline{a}$, $\overline{a} : T \to A$ are increasing functions that satisfy $\underline{a}(t) \leq t \leq \overline{a}(t)$, for all $t \in T$.

When choosing social position, there is a trade-off between *idealism* and *conformism*. On one hand, the individual wants the social status $a$ to be as close as possible to their identity $t$. Specifically, given some continuous, decreasing, and concave function $m$ : $[0, 1] \to \mathbb{R}$, the agent wants to maximize $m\big(|a - t|\big)$, that captures idealism. On the other hand, the player experiences discomfort when interacting with agents that have different social position from theirs. Whenever an agent of social position $a$ encounters an agent of social position $a'$, they receive utility $w\big(|a - a'|\big)$, for some continuous, decreasing, and concave function $w : [0, 1] \to \mathbb{R}$. This summarizes conformism.

Suppose that $\nu(t'|t)$ is a cumulative probability distribution determining the likelihood of an agent with identity $t$ meeting someone with identity $t'$. We assume it is continuous and first-order stochastically increasing in $t$. It captures the idea that *similar minds think alike* and players with similar identity are more likely to meet. Given the distribution of types-actions $\mu$, the one-period payoff of an agent of identity $t$, social position $a$ is

$$r(t, a, \mu) \ := \ m(|a - t|) + \int_T \int_A w(|a - a'|) d\mu(a'|t') d\nu(t'|t),$$

where $\mu(\cdot|t')$ is the distribution of actions of other players in the population conditional on $t'$. Therefore, payoff of an agent in a single period is the sum of their idealistic utility and expected payoff to conformity relative to their interactions with other agents. In particular, our specification implies that the social position can not be contingent on the social statuses of other agents. It is chosen before any interaction occurs.

Following the rule *you become whom you pretend to be*, we assume that the social position in a current period has a direct impact on the identity in the following period. Formally, the transition is governed by cumulative probability distribution $q(t'|a)$, that

points, including Bernheim (1994), Brock and Durlauf (2001), Bisin et al. (2011), and Blume et al. (2015). The model in this example is a dynamic extension of the static model formalized in Balbus et al. (2019).

determines the likelihood of the agent acquiring identity $t'$ in the next period, following their choice of $a$ at the current date. Specifically, we assume that function $a \to q(\cdot|a)$ is continuous and first order stochastically increasing in $a$.

It is straightforward to verify that the above game admits the greatest (and the least) MSDE. Indeed, function $r$ satisfies conditions (i), (ii) and (iv), (v) from Assumption 2. Moreover, since the transition kernel $q$ depends only on $a$, it satisfies Assumption 3. Finally, as long as functions $\underline{a}, \overline{a}$ are continuous, in addition to the previously stated assumptions, correspondence $\tilde{A}(t, \tau) = \left[\underline{a}(t), \overline{a}(t)\right]$ is continuous, compact-valued, and satisfy strong complementarity. Clearly, Assumption 1 holds as well.

In this example, it is crucial that the transition function $q$ depends *only* on action $a$. Following Remark 2, this allows to dispense the assumption that function $r$ and correspondence $\tilde{A}$ are increasing in $t$, which is critical for this application.

Apart from equilibrium existence, one can determine equilibrium comparative transitional dynamics in the model. It is clear that as the initial distribution of identities $\tau_0$ shifts in the first order stochastic sense, the equilibrium pair $(\mu^*, \Phi^*)$ increases as well. This implies an increase in the entire equilibrium transition path $\{\mu_n^*\}$.

### 5.3 Parenting and endogenous preferences for consumption

We now show how our tools can be applied to dynamic games with short-lived agents, where individuals make decisions in one period only, but their actions propel dynamics for future generations. This dynastic choice example is inspired by the literature on paternalistic bequests, keeping-up-with-the-Joneses, and growth with endogenous preferences.[43]

Consider a society populated with a measure space of single-parent single-child families. Each individual (a parent) lives for a single period and a parent-child sequence forms a dynasty. The type of a parent is determined by their lifetime income $y \in [0, 1]$ and a parameter $i \in [0, 1]$ that summarizes preferences of the individual toward consumption. So in this setting, the space of types will be $t = (y, i)$ is given by $T = [0, 1]^2$.

Each period, the income can be devoted to consumption $c$ and investment (savings) $s$.

[43] See Cole et al. (1992), Doepke and Zilibotti (2017) and Genicot and Ray (2017).

Thus, the constraint $y = c + s$ for each dynasty. Consumption yields immediate utility $u(c, g)$, where parameter $g$ represents *propensity to consume*. Formally, we assume the function $u$ is continuous and concave in $c$, and has increasing differences in $(c, g)$. That is, higher $g$ increases the marginal utility of consumption for the current generation.

We assume paternalistic preferences, where a parent evaluates the well-being of their child with a function $w(t', \tau')$, where $t' = (y', i')$ is the the future type of the child and $\tau'$ is a distribution of types in the next period. We assume $w$ is increasing in $t$, thus, the parent values high income and high propensity to consume of the child. Since the parent cares only about her immediate descendant, they want the child to consume as much as possible. Moreover, let $w$ have increasing differences in $(t', \tau')$, i.e., the higher is the future distribution of types the higher is the parent's incremental benefit of the child's type.[44]

Each parent devotes (e.g., educational) effort $e \in E = [0, 1]$ to shape preferences of their child (i.e., raise their aspiration level). The cost of effort is given by $C(e, \mu_E)$, where $\mu_E$ denotes the distribution of efforts in the population. We assume that the cost function is continuous and increasing with $e$, and has decreasing differences in $(e, \mu_E)$ — the higher effort in the population, the easier it is for an individual to influence their child.

Given our description, the action of an individual is $a = (s, e)$ and the action space is $A = [0, 1]^2$. Savings $s$ and effort $e$ affect both the future income and preferences of the child. Let the cumulative distribution $q(t'|s, e)$ determine the probability of the future type of the child being $t' = (y', i')$, where $q$ is stochastically increasing in both arguments and supermodular. Thus, investment $s$ and effort $e$ are complements. Indeed, from the parent's perspective higher effort (that skews preference of the child towards consumption) makes marginal investment/bequest more valuable. The higher amounts of child's income are devoted to consumption, the more it pleases the paternalistic parent.

Finally, the marginal propensity to consume $g$ is generated endogenously for each individual via keeping-up-with-the-Joneses effect. Formally, let $g = \theta\Gamma(t, \mu_C)$, for some positive parameter $\theta$ and an increasing function $\Gamma$, that depends both on the type of the

[44] This model is broadly related to issues raised in Echenique and Komunjer (2009) and Doepke et al. (2019) concerning endogenous transmission of preferences in dynastic models of household choice. Ours is a version of the model with quantile aspiration preferences and paternalism. This could be extended to altruistic dynastic choice, peer effects, or locational concerns as in Agostinelli et al. (2020).

player and the distribution of consumption levels across population. For example,

$$\Gamma(t, \mu_C) \ := \ \inf\Big\{c \in [0,1] : i \leq \mu_C(c'' \leq c)\Big\},$$

where $t = (y, i)$. That is, $\Gamma$ is equal to the $i$'th quantile of consumption in the population.

Given our description, the objective of a parent of type $t = (y, i)$ is to maximize

$$u\big(y - s, \theta\Gamma(t, \mu_C)\big) + \int_{[0,1]} w\big(t', \Phi_T(\mu)\big)q(dt'|s, e) - C(e, \mu_E),$$

with respect to $(s, e) \in \tilde{A}(t, \tau) = [0, y] \times [0, 1]$. Here, the mapping $\Phi_T(\mu)$ is the projected next-period distribution of types in the population. Note that, $w$ is not a value function in the sense discussed Section 2; rather, a paternalistic evaluation of child's welfare. Specifically, preferences of a parent may be *misaligned* with future preferences of the child.

To verify whether assumptions of our theorems are satisfied, consider an increasing Markov strategy: $\sigma : T \to A$, with $\sigma_s$ and $\sigma_e$ being its projections on both coordinates. Then for some measurable set $Z$, we have $\mu_C(Z) = \tau\big(\{t \in T : [y - \sigma_s(t)] \in Z\}\big)$, $\mu_E(Z) = \tau\big(\{t : \sigma_e(t) \in Z\}\big)$, and $\Phi_T(\mu)(Z) = \int_T q\big(Z|\sigma_s(t), \sigma_e(t)\big)\tau(dt)$. Then, higher $\sigma$ implies first order stochastic dominance increase of $\mu_E$ and $\Phi_T(\mu)$, but the first order stochastic dominance decrease in $\mu_C$.[45] Increasing differences of $u(c, g)$, $w(t', \tau')$, and $-C(e, \mu_E)$, together with assumptions on $q$ suffice to show that there exist the greatest MSDE $(\mu^*, \Phi^*)$, that can be computed using successive approximations.

