Is APR a Robust Measure of the Cost of Consumer Credit?

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Abstract

Most people have consumer loans during their lives, making it important that consumer credit legislation is effective. Legislation in many countries is based on the US Truth-in-Lending Act (TILA). Conventional financial analysis underlying the TILA argues the annual percentage rate (APR) is the best measure of credit cost, and therefore the legislation focuses on APR as a key policy variable. APR is a complicated concept, so the legislation is complex and research shows consumers find APR confusing. This article uses a new interpretation of the time value of money equation to challenge conventional analysis. A mathematical argument demonstrates that the simple rate of interest is a more effective policy variable than APR.

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Since the majority of people are likely to have consumer credit during their lives, and most borrowings will be in jurisdictions having consumer credit legislation, it is important that this legislation is effective because it can impact the welfare of hundreds of millions of people.

One of the earliest pieces of consumer credit legislation is the US Consumer Credit Protection Act of 1968, also known as the Truth-in-Lending Act (TILA). Its essence is captured in these words from the Office of the Comptroller of the Currency:

> *'The TILA is intended to ensure that credit terms are disclosed in a meaningful way so consumers can compare credit terms more readily and knowledgeably. ... The finance charge and APR, more than any other disclosures, enable consumers to understand the cost of credit and to comparison shop for credit.'*
> Comptroller's Handbook, Truth in Lending (2010, p. 6 and p. 15)

Credit legislation in other countries of the world is modelled on the TILA, placing the same emphasis on disclosure of APR and the finance charge (FC) to achieve similar objectives. Pre- and post-TILA comparisons of consumer awareness and understanding of the cost of credit suggest the legislation has resulted in greater awareness. However, there is evidence that shortcomings remain with respect to understanding, due to the inherent complexity of the subject.

The insistence on calculation and disclosure of APR is a major factor in this complexity. The need for a precise definition of this particular rate of interest results in legislation that is extraordinary because it contains mathematics explaining the time value of money (TVM) equation in which the rate is embedded. Credit legislation is not user-friendly, and evidence mounts that most consumers cannot compare credit terms '*readily and knowledgeably.*' See Appendix A for a summary of consumer credit legislation in the US, UK, and EU, and Appendix B for a summary of research on public awareness and understanding of the cost of credit.

This article argues that legislative emphasis on APR is misplaced. New analysis of the TVM equation leads to a hitherto unknown mathematical relationship between APR and the simple rate of interest (a derivative of the finance charge). The relationship demonstrates that banks are able to restructure the typical consumer loan in order to maintain or even increase profitability in the face of competitive pressure on APR. The analysis further demonstrates that the simple rate of interest is a more effective policy variable than APR, containing more information and inhibiting countervailing action by banks. Consumer credit legislation based on the simple rate of interest will also be more effective to the extent that it is more understandable.

2 Conventional analysis of APR and the TVM equation

The EU's Consumer Credit Directive of 2008 serves to present the analysis. An explanation of the APR equation is in Annex 1 of the Directive. The following extract illustrates the challenge presented by the legislation.

> *'The basic equation, which establishes the annual percentage rate of charge (APR), equates, on an annual basis, the total present value of drawdowns on the one hand and the total present value of repayments and payments of charges on the other hand, i.e.,*
>
> $$\sum_{k=1}^{m} C_k (1+X)^{-t_k} = \sum_{l=1}^{m^*} D_l (1+X)^{-s_l} \quad \textit{where}$$
>
> *- X is the APR,*
> *- m is the last drawdown,*
> *- k is the number of a drawdown, thus* $1 \leq k \leq m$*,*
> *-* C_k *is the amount of drawdown k,*
> *-* t_k *is the interval, expressed in years and fractions of a year, between the date of the first drawdown and the date of each subsequent drawdown,* t_1*=0,*
> *-* m^* *is the number of the last payment or payment of charges,*
> *- l is the number of a repayment or payment of charges,*
> *-* D_l *is the amount of a repayment or payment of charges,*
> *-* s_l *is the interval, expressed in years and fractions of a year, between the date of the first drawdown and the date of each repayment or payment of charges.'*
>
> EU Consumer Credit Directive (2208/48/EC, Annex 1)

Some simplifying assumptions are applied to this '*basic equation*' in order to develop the analysis. Assume a retail loan in which one drawdown of the principal amount is made in period zero, C_0, followed by a series of repayments D_t in periods t = 1 to n. These repayments (covering principal, interest and other charges) are not necessarily equal in amount. When the basic equation is modified to reflect these assumptions the result is Eq. (1).

$$C_0 = \sum_{t=1}^{n} \frac{D_t}{(1+X)^t} \tag{1}$$

The finance charge is the total amount repaid less the total amount advanced. In the context of the loan in Eq. (1), FC is the undiscounted sum of all repayments less the principal amount.[1] Eq. (2) captures this definition.

$$FC = \sum_{t=1}^{n} D_t - C_0 \tag{2}$$

Along with APR, the FC must be disclosed to the consumer in any credit agreement. The legislation defines the list of charges for inclusion in the FC, i.e., it defines the 'fully loaded' list of items to be covered by the repayments, D_t, in addition to the repayment of principal. The primary item in the list is interest paid, because it is usually the largest single charge. The total money charge for interest is based on a contract interest rate (CIR), different from APR. The list includes other charges, for example, arrangement fees. When all charges are included, a new interest rate is 'backed out' of the TVM equation, a rate higher than CIR because of the presence of the fees. This higher, 'inclusive-of-all-charges' interest rate is X = APR in Eq. (1).

The conventional interpretation of APR is expressed in the following quote from the EU technical document produced in support of the 2008 Consumer Credit Directive.

> *'... What distinguishes the APR from other cost measures is that it puts the credit, its costs and time together, thus recognizing that these three elements are relevant in determining a comparable and uniform measure of the cost of the credit. In this way, the APR presents significant advantages over other measures of cost.*
>
> *... Compared to a simple rate, ...* [APR] *... has in its favour the primacy of compound interest in finance and economics, a greater interpretability and a higher adaptation to situations where the amount of the credit varies, and the payments might adopt different and diverse patterns, as happens in consumer credit agreements.*
> Directorate General for Health & Consumer Protection (2009, p. 8) [2]

It is argued below that the conventional interpretation is not correct. The relative merits of the simple rate and APR are most effectively compared within a single equation containing both rates of interest. To the author's knowledge no equation containing both rates has been identified

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[1] In the EU the total amount repaid is known as the total charge for credit (TCC); the TCC rather than the FC is quoted to consumers alongside the APR.

[2] The source for this quote is chosen because it is a recent, well-written, and detailed statement of the conventional analysis of APR. Other documents, chosen from similar literature from other jurisdictions, could serve as an example.

in the financial literature. The remainder of this article derives and analyses such an equation. The equation demonstrates that the connection between APR and the simple rate is more subtle and powerful than conventional financial theory allows and that the simple rate of interest is a superior policy alternative to APR.

3. A deeper analysis of APR and the TVM equation

The FC normalized by loan amount and duration is more meaningful than the FC alone when comparing loans of different amounts and durations. When both sides of the expression for the FC are divided by the principal amount, C_0, the result is Eq. (3) defining 'FC-per-dollar-borrowed.'

$$\frac{FC}{C_0} = \frac{\sum_{t=1}^{n} D_t - C_0}{C_0} \tag{3}$$

When FC per dollar borrowed is further normalized for the duration of the loan, by dividing both sides of Eq. (3) by n, the result is Eq. (4) defining the simple rate of interest, S.

$$S = \frac{FC}{nC_0} = \frac{\sum_{t=1}^{n} D_t - C_0}{nC_0} \tag{4}$$

The simple rate is more easily understood than APR, and therefore in retail sales the rate is often used to explain the cost of credit to consumers. However, legislation in most countries deters use of the simple rate as a measure of credit cost for the reasons described earlier. Deterrence is achieved in two ways: first, disclosure of APR in all jurisdictions is compulsory while disclosure of the simple rate is not; and second, where both rates are disclosed, legislation insists APR is given pre-eminence over the simple rate in advertisements and documentation.

As suggested already, official preference for APR over the simple rate can be questioned on the grounds that conventional financial opinion does not recognize the true connection between them. It is to this connection that we turn now. The essence of the argument depends on the mathematics of the TVM equation.

Aleksandrov, Kolmogorov and Lavrent'ev (1969) summarize the fundamental theorem of algebra and its logical implications about factorization of polynomials.

‘*If we accept without proof the so-called fundamental theorem of algebra that every equation f(x)=0, where* $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$ *is a polynomial in x of given degree n and the coefficients* $a_1, a_2, \ldots, a_n$ *are given real or complex numbers, has at least one real or complex root, and take into consideration that all computations with complex numbers are carried out with the same rules as with rational numbers, then it is easy to show that the polynomial f(x) can be represented (and in only one way) as a product of first-degree factors* $f(x) = (x-a)(x-b)\ldots(x-l)$ *where a, b,…,l are real or complex numbers.’* Aleksandrov, Kolmogorov and Lavrent’ev (1969, Vol.1, pp. 271-272)

The loan equation, Eq. (1), is a polynomial rearranging and factorizing into n factors of the form $[(1+X)-(1+X_j)]$, each factor containing a root $(1+X_j)$, and each root containing an APR X_j. Eq. (5) is the result.

$$\begin{aligned} C_0(1+X)^n - D_1(1+X)^{n-1} - \ldots - D_{n-1}(1+X) - D_n = \\ C_0[(1+X)-(1+X_1)][(1+X)-(1+X_2)]\ldots[(1+X)-(1+X_n)] \end{aligned} \tag{5}$$

Eq. (5) demonstrates that any n^{th} order loan equation has n solutions for APR. One of these solutions is the orthodox solution, ‘orthodox’ referring to the APR calculated by a financial calculator or spread-sheet given values for C_0 and D_t for t = 1 to n. In this article we identify the orthodox value with the root $(1+X_1)$. The remaining roots $(1+X_j)$ for j = 2 to n are labelled ‘unorthodox’ because they are negative or complex numbers, the complex numbers having an imaginary component measured in units of $i = \sqrt{-1}$. These unorthodox solutions to the TVM equation have long been ignored; possibly because of pronouncements about them by some economists. Here is an early example from Boulding (1936):

‘Now it is true that an equation of the nth degree has n roots of one sort or another … Nevertheless, in the type of payments series with which we are most likely to be concerned, it is extremely probable that all but one of these roots will be either negative or imaginary, in which case they will have no economic significance.’