When considering ordered changes in the deep parameters of the model, we can apply our equilibrium comparative transitional dynamics and equilibrium approximation to these types of models. In particular, one can show the greatest (and the least) MSDE are *decreasing* with respect to the parameter $\theta$.

The above observations are true even though the payoff function is not necessarily increasing in $t$, nor it has increasing differences in $(t, \mu)$. In fact, whenever function $\Gamma$ is specified as above, the latter never holds. In our main argument the additional assumptions are crucial to show particular properties of the value function in the infinite horizon problem. In a game with short-lived agents, we may dispense such assumptions.

[45] Indeed, we have $\int_C f(c)\mu'_C(dc) = \int_T f(y - \sigma'_s(t))\tau(dt) \leq \int_T f(y - \sigma_s(t))\tau(dt) = \int_C f(c)\mu_C(dc)$, for any measurable and increasing function $f : [0, 1] \to \mathbb{R}$, where $\sigma'_s$ pointwise dominates $\sigma_s$.

## 5.4 Legal norms and public enforcement

Here we discuss a version of the model of social/legal norms an public enforcement as in Acemoglu and Jackson (2017). Suppose there is a continuum of agents, each endowed with a private type $t \in [0,1]$. Let the threshold $L \in [0,1]$ be the social/legal norm in the society. In each period, an individual randomly interacts with other members of the population. Before any interaction takes place, the individual of type $t$ must choose an action $a \in [0,t]$. We say that action $a$ is *legal* if $a \leq L$. Otherwise, it is *illegal*.

Whenever an agent of type $t$ playing action $a$ encounters an agent playing action $\tilde{a}$, the bilateral public enforcement takes place. If both actions $a$, $\tilde{a}$ are legal, the players are allowed to play the selected actions. If action $a$ is illegal, while $\tilde{a}$ is legal, the latter agent forces the former to abide the law, i.e., the former has to change their action to $L$. Analogously, if $a \leq L$ but $\tilde{a} > L$, the latter agent has to change their action to $L$. Finally, if both $a$, $\tilde{a}$ are illegal, the agents play their chosen actions, since none of the agents has the moral ground to enforce the legal action.[46]

In this game individuals agent care about two things. On one hand, they want their actual action (the one after a potential enforcement) to be as close to their type as possible, since it yields $u\big(|t - a + \mathbf{1}_{a>L}\mathbf{1}_{\tilde{a}\leq L}(a-L)|\big)$ for some continuous, decreasing, and concave function $u$, where $\mathbf{1}_{\tilde{a}\leq L}$ is the indicator function. Moreover, the agent wants their action to be as close as possible to the (potentially enforced) action of the other players, which yields utility $v\big(|a - \tilde{a} + \mathbf{1}_{a\leq L}\mathbf{1}_{\tilde{a}>L}(a-L)|\big)$, for some continuous, concave, and decreasing $v$. The one-period payoff of an agent of type $t$ choosing action $a$ is then given by

$$\begin{aligned} r(t,a,\mu) \ := \ \int_{[0,1]} \Big[ & u\big(|t - a + \mathbf{1}_{a>L}\mathbf{1}_{\tilde{a}\leq L}(a-L)|\big) \\ & + \ v\big(|a - \tilde{a} + \mathbf{1}_{a\leq L}\mathbf{1}_{\tilde{a}>L}(\tilde{a}-L)|\big) - \theta\mathbf{1}_{a>L}\mathbf{1}_{\tilde{a}\leq L}\Big]\mu_A(d\tilde{a}), \end{aligned}$$

where $\mu_A$ is the probability distributions over actions in the population, and $\theta$ is a fine that the individual has to pay when caught. The set of constraints is given by $\tilde{A}(t,\tau) := [0,t]$, and the type $t'$ of each player is drawn stochastically each period from $q(t'|a)$, that depends and stochastically increases in the action $a$ of the agent in the preceding period.

[46] It is straightforward to extend the above model in order to incorporate imperfect and/or exogenous (police) enforcement. In order to simplify notation, we discuss only the most basic form of the game.

One can easily check that the assumptions necessary for existence of (the greatest) MSDE are satisfied.[47] In particular, the equilibrium generates a transitional path of distributions of types and actions. Appealing to our monotone equilibrium comparative dynamics results (and approximation results), we can show that both the equilibria, transitional paths, and corresponding stationary equilibria decrease in the fine $\theta$.

### 5.5 Dynamics of large contests with coordination failures and learning

Consider a prototypical coordination game based on Angeletos and Lian (2016), with applications to beauty contests, bank runs, riot games, or currency attacks.[48] Here focus on a simple dynamic beauty contest. In this large dynamic game, each player receives a private signal $t$ and chooses an action $a$ every period. Action is costly and the cost depends on the type $t$, which is summarized in the utility function $u(t, a)$. Moreover, we assume $u$ is increasing in $t$ and has increasing differences in $(t, a)$. In addition to the utility $u$, the player's payoff depends on actions taken by other players, say $\int_A g(a, \tilde{a})\mu_A(d\tilde{a})$, where $g$ also has increasing differences between $a, \tilde{a}$.

As is standard in global games and dynamic coordination games with complementarities, we study symmetric monotone in type equilibria, where each player is using some increasing strategy $\sigma : T \to A$. The one-period payoff of an agent playing $a$ is

$$r(t, a, \mu) \;:=\; u(t, a) + \int_T g\big(a, \sigma(\tilde{t})\big)\mu_T(d\tilde{t}),$$

Such payoff satisfies assumptions of Theorem 2, and so there exists the greatest MSDE, where each player is using an increasing strategy $\sigma$.[49]

Similarly, the framework can applied to riot games with private types, where

$$r(t, a, \mu) \;:=\; a\left[\int_S (t_1 + L)\mathbf{1}_{\{R(\mu)\geq\tilde{s}\}}\nu(d\tilde{s}) - L\right] - c(a, t_2),$$

[47] Note that, function $r$ is upper semi-continuous in action $a$, rather that continuous. However, this can be show to be sufficient for our results to hold in this class of games.

[48] See Morris and Shin (2002) for an extensive discussion of this literature. See also Carmona et al. (2017) for an interesting recent application of mean-field methods to a related class of games.

[49] We may dispense monotonicity of $u$ with respect to $t$ as long as the transition function $q$ depends only on one-dimensional action $a$.

for some player type by $t = (t_1, t_2)$ and a compact interval $S \subseteq \mathbb{R}$. Thus, taking the risky action $a = 1$ allows the player to "win" $t_1$ if a sufficient number $R(\mu) := \mu\big(\{(t, a) : a = 1\}\big)$ of players takes a risky (and costly) action, or loose $L$ otherwise. The strength $s$ of the police is distributed according to measure $\nu$. Whenever the cost function is decreasing in $t_2$ and $c(0, t_2) = 0$ (normalization), the dynamic game can be solved for a general transition functions $q(\cdot|t, a, \mu)$, allowing to model inertia, habit formation, or dynamic social externalities. See also Morris and Yildiz (2016) applications.

### 5.6 Idiosyncratic risk under multidimensional production externalities and technological dynamics

Finally, our model can be applied to analyze dynamics of technological progress in large economies where agents face uninsurable private productivity risk. This includes the model of Romer (1986) in a Bewley-Huggett-Aiyagari type setting with ex-ante identical agents and ex-post heterogeneity in production and no borrowing.[50]

The economy is populated with a measure space of producers, each endowed with capital $t \in T = [0, 1]$, one unit of time, and a private technology $f$. The technology transforms private inputs into finished outputs. Moreover, its productivity depends on economy-wide externality summarized by the distribution of capital and labor in the economy. Specifically, each agent with $t$ units of capital and expending $l \in L = [0, 1]$ units of time is able to produce $y = f(t, l, \mu_{T\times L})$ units of a a finished output, where $\mu_{T\times L}$ is the distribution of capital-labor levels in the population. We assume the production function $f$ is continuous, increasing with respect to all arguments, and possess increasing differences in $(t, l)$, in $(t, \mu)$ and $(l, \mu)$.[51] In particular, the private technologies endowed to each agent need not be convex. In addition, our reduced form of technology allows for nontrivial interactions with market leaders, closely related companies, or a competitive fringe in both capital and labor dimensions.

The output can be devoted to consumption $c$ or investment $i$, hence, $c + i = y$.

[50] See also Angeletos and Calvet (2005) for a related study.

[51] For example, function $f(t, l, \mu_{T\times L}) := \int_{T\times L} g(t, k, \tilde{t}, \tilde{l})\mu_{T\times L}(d\tilde{t} \times d\tilde{l})$ would satisfy such conditions as long as $g$ is supermodular in all arguments jointly.

When $c$ units of the output are consumed and labor supply is $l$, the agent receives utility $U(c,l) = u(c) + v(1-l)$, where $u, v : \mathbb{R} \to \mathbb{R}$ are smooth, concave and strictly increasing. Whenever $i \in I := [0,1]$ units of the good are invested, the capital in the next period is determined stochastically with probability measure $q(\cdot|i)$.[52]

To preserve complementarity structure to the value functions, we require some known complementarity conditions for joint monotone controls (see Hopenhayn and Prescott, 1992 and Mirman et al., 2008). Along those lines, we assume the standard condition $-u''/u' \leq f''_{12}/(f'_1 f'_2)$. It requires that degree of complementarity between private capital and labour is high relatively to the curvature of the utility function. This suffices for payoffs to have increasing differences in $(t,l)$. To guarantee increasing differences in $(t, \mu_{T\times L})$, we require that $u'\big(f(t,l,\mu_{T\times L}) - c\big) f'_1(t,l,\mu_{T\times L})$ is increasing in $\mu_{T\times L}$.[53] Analogous conditions guarantee increasing differences in $(l, \mu_{T\times L})$.[54]

One can easily verify that the above conditions are sufficient for Theorem 2 to hold. Therefore, there exist extremal MSDE for this large dynamic nonmarket economy (interpreted as a large anonymous game). Moreover, the extremal equilibria can be approximated using iterative methods. This example highlights the difference between our results and those in the existing literature. Specifically, we consider Markov stationary transitional dynamics and comparative dynamics results (in additional to stationary equilibrium comparative statics). For example, Acemoglu and Jensen (2015) discuss stationary equilibria and comparative statics given single dimensional aggregates that summarize production externalities.[55] Our conditions on the primitives that guarantee each player's value function has increasing differences in $(t, \mu)$ are not crucial for their results.

[52] Our methods allow to analyze *two sector economies*. A consumption good sector with technology $f$ and investment good sector with stochastic technology $q(\cdot|t,i,l,\mu_{T\times L})$. In the example we consider a simple version of $q$ depending on investment $i$ only.

[53] Whenever the externality can be summarized with some increasing aggregate $G(\mu_{T\times L}) \in \mathbb{R}$, where $y = f\big(t,l,G(\mu_{T\times L})\big)$, the condition can be reduced to $-u''/u' \leq f''_{13}/(f'_1 f'_3)$.

[54] Notice, in our setting, the correspondence $A(t,l,\mu_{T\times L}) = \big[0, f(t,l,\mu_{T\times L})\big] \times L$ does not have strict complementarities. To assure that the value function $v^*$ in (4) preserves increasing differences in $(t,\mu)$ we need to use constructions of Mirman et al. (2008) (Lemmas 11, 12 and Theorems 3, 4). They show that under assumptions stated on $u$, $v$, and $f$ the value function posses increasing differences in $t$ and $\mu$.

[55] In Acemoglu and Jensen (2015), to identify positive shocks one would require additional structure on primitives to preserve increasing differences between individual states and shock parameters. Thus, more assumptions are needed than noted in their Lemma 1.

# 6 Related literature

This paper contributes to several strands of economics literature. First, our results are related to large anonymous sequential games that date back to Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), and Karatzas et al. (1994). Relative to these, we prove existence of *minimal state space stationary Markovian* distributional equilibrium.

Further, of independent interest, we also contribute to an important literature on the existence and characterization of *dynamic* exact law of large numbers (D-ELLN). Not only does such a result underpin all large anonymous stochastic games, but also is the foundation for many results in large dynamic economics (e.g., Bewley models). What is shared in all of these settings is that each from a measure space of agents draws private state each period, and hence the model's state variable must include a measure-valued sequence that summarize the distribution of states across players. To keep the environment tractable, this measure-valued process must be deterministic. Relative to this literature, we introduce a new characterization of a D-ELLN which provides a conditional *independence* of player types (relative to histories of the game), and a deterministic transition of aggregate distribution on types using rich Fubini extensions in saturated or super-atomless measure spaces of players. This is not a mere technical detail; rather, in our setting, given the strategic interaction between players, our equilibrium construction cannot even proceed without an appropriate D-ELLN. Our construction builds upon the important contributions of Sun (2006), Keisler and Sun (2009) and Podczeck (2010).

Additionally, our paper extends the class of games of strategic complementarities (GSC) to a dynamic setting with a measure space of players. Following the important work of Van Zandt (2010), in few recent papers including Balbus et al. (2015a, 2019, 2015b) and Bilancini and Boncinelli (2016), the class of supermodular games and GSC has been extended to situations of normal-form games with complete and incomplete information. Simultaneously, a number of papers studied dynamic GSC with complete and incomplete information.[56] This paper directly relates to this literature in many ways. First, the tools used in the current paper heavily extend that developed by Balbus et al.

[56] See the seminal papers of Curtat (1996), Amir (2005), or more recently Mensch (2020).

(2013, 2014) to study Markovian equilibria in the finite number of players games. In doing so, we provide sufficient conditions for preserving dynamic complementarities between the periods to player's value functions. The conditions allow one to avoid many of the issues related to the notion of extensive-form supermodular games as discussed in Vives (2009), Amir (2002), Echenique (2004), and Mensch (2020). Our new conditions imply that value functions have increasing differences between private types and the aggregate distribution summarizing agents types and actions. Very importantly, with our sufficient structure in place, our large stochastic supermodular games remains extensive-form supermodular over the *infinite horizon*. This fact is critical for all of our equilibrium comparative dynamic/statics results. In this sense, our work also relates to a recent literature on characterizing single-crossing differences over distributions studied in the recent papers of Quah and Strulovici (2012), Kartik et al. (2019), and Mensch (2020).

Given the distributional game specification, and the structural properties implied by our D-ELLN, we are able to avoid many of problems in characterizing dynamic complementarities in actions between periods and beliefs reported recently in Mensch (2020) for dynamic Bayesian games with a finite number of players. Finally, as in the work of Balbus et al. (2014), all our proofs are constructive and computable via simple successive approximations. In this sense, we are able to provide the applied researchers with tools allowing to approximate the equilibrium distributions.

Importantly, our paper also contributes to the recent literature on characterizing the equilibrium comparative statics and dynamics for large dynamic economies and games. The literature is extensive and we refer the reader to Acemoglu and Jensen (2015) and Light and Weintraub (2019) for an excellent discussion and citations. In particular, our results provide a foundation for a theory of equilibrium monotone comparative transitional *dynamics* relative to ordered perturbations of the space of games/economies.[57] Specifically, we provide sufficient conditions on payoffs and transition probabilities such that the sequence of equilibrium distributions, as well as the aggregate law of motion (specifying transition dynamics but also rational beliefs is our game), evolve monotonically in type for any positive shock to the game. Interestingly, our methods extend therefore equi-

[57] Our work complements the approach to transitional dynamics in large economies of Huggett (1997).

librium comparative *statics* results of Adlakha and Johari (2013), Acemoglu and Jensen (2015, 2018), and Light and Weintraub (2019), applied to comparative statics of invariant distributions or "stochastic steady states".

Further, in many papers on equilibrium comparative statics, the results only apply to equilibrium *aggregates*. Our approach contains this as a special case. Moreover, we are able to perform multidimensional equilibrium comparative static/dynamics relative to a (infinite dimensional) set of equilibrium distributions. Indeed, recall we compare distributions over $\mathbb{R}^n$. Set of such objects is (in general) *not* a lattice, hence the need to apply our new equilibrium comparative statics based upon Proposition 4 in the paper.

It also bears mentioning the assumptions of Acemoglu and Jensen (2015, 2018), and Light and Weintraub (2019) are not sufficient to obtain results of our paper. The key central difference between our work and these papers is that when studying stationary equilibrium (or mean-field equilibrium) comparative statics, one does *not* need conditions on the game that imply *single crossing in distribution* between private actions and aggregates.[58] This is because one is only characterizing the "steady state" structure of the sequential or Markovian equilibrium. For the results in the present paper on MSDE, one must deal with the influence of perturbations of dynamic interactions between players and their distributional counterparts via *the value function* that is needed to recursively define each player's stage game payoffs. In additional, one must study the equilibrium structure *away* from the fixed points of the equilibrium law of motion.

Finally, our monotone comparative statics/dynamics results are also shown to be *computable*, as we characterize the chain of parameterized equilibria converging to the one of interest for a particular parameter. This is of utmost importance for applied economists that calibrate the equilibrium invariant distributions' moments, or attempt to develop econometric methods for estimating equilibrium comparative statics/dynamics in data (e.g., via the quantile methods of Echenique and Komunjer, 2009, 2013).