With the exception of Dorfman (1981), exploration of the unorthodox solutions to the TVM equation is a twenty-first century phenomenon, and then only in a few works. Examples from capital budgeting are Hazen (2003), Osborne (2010), and Pierru (2010); an example from bond mathematics is Osborne (2005). The unorthodox solutions play a role in this article, and therefore Eq. (5) requires further discussion.

The variable (1+X) in Eq. (5) can roam over the complex plane. In this analysis several different salient values of (1+X) are inserted into Eq. (5) to demonstrate the equation's use and meaning.

In the first example, (1+X) takes the value (1+X_1) containing the orthodox APR. The right-hand side of Eq. (5) collapses to zero and the equation reverts to the conventional TVM equation, Eq. (6), which is a special case of Eq. (1).

$$C_0 = \sum_{t=1}^{n} \frac{D_t}{(1+X_1)^t} \tag{6}$$

This first example highlights the fact that (1+X) in Eq. (1) may take the value of any of the n roots (1+X_j) in Eq. (5), implying that n versions of Eq. (6) exist, each version containing an APR= X_j from j = 1 to n, all versions holding true simultaneously. This fact raises a question. To which version of Eq. (6), i.e. to which APR solving Eq. (1), does consumer credit legislation apply? Is it the orthodox APR X_1 defined earlier as the rate given by a financial calculator or spreadsheet? Or does it also apply to one or more of the unorthodox values? An answer to this question is deferred to the next section following further discussion of Eq. (5).

In the second example, X in Eq. (5) takes the value zero. Eq. (5) then reduces to Eq. (7).

$$\frac{\sum_{t=1}^{n} D_t - C_0}{C_0} = -\prod_{j=1}^{n} (-X_j) \tag{7}$$

The expression for 'FC-per-dollar-borrowed,' Eq. (3), substitutes into the left-hand side of Eq. (7). On the right-hand side of Eq. (7), the orthodox APR, X_1, is taken outside the product, its associated minus sign negating the overall minus sign. The result is Eq. (8).

$$\frac{FC}{C_0} = -\prod_{j=1}^{n} (-X_j) = \prod_{j=2}^{n} (-X_j) X_1 \tag{8}$$

Eq. (8) demonstrates that FC per dollar borrowed is the product of all possible APRs solving Eq. (1); in other words, it is a multiple of the orthodox APR X_1, where the multiple is the product of the (n-1) unorthodox APRs.

When the numerator and denominator on the left-hand side of Eq. (8) are both multiplied by loan duration n, and the equation for the simple rate, Eq. (4), is substituted into the left-hand side, the outcome is Eq. (9).

$$nS = \prod_{j=2}^{n}(-X_j)X_1 \tag{9}$$

Eq. (9) states that n simple rates are equal to a 'quantity' of the orthodox APR X_1, where the 'quantity' is the product of the unorthodox APRs. This equation is noteworthy for being the first in which the simple rate and APR appear together. The equation justifies the earlier assertion that the conventional interpretation of APR is not correct. The simple rate (no compounding) is intimately connected with all conceivable APRs (every one of which involves compounding). It appears that the connection between the simple rate in Eq. (4) and the orthodox APR in Eq. (6) is not straightforward. The financial significance of Eq. (9) is not clear, however, unless meaning can be attributed to the 'quantity' -- the unorthodox product.

It is stated here, and proved in Appendix C, that the unorthodox product is the number of times $(1+X_1)$ is applied to a borrowed dollar during amortization of the loan in Eq. (6). This result implies that the orthodox APR X_1 is a unit of measurement, and the product of all the other simultaneously determined APRs enumerates the number of times the orthodox unit is applied. The entire product is a unit multiplied by a number of units, the product measuring the finance charge per borrowed dollar. It follows that Boulding's assertion quoted earlier is incorrect. The product of the unorthodox interest rates possesses meaning, and this fact justifies giving the product a label.[3] In Eq. (10) the 'quantity' is labelled N.

$$nS = NX_1 \tag{10}$$

Eq. (10) states that n flat rates are equal to N APRs. This equation is new, not obvious, and, as will be demonstrated, is significant. Because of its significance the equation is given a name – the charge equation. The policy implications of the charge equation are discussed in the next section.[4]

[3] Boulding's assertion remains true to the extent that the meaning of an unorthodox interest rate considered in isolation from its companions remains an open question.

[4] Examination of Eq. (5) can be taken a step further by assuming a third meaningful value of X to enter into the equation, namely, the cost of funds. However, this particular analysis is relegated to Appendix D because, although the analysis is illuminating, it is not vital to the main argument.

4. The charge equation

Consumer credit legislation forces disclosure of APR in order that competitive pressure drives down this measure of the cost of borrowing. At this point the question posed in the previous section is asked again: which of the *n* APRs solving Eq. (1) is the object of the legislation? Which APR is to be driven down? There is no mention of the unorthodox APRs in the legislation and the surrounding research. The neglect of the unorthodox APRs makes it reasonable to assume that the object of the legislation is the orthodox APR, X_1.

The fundamental theorem of algebra implies that a given loan -- characterized by a specific set of cash flows C_0 and D_t from $t = 1$ to n -- is associated with a unique set of APRs, X_j. Any change in the cash flows is associated with a change in all APRs. For a given borrowing C_0, a financial institution can restructure the repayments D_t such that the publicly stated APR X_1 falls while the product of the unorthodox APRs rises. The product of the unorthodox APRs counts the number of times $(1+X_1)$ is applied to a borrowed dollar during the life of the loan. The way to raise this number is to restructure the repayments away from a series of even payments towards a 'back-loaded' structure in which a significant portion of the total repayment is concentrated into one or more large payments towards the end of the term. Examples include a payment holiday at the start of a furniture loan, a car loan having a final balloon payment, and an interest-only house mortgage having another investment vehicle to pay off the capital sum. Back-loading means that borrowed dollars remain on the books to be marked up again and again. Thus, in Eq. (10), downward pressure on the single value of APR, X_1, is offset by an increase in *N*, the product of the unorthodox APRs. The overall product, the finance charge per borrowed dollar (and, by implication, profit per dollar), remains the same or even increases. Back-loading is possible because the unorthodox APRs are neglected, and their product, *N*, which reflects the repayment structure, is not a variable amenable to legislation.

This article recommends that legislators switch attention to the left-hand side of the charge equation. The duration of the loan, *n*, can be controlled -- it can be capped. For example, the cap on *n* could be four or five years for a loan on a new car, or 25 to 30 years for a mortgage on a house. If the simple rate of interest *S* supplants the APR as a policy variable, then competitive pressure on *S*, combined with the cap on *n*, produces downward pressure on the overall product (finance charge per dollar per period). In contrast with the impact of current legislation, this policy would place downward pressure on <u>all APRs simultaneously</u>, affecting the size of the mark-up X_1 and, at the same time, inhibiting back-loading.

The EU technical document argues that APR, meaning the orthodox APR, '*has in its favour ... a higher adaptation to situations where ... the payments might adopt different and diverse patterns...'* Directorate General for Health & Consumer Protection (2009, p. 8). Higher adaptation is indeed a feature of the orthodox APR, but it is not a virtue. Under current legislation, financial institutions can take advantage of the adaptability, restructuring loans to keep the headline APR down while maintaining or even increasing charges and profits.

Financial data will show whether the argument is sound and whether financial institutions have exploited the situation during the post-TILA period. Two research hypotheses come to mind, one involving time-series data and the other cross-section. The first hypothesis is that during the period when APR legislation is enacted in a single jurisdiction, takes effect, and becomes entrenched, we should be able to observe an increasing supply of back-loaded products compared with loans having an even stream of repayments.[5] The second hypothesis is that, at a moment in time in a large sample of countries such as the EU, we should be able to observe a higher prevalence of back-loaded products in countries with strong implementation of APR legislation than in countries with weak implementation.[6]

5 Numerical examples

A consumer loan of $20,000 is repayable in four annual instalments of $6,309.42. Eq. (11) describes this arrangement for loan A. Eq. (12) shows the profit in dollars when the cost of funds is 3%. Table 1 contains the conventional financial statistics relating to (11) and (12).

$$20{,}000 = \frac{6{,}309.42}{(1+X)} + \frac{6{,}309.42}{(1+X)^2} + \frac{6{,}309.42}{(1+X)^3} + \frac{6{,}309.42}{(1+X)^4} \tag{11}$$

$$3{,}452.74 = -20{,}000 + \frac{6{,}309.42}{(1+0.03)} + \frac{6{,}309.42}{(1+0.03)^2} + \frac{6{,}309.42}{(1+0.03)^3} + \frac{6{,}309.42}{(1+0.03)^4} \tag{12}$$

[Table 1 about here]

The unconventional results described earlier and in Appendix D are applied to this example in the following way. Eq. (11) is solved for all four values of *(1+X)* that satisfy it. The values are listed in Col. 2 of Table 2.[7] The values of X = APR implied by these solutions are in Col. 3. The product of the APRs is 0.2619, the ratio of the FC to the loan amount. Col. 4 contains

[5] A source of information about auto loans is the local press. Newspaper adverts through the years could be a source of data to include in a test of the hypothesis.

[6] The research conducted for the EU Commission mentioned in the final paragraph of Appendix A is a likely source of data for this test.

[7] The calculation requires a specialized math program such as Maple, Mathcad, Mathematica or Matlab.

the multiplicative mark-ups of all APRs over the assumed cost of funds of 3%. The product of these mark-ups is 0.1726, the profit per dollar on loan A when the cost of funds is 3%.