Our paper is also related to the recent work on oblivious (or stationary) equilibrium (OE) and mean-field games (MFGs). This is a large and important growing literature

[58] Characterizing sufficient single crossing conditions with respect to beliefs in static large, Bayesian games with strategic complementarities is a challenge. See Balbus et al. (2015a); Liu and Pei (2017).

that includes papers by Ifrach and Weintraub (2016); Weintraub et al. (2008), Adlakha and Johari (2013), Adlakha et al. (2015), Doncel et al. (2016), Lacker (2018), and Light and Weintraub (2019), among many others.[59] This work in OE and MFGs is motivated primarily by computability and complexity considerations, and many of these papers build methods for games in continuous time, with finitely many states, finite actions sets, symmetric equilibrium in mixed strategies, where games externalities are characterized by distributions or aggregates on states only (so not on actions). Equilibrium of such games are stationary distributions on players states. Such mean field equilibrium implies a best response *oblivious* strategy, i.e. distribution on action sets, where each players' action is optimal taking the invariant mean field distribution as given. For some recent progress on this line of literature we refer the reader to e.g. Adlakha et al. (2015).

In a related context, we also extend a very interesting result of Kalai and Shmaya (2018) on foundations of epsilon Bayesian Nash equilibrium of a finite number of players game via *imagined-continuum* equilibrium. An imagined-continuum is a powerful, and tractable, tool that as itself is a behavioral concept of equilibrium in a Bayesian game, where although the players are playing a game with a finite number of players, they view the equilibrium interaction and learning (and in particular, their belief formation) as in a game with a continuum of players. For this setting, we show that the equilibrium of the imagine-continuum version of the Bayesian game converges to the stationary Markovian equilibrium of the actual game. Our paper extends Kalai-Shmaya setting to non-stationary equilibria without imposing the aggregative structure.

# A Auxiliary fixed point results

Here we present two theorems that are critical in proving Theorems 2 and 3 in Section 3. Recall that a *chain* is a completely ordered set. A poset $X$ is (countably) *lower chain*

[59] For related work on large dynamic supermodular games see Wiecek (2017), who analyses a supermodular game in continuous time, where each player moves in a discrete but different period of time. Moreover, Adlakha and Johari (2013), study a mean-field version on our large dynamic supermodular game with one-dimensional actions and strategic interaction via distribution on types. For such environment they show existence of a mean-field equilibrium, i.e., an oblivious strategy and invariant distribution.

*complete* if any (countable) chain $A \subseteq X$ has its infimum in $X$. The poset is (countably) *upper chain complete* if any such chain has its supremum in $X$. Given posets $X$ and $Y$, function $f : X \to Y$ is *increasing* if $x' \geq_X x$ implies $f(x') \geq_Y f(x)$. Below is a useful generalization of Theorem 9 in Markowsky (1976).

**Proposition A.1.** *Let $(X, \geq_X)$ be a lower chain complete poset with the greatest element. The set of fixed points of an increasing function $f : X \to X$ is a nonempty lower chain complete poset. Moreover, its greatest fixed point is given by $\bigvee \{x \in X : f(x) \geq_X x\}$.*[60] *We prove it in the online appendix.*

Given posets $X$ and $Y$, function $f : X \to Y$ is *monotone sup-preserving* if, for any increasing sequence $\{x_k\}_{k\in\mathbb{N}}$, we have $f\big(\bigvee\{x_k\}_{k\in\mathbb{N}}\big) = \bigvee \{f(x_k)\}_{k\in\mathbb{N}}$. It is *monotone inf-preserving* if $f\big(\bigwedge\{x_k\}_{k\in\mathbb{N}}\big) = \bigwedge \{f(x_k)\}_{k\in\mathbb{N}}$, for any decreasing sequence $\{x_k\}_{k\in\mathbb{N}}$. The second theorem extends the classic fixed point comparative statics results of Veinott (1992) and Topkis (1998) to countably chain complete posets. It is based on the Tarski-Kantorovich theorem. See Balbus et al. (2015c) for a proof.[61]

**Proposition 4.** *Let $X$ be a lower countably chain complete poset with the greatest element, and $\Theta$ be a poset. For any function $f : X \times \Theta \to X$ and $\theta \in \Theta$ such that $f_\theta$ is increasing and monotone inf-preserving over $X$, the greatest fixed point of $f_\theta$ is given by $\bigwedge \{f_\theta^n(\bigvee X)\}_{n\in\mathbb{N}}$.*[62] *In addition, if $f$ is increasing in the product order and $f_\theta$ is monotone inf-preserving, for all $\theta \in \Theta$, then the greatest fixed point is increasing over $\Theta$.*

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[60] There is an obvious order dual to this result for increasing functions defined on an upper chain complete domain $X$ that implies existence of the least fixed point theorem.

[61] There is a dual version of this theorem for the least fixed point of the monotone sup-preserving function defined over an upper countably chain complete domain $X$.

[62] By $f^n$ we denote the $n$'th composition of $f$, i.e., $f^n = f \circ f \circ \ldots \circ f$ ($n$ times).

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# Supplement to

# Markov distributional equilibrium dynamics in games with complementarities and no aggregate risk

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**Abstract**

This supplement contains the proofs omitted from the main text of the paper as well as preliminaries on the law of large numbers and lattice theory.

## A Preliminaries

In this section we introduce some mathematical notions in measure and lattice theory that are employed in our main analysis.

### A.1 Fubini extensions and the law of large numbers

We begin by defining the notion of *super-atomless* probability space.[1] Let $(\Lambda, \mathcal{L}, \lambda)$ be a probability space. For any $E \in \mathcal{L}$ such that $\lambda(E) > 0$, let $\mathcal{L}^E := \big\{E \cap E' : E' \in \mathcal{L}\big\}$

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[1] The following definition is by Podczeck (2009, 2010), which we find to be the most convenient for our purposes. However, equivalent definitions are provided in Hoover and Keisler (1984), who call such spaces $\aleph_1$-*atomless*, and Keisler and Sun (2009), who dubbed such spaces *rich*.

and $\lambda^E$ be the re-scaled measure from the restriction of $\lambda$ to $\mathcal{L}^E$. Let $\mathcal{L}^E_\lambda$ be the set of equivalence classes of sets in $\mathcal{L}^E$ such that $\lambda^E(E_1 \triangle E_2) = 0$, for $E_1, E_2 \in \mathcal{L}^E$.[2] We endow the space with metric $d^E : \mathcal{L}^E_\lambda \times \mathcal{L}^E_\lambda \to \mathbb{R}$ given by $d^E(E_1, E_2) := \lambda^E(E_1 \triangle E_2)$.

**Definition 1** (Super-atomless space)**.** A probability space $(\Lambda, \mathcal{L}, \lambda)$ is super-atomless if for any $E \in \mathcal{L}$ with $\lambda(E) > 0$, the space $(\mathcal{L}^E_\lambda, d^E)$ is non-separable.

Classical examples of super-atomless probability spaces include: $\{0,1\}^I$ with its usual measure when $I$ is an uncountable set; the product measure $[0,1]^I$, where each factor is endowed with Lebesgue measure and $I$ is uncountable;[3] subsets of these spaces with full outer measure when endowed with the subspace measure, or an atomless Loeb probability space. Furthermore, any atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure (see Podczeck, 2009).

Given a probability space $(\Lambda, \mathcal{L}, \lambda)$, a collection of random variables $(X_\alpha)_{\alpha \in \Lambda}$ is *essentially pairwise independent*, if for $(\lambda \otimes \lambda)$-almost every $(\alpha, \alpha') \in \Lambda \times \Lambda$, random variables $X_\alpha$ and $X_{\alpha'}$ are independent. For any set $\Omega$ and $E \subseteq (\Lambda \times \Omega)$, we denote its sections by $E_\alpha := \big\{\omega \in \Omega : (\alpha, \omega) \in E\big\}$ and $E_\omega := \big\{\alpha \in \Lambda : (\alpha, \omega) \in E\big\}$, for any $\alpha \in \Lambda$ and $\omega \in \Omega$. Similarly, for any function $f$ defined over $\lambda \times \Omega$, let $f_\alpha$ and $f_\omega$ denote the section of $f$ for a fixed $\alpha$, $\omega$, respectively. Consider the following definition.

**Definition 2** (Fubini extension)**.** The probability space $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a *Fubini extension* of the natural product of probability spaces $(\Lambda, \mathcal{L}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ if:

(i) $\mathcal{L} \boxtimes \mathcal{F}$ includes all sets from $\mathcal{L} \otimes \mathcal{F}$;

(ii) for an arbitrary set $E \in \mathcal{L} \boxtimes \mathcal{F}$ and $(\lambda \otimes P)$-almost every $(\alpha, \omega) \in \Lambda \times \Omega$, the sections $E_\alpha$ and $E_\omega$ are $\mathcal{F}$- and $\mathcal{L}$-measurable, respectively, while

$$(\lambda \boxtimes P)(E) \;=\; \int_\Omega \lambda(E_\omega) P(d\omega) \;=\; \int_\Lambda P(E_\alpha) \lambda(d\alpha).$$

A Fubini extension is *rich*, if there is a $(\mathcal{L} \boxtimes \mathcal{F})$-measurable function $X : \Lambda \times \Omega \to \mathbb{R}$ such that the random variables $(X_\alpha)_{\alpha \in \Lambda}$ is essentially pairwise independent and the random variable $X_\alpha$ has the uniform distribution over $[0,1]$, for $\lambda$-almost every $\alpha \in \Lambda$.