[Table 2 about here]

Assume a second lender offers another product, loan B, for the same amount of $20,000, but asks for repayment in three instalments of $4,000 and a final balloon payment of $14,000. Eq. (13) captures the new arrangement. Eq. (14) shows the dollar profit when the cost of funds is 3%. Table 3 contains the conventional financial statistics for (13) and (14).

$$20{,}000 = \frac{4{,}000}{(1+X)} + \frac{4{,}000}{(1+X)^2} + \frac{4{,}000}{(1+X)^3} + \frac{14{,}000}{(1+X)^4} \tag{13}$$

$$3{,}753.24 = -20{,}000 \frac{4{,}000}{(1+0.03)} + \frac{4{,}000}{(1+0.03)^2} + \frac{4{,}000}{(1+0.03)^3} + \frac{14{,}000}{(1+0.03)^4} \tag{14}$$

[Table 3 about here]

The unconventional results for loan B are calculated in the same way as for loan A. Eq. (13) is solved for all values of *(1+X)* that satisfy it. They are listed in Col. 2 of Table 4. The values of X = APR implied by these solutions are in Col. 3. The product of these APRs is 0.30, the ratio of the FC to the loan amount. Col. 4 contains the multiplicative mark-ups of all APRs over the 3% cost of funds. The product of the mark-ups is 0.1877, the profit per dollar on loan B when the cost of funds is 3%.

[Table 4 about here]

Comparison of these summary statistics for the two loans shows the orthodox APR for loan B is lower than that for loan A by almost a full percentage point. Therefore, according to TILA-like legislation, loan B is 'cheaper' for the consumer and should be preferred to loan A. Furthermore, the even payments for loan B are lower than those for loan A, the terms for loan B inviting the consumer to take on a debt they might refuse under the terms for loan A. Nearing termination of loan B, however, the consumer remains indebted; there is the balloon payment to take care of. This is often done with a second loan, effectively extending the duration of the financial arrangement.

The last point demonstrates there is an alternative perspective on this financial story, a perspective further developed by the analysis in this article. The lender's profit per dollar is higher for loan B than for loan A. Every dollar lent via loan B earns the lender an extra 1.5 cents profit compared with a dollar lent via loan A ($0.1877 compared with $0.1726). Moreover, the simple rate for loan B is 1% higher than for loan A. Given that loan duration is the same for both loans, we know from the charge equation that a higher simple rate implies the product of all APRs for loan B must be higher than the product of all APRs for loan A (the FC per borrowed dollar is 30 cents compared with 26.2 cents). When judged by lender profitability, and by all APRs combined, loan B is actually more expensive.

6 Conclusion

This analysis undermines the law's insistence that consumers are told APR, conventionally conceived, is *the* measure of the relative cost of loans. Appendix B summarises research into the psychology of borrowers showing they find FC and, by implication, the simple rate of interest better measures of the cost of a loan than orthodox APR. The mathematical argument in this article is in agreement with consumer intuition.

We revert to the title of this article: Is APR a robust measure of the cost of consumer credit? The answer is yes, but only if all APRs are taken into account at once. Only a policy variable incorporating all APRs provides the whole 'truth-in-lending' and, surprising as it may be, the simple rate of interest is such a variable.

The current emphasis on the orthodox APR is misplaced; the simple rate of interest and loan duration should supplant the orthodox APR and the FC as policy variables in consumer credit legislation.

There is one final, disturbing reflection: the analysis implies that a factor in the build-up of consumer debt contributing to the 2007 financial crisis may have been official policy founded on conventional financial advice. It follows that reform of consumer credit legislation along the lines indicated may be one small step towards the prevention of future crises.

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Appendix A: Consumer credit legislation

The unfolding of consumer credit legislation follows a pattern. Initial, enabling legislation is followed by detailed regulations to implement it. Legislation and regulations are followed by amendments based on feedback from interested parties. Throughout, numerous explanatory documents appear, including manuals clarifying the legislation for employees of financial institutions, and booklets for the general public. Common to all legislation, regulations and most explanatory documents, is a math component devoted to the algebra and arithmetic of retail loans.

US legislation is the 1968 Consumer Credit Protection Act, otherwise known as the Truth-in-Lending Act. Enabling regulation is 'Regulation Z', introduced in 1969. Three appendices to Regulation Z describe APR computations in detail. Appendix J, describing APR computations for closed-end credit transactions, is particularly pertinent to the analysis in this article; the appendix contains 10 pages of mathematical explanation. '*Truth in Lending*,' the Comptroller's Handbook issued by the Comptroller of the Currency (2010), is an example of a clarifying document for financial institutions. A web search for 'truth in lending' or 'annual percentage rate' testifies to the large number of explanatory documents written for the public.

In the UK, the Office of Fair Trading (OFT) issues a booklet called '*Credit Charges and APR'* explaining '*how to calculate the total charge for credit and the annual percentage rate'* on a retail loan (OFT, 2007). The booklet is a clarification of the Consumer Credit Act 1974 and the accompanying regulatory detail contained in the Consumer Credit (Total Charge for Credit) Regulations of 1980. The many amendments made to the Act and Regulations since 1980 are detailed in the OFT booklet. The 1980 Regulations consist mostly of explanations of the TVM equation and APR. The OFT's explanatory booklet is similar to the Regulations in that three-fourths are an explanation of the math of retail loans. New regulations are imminent in 2013.

EU consumer credit legislation first appeared in the 1987 Directive (87/102/EEC) followed by clarifying directives in 1990 and 1998. The intention was to harmonize national credit legislation across the EU Member States. The EU Consumer Credit Directive of 2008 (2008/48/EC) quotes research conducted for the European Commission a decade after the 1987 Directive, stating that *'substantial differences* [remained] *between the laws of the various Member States in the field of credit,* […] *consumer credit in particular.'* The research prompted a 2002 proposal for new laws. The 2008 Directive (2008/48/EC) is another attempt to produce harmonization. Overall, it is a smaller and less intimidating legal document than the US and UK legislation; the mathematical part of the Directive, Annex 1, is just over one page long. However, there is a separate technical document supporting the legislation that is over 200 pages long (Directorate General, Health and Consumer Protection 2009), large parts of which are devoted to explaining the math of the TVM equation.

Appendix B. The effectiveness of consumer credit legislation

An early pre-TILA study is Due (1955), which looked at *'consumer knowledge of instalment credit charges'* and found significant lack of understanding. Other pre-TILA studies with similar findings were Juster & Shay (1964) and Mors (1965). An early summary of the situation pre-TILA, documenting studies leading to the introduction of the legislation, is Parker & Shay (1974).

Parker & Shay (1974) also document post-TILA studies describing improvements in awareness of credit costs, examples being Shay & Schober (1972) and Day & Brandt (1972). These studies showed a majority of consumers still displayed a lack of understanding about the true cost of borrowing. These early studies identified a problem - lack of understanding - without identifying reasons for it.

Mandell (1971) and Parker & Shay (1974) were among the first studies to attempt identification of the factors contributing to understanding, the most important being education levels and the total debt of a borrower. Later researchers have investigated the issue in detail. For example, Lee & Hogarth (1999) test a number of hypotheses about particular areas of misunderstanding and find that 40% of consumers do not understand the difference between contract interest rate (CIR) and APR.

More recent research suggests clarifications to the legislation, focusing on variations in the information to be disclosed. Following an analysis of the UK and US experience, Buch *et al.* (2002) propose a revised definition of APR incorporating standardized assumptions about major inputs to the APR calculation. Ramsay & Oguledo (2006) make a similar proposal.

Renuart & Thompson (2008) argue that the value of APR disclosure in the US has been diminished over the years by regulatory exclusion of numerous fees from the definition of FC. They urge regulatory change to restore the wide definition of FC envisaged in the original TILA.

> '*The* [Federal Reserve] *Board has already recognized that the APR is weakened by the unbundling of fees. If the Board is serious about financial literacy and informed consumer choice, it should embrace a "fully loaded" APR.'* (Renuart & Thompson, 2008)

Recent research further probes consumer credit behaviour. For example, Yard (2004) issued a sample of individuals with repayment schedules (and only repayment schedules) for a selection of loans and asked them to rank the loans by cost. Yard concludes:

> '*the respondents based their estimates of loan cost levels on some type of FC measure rather than on some kind of relative measure, such as APR.*'

Ranyard *et al.* (2006) interviewed adults in the UK and reported similar findings.

> '... [It] *is clear that consumers want additional information, not all of which is routinely available in the credit market. In particular, for longer-term planning they needed clear information on the duration and total cost of a loan* ...'

Thus, research demonstrates measures of loan cost other than APR are important decision variables for consumers. On this basis, Yard (2004) makes a pertinent proposal.

> *'If the FC per annum (FCA) is disclosed ... then the bias against loans of long duration can be avoided. The FCA can also be developed into a useful approximation of the APR by dividing it* [FCA] *by half the initial loan. This accounts for the effect of loan size almost as well as does the APR. Once this approximate APR (AAPR) is understood, the exact APR may become more understandable and accepted.'* Yard (2004)

Yard is making a similar policy proposal to this article, suggesting FC normalized by loan size and duration should be used as a decision variable. A difference between the two proposals is that this article uses the mathematics of the TVM equation to demonstrate that the lone, orthodox APR is an unsuitable policy variable, and therefore proposes that the simple rate of interest *supplant* APR, not supplement it.

Appendix C: Proof that the product of the unorthodox APRs is the number of times (1+X_1) is applied to a borrowed dollar during amortization of the loan

The proof requires a demonstration that the entity $\prod_{j=2}^{n}(-X_j)$ in Eq. (8) is equal to the number of times $(1+X_1)$ is applied during amortization of the loan described by Eq. (6). These two equations are repeated below for convenience.

$$C_0 = \sum_{t=1}^{n} \frac{D_t}{(1+X_1)^t} \tag{6}$$

$$\frac{FC}{C_0} = \prod_{j=2}^{n}(-X_j)X_1 \tag{8}$$

The amortization of a loan described by Eq. (6) and restricted to four repayments is displayed in Table C1.