[2] We denote $E_1 \triangle E_2 := (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$.

[3] Indeed, Maharam's theorem shows that the measure algebra of every super-atomless probability spaces must correspond to the countable convex combination of such spaces. See Maharam (1942).

Existence of a rich Fubini extension is proven in Proposition 5.6 of Sun (2006), for $\Lambda = [0,1]$. Moreover, $\mathcal{L}$ can not be a collection of Borel subsets of $\Lambda$ (see Proposition 6.2 in Sun, 2006). In fact, Podczeck (2010) there exists a rich Fubini extension if and only if the space is super-atomless. Moreover, without loss, one may assume the random variables $(X_\alpha)_{\alpha\in\Lambda}$ to be independent, rather than pairwise-independent.

A *process* is a $(\mathcal{L} \boxtimes \mathcal{F})$-measurable function with values in a Polish space. For any process $f$ and set $E \in \mathcal{L}$ such that $\lambda(E) > 0$, we denote the restriction of $f$ to $E \times \Omega$ by $f^E$. Naturally, $\mathcal{L}^E \boxtimes \mathcal{F} := \{W \in \mathcal{L} \boxtimes \mathcal{F} : W \subseteq E \times \Omega\}$ and $(\lambda^E \boxtimes P)$ is a probability measure re-scaled from the restriction of $(\lambda \boxtimes P)$ to $(\mathcal{L}^E \boxtimes \mathcal{F})$. The following version of (exact) Law of Large Numbers is by Sun (2006).

**Proposition 1** (Law of Large Numbers)**.** *Suppose that $f$ is a process from a rich Fubini extension $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to some Polish space. Then, for all $E \in \mathcal{L}$ such that $\lambda(E) > 0$ and $P$-almost every $\omega \in \Omega$, we have $\lambda(f^E_\omega)^{-1} = (\lambda^E \boxtimes P)(f^E)^{-1}$.*[4]

## A.2 Lattices, chains, and fixed points

A *partial order* $\geq_X$ over a set $X$ is a reflexive, transitive, and antisymmetric binary relation. A partially ordered set, or a *poset*, is a pair $(X, \geq_X)$ consisting of a set $X$ and a partial order $\geq_X$. Whenever it causes no confusion, we denote $(X, \geq_X)$ with $X$.

For any $x$, $x' \in X$, their *infimum* (the greatest lower bound) is denoted by $x \wedge x'$, and their *supremum* (the least upper bound) by $x \vee x'$. The poset $X$ is a *lattice* if for any $x$, $x' \in X$ both $x \wedge x'$ and $x \vee x'$ belong to $X$. Set $A$ is a *sublattice* of $X$, if $A \subseteq X$ and it is a lattice with the induced order, with $x \wedge x'$ and $x \vee x'$ defined with $\geq_X$.[5]

For any subset $A$ of a poset $X$, we denote the *supremum* and *infimum* of $A$ by $\bigvee A$ and $\bigwedge A$, respectively.[6] A lattice $X$ is *complete*, if each both $\bigvee A$ and $\bigwedge A$ belong to $X$,

[4] Given the probability space $(\Lambda, \mathcal{L}, \lambda)$ and a measurable function $f : \Lambda \to Y$, we denote measure $\lambda f^{-1}(U) := \lambda(\{\alpha \in A : f(\alpha) \in U\})$, for any measurable subset $U$ of $Y$.

[5] A basic example of a lattice is the Euclidean space $\mathbb{R}^\ell$ endowed with the natural product order $\geq$, i.e., we have $x' \geq x$ if $x'_i \geq x_i$, for all $i = 1, \ldots, \ell$. In this case, we have $x \wedge x'$ and $x \vee x'$ are given by $(x \wedge x')_i = \min\{x_i, x'_i\}$ and $(x \vee x')_i = \max\{x_i, x'_i\}$, for all $i = 1, \ldots, \ell$.

[6] This is to say that, $\bigvee A$ is the least element of $X$ such that $\bigvee A \geq a$, for all $a \in A$. Clearly, by definition, we have $x \vee x' = \bigvee\{x, x'\}$. We define $\bigwedge A$ analogously.

for any $A \subseteq X$. We define a *complete sublattice* analogously.

A function $f : X \to \mathbb{R}$ over a lattice $X$ is *supermodular* in $x$ if $f(x \wedge x') + f(x \vee x') \geq f(x)+f(x')$. If $X$ and $T$ are posets, then function $f : X \times T \to \mathbb{R}$ has *increasing differences* in $(x,t)$ if, for any $x' \geq_X x$ and $t' \geq_T t$, we have $f(x',t') - f(x,t') \geq f(x',t) - f(x,t)$.

Finally, correspondence $\Gamma : X \times Y \to Z$, where $X$ and $Y$ are posets and $Z$ is a lattice, satisfies *strict complementarities* if for any $x' \geq x$, $y' \geq y$, $z \in \Gamma(x,y')$, and $z' \in \Gamma(x',y)$, we have $z \wedge z' \in \Gamma(x,y)$ and $z \vee z' \in \Gamma(x',y')$.

# B Auxiliary results

**Lemma B.1.** *Let $(\Xi, \geq)$ be a poset with its order topology, and $\{f_k\}$ be a sequence of increasing and monotone inf-preserving functions $f_k : \Xi \to \mathbb{R}$. Whenever $x_k \downarrow x$ in $\Xi$ and $f_k \downarrow f$ (pointwise), then $f_k(x_k) \to f(x)$.*

*Proof.* Let $n \in \mathbb{N}$. Since $\{f_k\}$ is decreasing sequence of increasing functions and $x_k \downarrow x$, then $k \geq n$ implies $f(x) \leq f_k(x_k) \leq f_k(x_n)$. Thus, we have $f(x) \leq \liminf_{k\to\infty} f_k(x_k) \leq \limsup_{k\to\infty} f_k(x_k) \leq f(x_n)$. To finish the proof, let $n \to \infty$. □

**Lemma B.2.** *Let $\{\nu_k\}$ be a sequence of probability measures on a Polish space $S$, and $\{h_k\}$ be a sequence of bounded, measurable functions $h_k : S \to \mathbb{R}$. If $\nu_k \downarrow \nu$ (stochastically and in weak topology) and $h_k \downarrow h$, then $\lim_{k\to\infty} \int h_k d\nu_k = \int h d\nu$.*

*Proof.* It is a consequence of Lemma B.1, where $\Xi$ is a space of bounded, measurable, real valued functions on $S$, and $f_k(x) := \int_S x(s)\nu_k(ds)$, $x_k(s) = h_k(s)$. □

**Lemma B.3.** *Let $S_1$, $S_2$ be topological spaces and $f : S_1 \times S_2 \mapsto \mathbb{R}$ be a continuous function. Let $\Gamma : S_1 \rightrightarrows S_2$ be a continuous, compact-valued correspondence and $\Gamma^*(x) := \arg\max_{y \in \Gamma(x)} f(x,y)$. If $x_k \to x$ in $S_1$, $y_k \to y$ in $S_2$, and $y_k \in \Gamma^*(x_k)$, then $y \in \Gamma^*(x)$.*

*Proof.* Let $y' \in \Gamma(x)$. By continuity of $\Gamma$, for any $k \in \mathbb{N}$, there is $y'_k \in \Gamma(x_k)$ such that $y'_k \to y'$. Since $y_k \in \Gamma^*(x_k)$, we have $f(x_k, y_k) \geq f(x_k, y'_k)$, for all $k \in \mathbb{N}$. By continuity of $f$, we have $f(x,y) \geq f(x,y')$. Since $y' \in \Gamma(x)$ is arbitrary, hence $y \in \Gamma^*(x)$. □

# C Omitted proofs

**Proof of Proposition 4.** This argument is analogous to Echenique (2005). Let $\bar{x}$ be the greatest element of $X$. Let $\mathscr{I}$ be a set of ordinal numbers with cardinality strictly greater than $X$. Define the following transfinite sequence with the initial element $x_0 = \bar{x}$ and $x_i = \bigwedge \{f(x_j) : j < i\}$, for $i \in \mathscr{I} \setminus \{0\}$. We claim that $\{x_i\}$ is a well-defined decreasing sequence. Clearly $x_1 = f(x_0) \leq x_0$. Suppose that $\{x_j\}_{j<i}$ is well-defined and decreasing for some $i$. Then $\{f(x_j)\}_{j<i}$ is a decreasing sequence, that has an infimum equal to $x_i$. Consequently $x_j$ is well defined and decreasing on $[0, i]$. By transfinite induction, the transfinite sequence $\{x_i\}_{i\in\mathscr{I}}$ is well defined and decreasing. Since $\mathscr{I}$ has the cardinality strictly greater than $X$, there is no one-to-one mapping between $\mathscr{I}$ and $X$. Consequently, take the least element $\bar{i}$ in $\{i \in \mathscr{I} : x_i = x_{i+1}\}$. Then $x_{\bar{i}} = x_{\bar{i}+1} = f(x_{\bar{i}})$, and $e^* := x_{\bar{i}}$ is a fixed point of $f$. To show that $e^* = \bigvee \{x \in X : f(x) \geq x\}$, set $\mathcal{X} := \{x \in X : f(x) \geq x\}$. Obviously, we have $e^* \in \mathcal{X}$. For any other $y \in \mathcal{X}$, we have $y \leq x_0$. Suppose there is $i \in \mathscr{I}$ such that $y \leq x_j$, for any $j < i$. Since $y \in \mathcal{X}$, by transfinite induction, we have $y \leq f(y) \leq f(x_j)$. Thus, $y \leq \bigwedge \{f(x_j) : j \leq i\}$ and $y \leq x_i$, for any $i \in \mathscr{I}$, including $\bar{i}$. □

**Proof of Theorem 1.** By Proposition 5.6 of Sun (2006) and Theorem 1 in Podczeck (2010) there is a probability space $(\Omega, \mathcal{F}, P)$ and a rich Fubini extension of a natural product space on $\Lambda \times \Omega$, denoted by $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Consequently, we can find a process $\eta : \Lambda \times \Omega \to [0, 1]$ such that the family $(\eta_\alpha)_{\alpha\in\Lambda}$ is essentially pairwise independent with the uniform distribution on $[0, 1]$. Define $(\eta_n)_{n\in\mathbb{N}}$ as a set of independent copies of $\eta$. Construct a sequence $(X_n)_{n=1}^{\infty}$ satisfying theses (i)–(iii). Let $(I, \mathcal{I}, \iota)$ be the standard interval $I = [0, 1]$, with Borel sets $\mathcal{I}$, and the Lebesgue measure $\iota$. For any $\mu \in \mathcal{M}$, there is a $(\mathcal{I} \otimes \mathcal{T} \otimes \mathcal{A})$-measurable function $G^\mu : I \times T \times A \mapsto T$ such that

$$\iota\big(G^\mu_{(t,a)}\big)^{-1}(Z) \;=\; \iota\Big(\big\{l \in I : G^\mu(l, t, a) \in Z\big\}\Big) \;=\; q(Z|t, a, \mu),$$

for any $Z \in \mathcal{T}$.[7] For any initial distribution $\tau_1 \in \mathcal{M}_T$, there exists a $T$-valued $(I \otimes \mathcal{T})$-measurable function $\tilde{G}$ such that $\tau_0 = \iota\tilde{G}^{-1}$.[8] Put $X_1 := \tilde{G}(\eta_1)$. Having the initial random

[7] For example, see Lemma A5 in Sun (2006).
[8] Again, see Lemma A5 in Sun (2006).

variable $X_1$, define the following process $X_{n+1} = G^{\mu_n}(\eta_{n+1}, K_n)$, for $n > 1$, where $K_n := \big(X_n, \sigma(X_n, \tau_n)\big)$, $\tau_n := (\lambda \boxtimes P)X_n^{-1}$, and $\mu_n := (\lambda \boxtimes P)K_n^{-1}$. As usual, put $(K_n)_\alpha(\omega) := K_n(\alpha, \omega)$ for $(\alpha, \omega) \in \Lambda \times \Omega$. Let $\mathcal{S}_n$ by the sigma field generated by $\{\eta_k : k \le n\}$. By definition of $X_1$ and $X_{n+1}$, we conclude that $X_n$ is $\mathcal{S}_n$-measurable. Hence, $(X_n)_\alpha$ and $(\eta_{n+1})_\alpha$ are independent, for $\lambda$-almost every $\alpha \in \Lambda$. We show that (i)–(ii) are satisfied by induction on $n$. For $n = 1$, the claim holds by essential independence of $\eta_1$ and $X_1$. Moreover, by Proposition 1, for $P$-almost every $\omega \in \Omega$ the sampling distribution $\lambda(X_1)_\omega^{-1}$ of $X_1$, i.e., satisfies $\lambda(X_1)_\omega^{-1} = (\lambda \boxtimes P)X_1^{-1} = \tau$. Again by Proposition 1, for $P$-almost all $\omega \in \Omega$, we have $\lambda(K_1)_\omega^{-1} = (\lambda \boxtimes P)K_1^{-1} := \mu_1$. Hence, (ii) is satisfied for $n = 1$. Suppose that both (i) and (ii) hold, for some $n \ge 1$. Observe that $\big((\eta_{n+1})_\alpha, (X_n)_\alpha\big)_{\alpha\in\Lambda}$ is a family $(\lambda \otimes \lambda)$-almost everywhere pairwise conditionally independent random variables. This follows from induction hypothesis for $(X_n)_\alpha$, and the previous observation that random variables $(X_n)_\alpha$ and $(\eta_{n+1})_\alpha$ are independent $\lambda$-almost surely. Hence, by construction of $X_{n+1}$, the family $\big((X_{n+1})_\alpha\big)_{\alpha\in\Lambda}$ is $(\lambda\otimes\lambda)$-almost surely pairwise conditionally independent. Hence the property (i) is satisfied for $(n+1)$. By Proposition 1, we obtain (ii) for $(n+1)$. Thus, (i) and (ii) hold for all $n \ge 1$. To show (iii), let $(\mathcal{S}_n)_\alpha$ be the sigma field generated by $\{(\eta_k)_\alpha : k \le n\}$ and similarly $(\Sigma_n)_\alpha$ by $\{(X_k)_\alpha : k \le n\}$. By definition of $X_n$ and $(\Sigma_n)_\alpha$ we conclude that $\sigma\big((X_n)_\alpha\big) \subseteq (\Sigma_n)_\alpha \subset (\mathcal{S}_n)_\alpha$. Let $E$ be the standard expectation with respect to $P$. Hence the conditional distribution of $(X_{n+1})_\alpha$ with respect to $(\Sigma_n)_\alpha$ satisfies

$$\begin{aligned} P\big((X_{n+1})_\alpha \in Z\big|(\Sigma_n)_\alpha\big) &= E\Big[P\big((X_{n+1})_\alpha \in Z\big|(\mathcal{S}_n)_\alpha\big)|(\Sigma_n)_\alpha\Big] \\ &= E\Big[P\big(G^{\mu_n}((\eta_{n+1})_\alpha, (K_n)_\alpha) \in Z\big|(\mathcal{S}_n)_\alpha\big)|(\Sigma_n)_\alpha\Big] \\ &= E\Big[q\big(Z|(K_n)_\alpha, \mu_n\big)\big|(\Sigma_n)_\alpha\Big] = q\Big(Z\big|(X_n)_\alpha, \sigma^*\big((X_n)_\alpha, \tau_n\big), \mu_n\Big) \end{aligned}$$

for $\lambda$-almost all $\alpha \in \Lambda$ and all $Z \in \mathcal{T}$, where the last equality follows from independence of $(\eta_{n+1})_\alpha$ and $(X_n)_\alpha$. Hence, property (iii) is satisfied. □

**Proof of Lemma 1.** Suppose that $v_n \in \mathcal{V}$, for all $n \in \mathbb{N}$, and $v_n \to v$. Furthermore, let $(\mu_k)$ and $(\Phi_k)$ be decreasing sequences in $\mathcal{M}$ and $\mathcal{D}$, respectively, such that $\mu_k \to \mu$ (weakly) and $\Phi_k \to \Phi$ (pointwise). Take any $t \in T$ and $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$ and $n \geq n_0$, we have

$$|v(t,\mu_k,\Phi_k) - v(t,\mu,\Phi)| \;\leq\; |v(t,\mu_k,\Phi_k) - v_n(t,\mu_k,\Phi_k)| + |v_n(t,\mu_k,\Phi_k) - v_n(t,\mu,\Phi)| \\ + |v_n(t,\mu,\Phi) - v(t,\mu,\Phi)| \;\leq\; \frac{2}{3}\epsilon + |v_n(t,\mu_k,\Phi_k) - v_n(t,\mu,\Phi)| \quad (1)$$