[Table C1 about here]

The cash flows are in Col. 2 and the amounts outstanding at each moment in time are in Col. 3. The elements in Col. 4 are extracted from the corresponding elements in Col. 3; each element in Col. 4 contains the number of dollars marked up by $(1+X_1)$. At the foot of Col. 4 is the sum of the elements; this sum is labelled Y. The sum Y is the total number of times a dollar is marked up by $(1+X_1)$ during the entire amortization process. The sum Y divided by the borrowed amount C_0 is the total number of times a *borrowed* dollar is marked up by $(1+X_1)$ during amortization. The desired proof for the fourth order case requires that Eq. (A1) is true.

$$\frac{Y}{C_0} = \prod_{j=2}^{4}(-X_j) \tag{A1}$$

To prove the result, the matrix (M1) is formed. The first row of matrix (M1) is Eq. (6) having order four. The second row is the previous row multiplied by $(1+X_1)$, and similarly for the third and fourth rows. Every equation sums to zero, and therefore the matrix sums to zero.

$$\begin{Bmatrix} C_0 & & & -\dfrac{D_1}{(1+X_1)} & -\dfrac{D_2}{(1+X_1)^2} & -\dfrac{D_3}{(1+X_1)^3} - \dfrac{D_4}{(1+X_1)^4} = 0 \\ C_0(1+X_1) & - & & D_1 & -\dfrac{D_2}{(1+X_1)} & -\dfrac{D_3}{(1+X_1)^2} - \dfrac{D_4}{(1+X_1)^3} = 0 \\ C_0(1+X_1)^2 - D_1(1+X_1) & - & & D_2 & & -\dfrac{D_3}{(1+X_1)} - \dfrac{D_4}{(1+X_1)^2} = 0 \\ C_0(1+X_1)^3 - D_1(1+X_1)^2 - D_2(1+X_1) & - & & D_3 & & -\dfrac{D_4}{(1+X_1)} = 0 \end{Bmatrix} \tag{M1}$$

The ten elements comprising Col. 4 in Table C1 are identical to the ten elements in the lower left-hand triangle of matrix (M1). The triangle at the lower left-hand corner is labelled Y, and therefore the triangle at the upper right-hand corner equals $-Y$. The elements of the second triangle have their signs reversed and form the following equation.

$$\left\{\begin{array}{llll}\dfrac{D_1}{(1+X_1)} & +\dfrac{D_2}{(1+X_1)^2} & +\dfrac{D_3}{(1+X_1)^3} & +\dfrac{D_4}{(1+X_1)^4} \\ & +\dfrac{D_2}{(1+X_1)} & +\dfrac{D_3}{(1+X_1)^2} & +\dfrac{D_4}{(1+X_1)^3} \\ & & +\dfrac{D_3}{(1+X_1)} & +\dfrac{D_4}{(1+X_1)^2} \\ & & & +\dfrac{D_4}{(1+X_1)}\end{array}\right\} = Y$$

The last equation is rearranged.

$$D_1\left[\frac{1}{(1+X_1)}\right]+D_2\left[\frac{1}{(1+X_1)}+\frac{1}{(1+X_1)^2}\right]+D_3\left[\frac{1}{(1+X_1)}+\frac{1}{(1+X_1)^2}+\frac{1}{(1+X_1)^3}\right]$$
$$+D_4\left[\frac{1}{(1+X_1)}+\frac{1}{(1+X_1)^2}+\frac{1}{(1+X_1)^3}+\frac{1}{(1+X_1)^4}\right]=Y$$

The summations in square brackets are re-expressed using the following well-known result.

$$\sum_{t=1}^{n}\frac{1}{(1+X_1)^t}=\frac{1}{X_1}\left[1-\frac{1}{(1+X_1)^n}\right]$$

The equation for Y then simplifies as follows.

$$\left[\sum_{t=1}^{4}D_t\right]-\left[\sum_{t=1}^{4}\frac{D_t}{(1+X_1)^t}\right]=YX_1$$

Fourth order versions of Eq. (1) and Eq. (2) for FC and C_0 are substituted into the last equation.

$$\left[(FC+C_0)-C_0\right]=FC=YX_1 \tag{A2}$$

Both sides of Eq. (A2) are divided by C_0 to produce Eq. (A3).

$$\frac{FC}{C_0}=\left(\frac{Y}{C_0}\right)X_1 \tag{A3}$$

Eq. (A3) is juxtaposed with the fourth order version of Eq. (8) and their elements are compared.

$$\frac{FC}{C_0} = \prod_{j=2}^{4}(-X_j)X_1 \tag{8}$$

By inspection Eq. (A1) is true. This proof for the fourth-order equation generalizes to order *n*.

Appendix D: An examination of Eq. (5) when *X* is the cost of funds

This appendix contains a third example of the behaviour of Eq. (5) under different assumptions about the variable *X*. In this example *X* takes the value of the lender's cost of funds, or required rate of return, labelled *r*. In this situation Eq. (5) rearranges as follows.

$$C_0 - \sum_{t=1}^{n}\frac{D_t}{(1+r)^t} = C_0\frac{\prod_{j=1}^{n}(r-X_j)}{(1+r)^n} \tag{D1}$$

Both sides of Eq. (D1) can be interpreted.