Take any $n \in \mathbb{N}$ satisfying (1). Therefore, since $v_n \in \mathcal{V}$, for large enough $k$, we obtain $\big|v_n(t,\mu_k,\Phi_n) - v_n(t,\mu,\Phi)\big| \leq \epsilon/3$. Given (1), this implies $|v(t,\mu_k,\Phi_k) - v(t,\mu,\Phi)| < \epsilon$, for large $k$. Hence $v$ is monotonically sup- and inf-preserving. Thus, $v \in \mathcal{V}$. □

**Continuation of the proof to Lemma 4.** We prove (vi). Using Assumption 2, definition of $\mathcal{V}$, and Lemma 4, one can show that $F$ is a Carathéodory function in $(t,a)$, i.e., measurable in $t$ and continuous in $a$. Hence, by Assumption 1 and Measurable Maximum Theorem (Theorem 18.19 in Aliprantis and Border, 2006) the correspondence $\Gamma(t,\mu;v,\Phi)$ is measurable in $t$, hence, weakly measurable.[9] For each $j = 1,2,\ldots,k$, the function $\pi_j(t) := \max_{a\in\Gamma(t,\mu;v,\Phi)} a_j$ is measurable (again, by Measurable Maximum Theorem). Thus, $t \to \overline{\gamma}(t,\mu,\Phi;v) = \big(\pi_1(t),\pi_2(t),\ldots,\pi_k(t)\big)$ is measurable. □

**Proof of Lemma 8.** Suppose that $f : T \times A \mapsto \mathbb{R}$ belongs to the space of bounded and continuous function $C(T\times A)$. Clearly, we have $(1/N)f\big(\xi^N(\omega),\eta^N(\omega)\big) \to 0$, for all $\omega \in \Omega$. By the standard Kolmogorov Law of Large Numbers Theorem, we obtain

$$\lim_{N\to\infty} \frac{1}{N-1}\sum_{l\neq j} f\big(\tilde{T}_l,\sigma_n(\tilde{T}_l)\big) \;=\; \int_T f\big(t,\sigma_n(t)\big)\tau_n(dt) = \int_{T\times A} f(t,a)(\tau_n \star \sigma_n)(dt\times da),$$

$\mathbb{P}$-almost surely. Consequently, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\lim_{N\to\infty}\int_{T\times A} f(t,a)\hat{\mu}_n^N\big((\tilde{T}_{-j},\xi^N),\eta^N\big) \;=\; \int_{T\times A} f(t,a)(\tau_n\star\sigma_n)(dt\times da). \quad (2)$$

Let $\mathbf{F}$ be a countable, dense set in $C(T\times A)$. Let $\tilde{\Omega} \subseteq \Omega$ be such that any element of $\mathbf{F}$ obeys (2). Then, $\mathbb{P}(\tilde{\Omega}) = 1$. We claim that (2) holds for any $f \in C(T\times A)$ whenever $\omega \in \tilde{\Omega}$. Take any $\epsilon > 0$. Since $\mathbf{F}$ is dense in $C(T\times A)$, take $f_0 \in \mathbf{F}$ such that $\|f - f_0\|_\infty < \frac{\epsilon}{3}$. Then, $\int_{T\times A}|f(t,a) - f_0(t,a)|\hat{\mu}_n^N\big((\tilde{T}_{-j},\xi^N),\eta^N\big)(dt\times da) \leq \frac{\epsilon}{3}$ as well as

[9] See, e.g., Lemma 18.2 in Aliprantis and Border (2006).

$\int_{T\times A}|f(t,a)-f_0(t,a)|(\tau_n\star\sigma_n)(dt\times da)\leq\frac{\epsilon}{3}$. This implies

$$\left|\int_{T\times A} f(t,a)\hat{\mu}_n^N\big((\tilde{T}_{-j},\xi^N),\eta^N\big)(dt\times da)-\int_{T\times A} f(t,a)(\tau_n\star\sigma_n)(dt\times da)\right|$$
$$\leq \int_{T\times A}|f(t,a)-f_0(t,a)|\hat{\mu}_n^N\big((\tilde{T}_{-j},\xi^N),\eta^N\big)(dt\times da)$$
$$+\int_{T\times A}|f(t,a)-f_0(t,a)|(\tau_n\star\sigma_n)(dt\times da)$$
$$+\left|\int_{T\times A} f_0(t,a)\hat{\mu}_n^N\big((\tilde{T}_{-j},\xi^N),\eta^N\big)(dt\times da)-\int_{T\times A} f_0(t,a)(\tau_n\star\sigma_n)(dt\times da)\right|\leq$$
$$\frac{2}{3}\epsilon+\left|\int_{T\times A} f_0(t,a)\hat{\mu}_n^N\big((\tilde{T}_{-j},\xi^N),\eta^N\big)(dt\times da)-\int_{T\times A} f_0(t,a)(\tau_n\star\sigma_n)(dt\times da)\right|. \quad (3)$$

Since $\omega\in\tilde{\Omega}$, there exists an integer $N_0$ such that, for any $N>N_0$,

$$\left|\int_{T\times A} f_0(t,a)\hat{\mu}_n^N((\tilde{T}_{-j},\xi^N),\eta^N)(dt\times da)-\int_{T\times A} f_0(t,a)\tau_n\star\sigma_n(dt\times da)\right| < \frac{\epsilon}{3}. \quad (4)$$

Combining (3) and (4), for $N>N_0$, we have

$$\left|\int_{T\times A} f(t,a)\hat{\mu}_n^N((\tilde{T}_{-j},\xi^N),\eta^N)(dt\times da)-\int_{T\times A} f(t,a)\tau_n\star\sigma_n(dt\times da)\right| < \epsilon. \quad (5)$$

Since $\epsilon>0$, the (5) implies that (2) holds for $f$ and $\omega\in\tilde{\Omega}$. Given that $f\in C(T\times A)$ is arbitrary and $\mathbb{P}(\tilde{\Omega})=1$, we have $\hat{\mu}_n^N\big((\tilde{T}_{-j},\xi^N),\eta^N\big)\to(\tau_n\star\sigma_n)$, almost surely. □

Recall that $\tilde{v}_1^N(t):=\sup_{\pi\in\Sigma}\mathcal{R}(\sigma^{-j},\pi)(t)$. Then, the Bellman equation for optimal value $\tilde{v}_n^N$, updated for any $n\in\mathbb{N}$, take the form of

$$\tilde{v}_n^N(t) = \max_{a\in\tilde{A}(t,\tau_n)}\left\{(1-\beta)r_n^N(t,a)+\beta\int_T\tilde{v}_{n+1}^N(t')q_n^N(ds'|t,a)\right\}. \quad (6)$$

Let $\mathbf{C}$ be the set of continuous real-valued functions on $T$, uniformly bounded by $\bar{r}$, which is a closed subset of a Banach space. The metric in product space $\mathcal{C}:=\mathbf{C}^\infty$ is embedded in the natural Banach space the following norm: For $v=(v_n)_{n\in\mathbb{N}}$, define

$$\|v\|^\zeta := \sum_{n=1}^{\infty}\frac{1}{\zeta^{n-1}}\sup_{t\in T}|v_n(t)|,$$

where $\zeta\in(0,1/\beta)$ is a fixed value. Clearly, $v^N\to v$ in $\|\cdot\|^\zeta$ if and only if $v_n^N\to v_n$, for any $n\in\mathbb{N}$. Let $v\in\mathcal{C}$, $t\in T$, and $B^N(v)(t):=\big(B_n^N(v)(t)\big)_{n\in\mathbb{N}}$ where

$$B_n^N(v)(t) := \max_{a\in\tilde{A}(t,\tau_n)}\left\{(1-\beta)r_n^N(t,a)+\beta\int_T v_{n+1}(t')q_n^N(dt'|t,a)\right\}.$$

Similarly, define $\mathcal{B}^N(v)(t) := \big(\mathcal{B}^N_n(v)(t)\big)_{n\in\mathbb{N}}$ where

$$\mathcal{B}^N_n(v)(t) \;:=\; (1-\beta)r^N_n\big(t,\sigma_n(t)\big) + \beta\int_T v_{n+1}(t')q^N_n\big(dt'|t,\sigma_n(t)\big).$$

For $v \in \mathcal{C}$, let $B^\infty(v)(t) := (B^\infty_n(v)(t))_{n\in\mathbb{N}}$ where

$$B^\infty_n(v) \;:=\; \max_{a\in\tilde{A}}\left\{(1-\beta)r_n(t,a) + \beta\int_T v_{n+1}(t')q_n(dt'|t,a)\right\},$$

where $r_n(t,a) := r(t,a,\tau_n \star \sigma_n)$ and $q_n(\cdot|t,a) := q(\cdot|t,a,\tau_n\star\sigma_n)$, for $(t,a) \in Gr\big(\tilde{A}(\cdot,\tau_n)\big)$. Similarly define $\mathcal{B}^\infty(v)(t) := \big(\mathcal{B}^\infty_n(v)(t)\big)_{n\in\mathbb{N}}$ where

$$\mathcal{B}^\infty_n(v)(t)' := \;(1-\beta)r_n\big(t,\sigma_n(t)\big) + \beta\int_T v_{n+1}(t')q_n\big(dt'|t,\sigma_n(t)\big).$$

Now we prove basic properties of $B^N$ and $B^\infty$.

**Lemma C.1.** *Let $\sigma$ be a Borel measurable function. Then,*

*(i) mappings $B^N, \mathcal{B}^N_n$, $B^\infty$, and $\mathcal{B}^\infty_n$ map $\mathcal{C}$ into itself;*

*(ii) $B^N$, $\mathcal{B}^N_n$, $B^\infty$, and $\mathcal{B}^N_n$ are $\beta\zeta$-contraction mappings on $\mathcal{C}$;*

*(iii) if $v^N \to v$ in $\mathcal{C}$, then $B^N(v^N) \to B^\infty(v)$ and $\mathcal{B}^N(v^N) \to \mathcal{B}^\infty(v)$ in $\mathcal{C}$;*

*(iv) we have $\|\tilde{v}^N - \tilde{v}^\infty\|_\infty \to 0$, where $\tilde{v}^N$, $\tilde{v}^\infty$ in $\mathbf{C}$ is a fixed point of $B^N$, $B^\infty$;*

*(v) we have $\|\check{v}^N - \check{v}^\infty\|_\infty \to 0$, where $\check{v}^N$, $\check{v}^\infty$ in $\mathbf{C}$ is a fixed point of $\mathcal{B}^N$, $\mathcal{B}^\infty$.*

*Proof.* In order to prove (i), take any $v \in \mathcal{C}$. Given Assumptions 6, for any $n$ and $N$, the following functions $\Pi^N_n(t,a,v) = (1-\beta)r^N_n(t,a) + \beta\int_T v_{n+1}(t')q^N_n(dt'|t,a)$ and $\Pi^\infty_n(t,a,v) = (1-\beta)r_n(t,a) + \beta\int_T v_{n+1}(t')q_n(dt'|t,a)$, are both continuous in $(t,a)$. Since $B^N_n(v)(t) = \max_{a\in\tilde{A}(t,\tau_n)}\Pi^N_n(t,a,v)$ and $B^\infty_n(v)(t) = \max_{a\in\tilde{A}(t,\tau_n)}\Pi^\infty_n(t,a,v)$, statement (i) follows immediately from Berge Maximum Theorem. We show (ii). It is routine to verify $\|B^N_n(v) - B^N_n(w)\|_\infty \le \beta||v_{n+1} - w_{n+1}||_\infty$, for $v,w \in \mathcal{C}$. By dividing both sides by $\zeta^{n-1}$ and summing over $n$, we obtain

$$\|B^N_n(v) - B^N_n(w)\|^\zeta \;=\; \sum_{n=1}^\infty \frac{\|B^N_n(v) - B^N_n(w)\|_\infty}{\zeta^{n-1}} \;\le\; \beta\zeta\sum_{n=1}^\infty \|v_n - w_n\|_\infty \;=\; \beta\zeta\|v_n - w_n\|_\infty.$$

An analogous argument can be applied to prove the property for $B^\infty$. In order to show (iii), suppose that $v^N \to v$ in $(\mathcal{C}, ||\cdot||_\infty)$ and $(t^N, a^N) \to (t,a)$, for $(t^N,a^N) \in \tilde{A}(t^N,\tau_n)$.

We claim that $\Pi_n^N(t^N, a^N, v^N) \to \Pi_n^\infty(t, a, v)$. By Lemma 8and Assumption 6 we have that $r_n^N(t^N, a^N) \to r_n(t, a)$ and $q_n^N(\cdot|t^N, a^N) \to q_n(\cdot|t, a)$. This proves the claim. Furthermore, by (i), there is $t^N$ such that

$$\sup_{t \in T} |B_n^N(v^N)(t) - B_n^\infty(v)(t)| \;=\; \|B_n^N(v^N)(t^N) - B_n^\infty(v)(t^N)\|.$$

Without loss of generality suppose that $t^N \to t$. Combining the definition of $r_n$ and $q_n$, Lemma 8, and the above claim, it follows that the right hand-side above tends to 0. Hence, $\|B^N(v^N) - B^\infty(v^\infty)\|^\zeta \to 0$. Finally, to prove (iv), observe that

$$\begin{aligned} \|\tilde{v}^N - \tilde{v}^\infty\|^\kappa &= \|B^N(\tilde{v}^N) - B^\infty(\tilde{v}^\infty)\|^\zeta \\ &\leq \|B^N(\tilde{v}^N) - B^N(\tilde{v}^\infty)\|^\zeta + \|B^N(\tilde{v}^\infty) - B^\infty(\tilde{v}^\infty)\|^\zeta \\ &\leq \beta\zeta||\tilde{v}^N - \tilde{v}^\infty||^\zeta + ||B^N(\tilde{v}^\infty) - B^\infty(\tilde{v}^\infty)||^\zeta, \end{aligned}$$

where the last inequality is by (ii). Thus, $\|\tilde{v}^N - \tilde{v}^\infty\|^\kappa \leq \|B^N(\tilde{v}^\infty) - B^\infty(\tilde{v}^\infty)\|^\zeta/(1-\beta\zeta)$. To finish the proof, we only take $N \to \infty$, since by (iii) the right hand-side above tends to 0. The proof of (v) is analogous to (iv). □

**Lemma C.2.** *Consider MDP, where $(\tau_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ are implied by sequences of distribution on types-policies for some MSDE $(\mu^*, \Phi^*)$. Then, the sequences of value functions $\bar{v}$ for $(\mu^*, \Phi^*)$ is a common fixed point of $B^\infty$ and $\mathcal{B}^\infty$. As a result, $\bar{v} = \tilde{v}^\infty = \check{v}^\infty$.*

*Proof.* By Lemma C.1, it follows that $B^\infty$ and $\mathcal{B}^\infty$ are both contractions on $\mathcal{C}$. Hence, we only need to show $\bar{v}$ is the fixed point of $B^\infty$ and $\mathcal{B}^\infty$. By definition of $\bar{v}$, $v^*$, $\mu_n$, and $\tau_n$, for any $t \in T$, we have $\bar{v}_n(t) = v^*(t, \tau_n, \Phi^*)$ and

$$\begin{aligned} \bar{v}_n(t) &= \max_{a\in\tilde{A}(t,\tau_n)} \left\{(1-\beta)r(t,a,\mu_n) + \beta\int_T v^*(t',\mu_{n+1},\Phi^*)q(dt'|t,a,\mu_n)\right\} \\ &= \max_{a\in\tilde{A}(t,\tau_n)} \left\{(1-\beta)r(t,a,\mu_n) + \beta\int_T \bar{v}_{n+1}(t')q(dt'|t,a,\mu_n)\right\} \\ &= \max_{a\in\tilde{A}(t,\tau_n)} \left\{(1-\beta)r(t,a,\tau_n\star\sigma_n) + \beta\int_T \bar{v}_{n+1}(t')q(dt'|t,a,\tau_n\star\sigma_n)\right\} = B_n^\infty(\bar{v}_{n+1})(t). \end{aligned}$$

Hence $\bar{v} = B^\infty(\bar{v})$ and by uniqueness of the fixed point of $B^\infty$, $\bar{v} = \tilde{v}^\infty$. By the same argument we obtain $\bar{v} = \mathcal{B}^\infty(\bar{v})$, and $\bar{v} = \check{v}$. □

**Proof of Theorem 4.** Let $\epsilon > 0$ and $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ be a sequential policy function associated with $(\mu^*, \Phi^*)$. If player $j$ unilaterally deviates from $\sigma$ using $\pi$ then, for any $t \in T$, we have $\mathcal{R}^N((\sigma)^{-j}, \pi)(t) - \check{v}_1^N(t) \leq \tilde{v}_1^N(t_1^j) - \check{v}_1^N(t_1^j) \leq \|\tilde{v}_1^N - \check{v}_1^N\|_\infty$. By Lemma C.1, $\tilde{v}_1^N \to v_1^\infty$ and $\check{v}_1^N \to \check{v}_1^\infty$. Since the policy is $\sigma = \sigma^*$ and the initial state is $\tau_1 = \tau^*$, then $\check{v}_1^\infty = v^\infty$, by Lemma C.2. Thus, for large enough $N$, $\|\tilde{v}_1^N - \check{v}_1^N\|_\infty < \epsilon$. □

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