First, the left-hand side of Eq. (D1) is given meaning. If, in Eq. (1), *X* is replaced by the cost of funds, then Eq. (1) can be rewritten as Eq. (D2), in which π is the lender's profit on the loan (positive or negative depending on whether *r* is less than or greater than *X*).

$$\pi = \sum_{t=1}^{n}\frac{D_t}{(1+r)^t} - C_0 \tag{D2}$$

Second, the right-hand side of Eq. (D1) is simplified. Assume the relationship between the cost of funds *r* and the APR *X* is in the form of a mark-up *m*. If there are *n* APRs and a single cost of funds than there must be *n* mark-ups as in Eq. (D3).

$$(1+r)(1+m_j)=(1+X_j) \text{ for } j = 1 \text{ to } n. \tag{D3}$$

This last equation rearranges into Eq. (D4).

$$m_j = \frac{(X_j - r)}{(1+r)} \quad \text{or} \quad -m_j = \frac{(r - X_j)}{(1+r)} \tag{D4}$$

Eq. (D2) and Eq. (D4) substitute into Eq. (D1) and the result rearranges into Eq. (D5).

$$\frac{\pi}{C_0} = -\prod_{j=1}^{n}(-m_j) = \prod_{j=2}^{n}(-m_j)m_1 = \frac{\prod_{j=2}^{n}(r - X_j)}{(1+r)^{n-1}}\frac{(X_1 - r)}{(1+r)} \tag{D5}$$

Eq. (D5) demonstrates that the lender's profit per loaned dollar is equal to the product of the mark-ups of all APRs over the cost of funds. It could be argued that consumer credit legislation should target excessive profit and bear down on this variable. However, it is argued here that borrowers are indifferent to variation in the costs of funds between different lenders. When comparing loans, borrowers are interested in the payments they will make to service the loans. These payments comprise lenders' gross revenues; borrowers are not interested in lenders' profits. Therefore the cost of funds can be set to zero, in which case Eq. (D5) reverts to Eq. (8) which leads to the charge equation, Eq. (10).

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Table 1 Conventional financial statistics for loan A

Finance charge	$FC = (6{,}309.42 \times 4) - 20{,}000 = 5{,}237.68$
Finance charge per $ borrowed	FC / C_0 = 5,237.68 / 20,000 = 26.19 or 26.2 cents
Simple rate	$F = 5{,}237.68/(4 \times 20{,}000) = 0.06547 = 6.5\%$
Profit when cost of funds is 3%	3,452.74
Profit per $ loaned (cost of funds = 3%)	3,452.74 / 20,000 = 0.1726 or 17.3 cents
Profit per $ loaned p.a. (cost of funds = 3%)	0.1726 / 4 = 0.0432 = 4.32% or 4.3 cents
Orthodox APR	X = 10%

Table 2 All APRs and their mark-ups over the cost of funds for loan A

j	$(1+X_j)$	Implied value of $\lvert X_j \rvert$ or X_j	Implied value of $\lvert m_j \rvert$ or m_j
1	1.1000	X_1 = 0.1	m_1 = 0.0680
2	-0.6342	X_2 = -1.6342	m_2 = -1.6158
3	-0.0751 + 0.6682.i	$\lvert X_3 \rvert$ = 1.2659	$\lvert m_3 \rvert$ = 1.2539
4	-0.0751 - 0.6682.i	$\lvert X_4 \rvert$ = 1.2659	$\lvert m_4 \rvert$ = 1.2539
		From Eq. (8): $FC/C_0 = -(-X_1)(-X_2)\lvert X_3 \rvert\lvert X_4 \rvert$ = 0.2619 = 26.2 cents	From Eq. (D5): $\pi/C_0 = -(-m_1)(-m_2)\lvert m_3 \rvert\lvert m_4 \rvert$ = 0.1726 = 17.3 cents

Notes to Table 2

a. The product of the implied values may not agree with the true product because of rounding errors. Calculated to full precision the figures agree to as many decimal places as hardware and software allow.

b. In Eq. (5), when a root $(1+X_j)$ is complex-valued, the absolute value of the difference $\left|(1+X)-(1+X_j)\right| = \left|X - X_j\right|$ is a positive, real number measuring a distance in complex space. When $X = 0$ the difference is $|X_j|$. When $(1+X_j)$ lies on the real number line absolute values are unnecessary; the sign (+/-) of a real-valued difference $(X-X_j)$ is determined. When $X = 0$ the difference is $(-X_j)$.

c. The calculation of the variable m_j is explained in Appendix D.

These comments also apply to Table 4.

Table 3 Conventional financial statistics for loan B

Finance charge	$FC = (4{,}000 \times 3 + 14{,}000) - 20{,}000 = 6{,}000$
Finance charge per $ borrowed	FC / C_0 = 6,000 / 20,000 = 0.3 or 30 cents
Simple rate	$F = 6{,}000/(4 \times 20{,}000) = 0.075 = 7.5\%$
Profit when cost of funds is 3%	3753.24
Profit per $ loaned (cost of funds = 3%)	3753.24 / 20,000 = 0.1877 or 18.8 cents
Profit per $ loaned p.a. (cost of funds = 3%)	0.1877 / 4 = 0.0469 = 4.69% or 4.7 cents
Orthodox APR	X = 9.0794%

Table 4 All APRs and their mark-ups over the cost of funds for loan B

j	$(1+X_j)$	Implied value of $\lvert X_j \rvert$ or X_j	Implied value of $\lvert m_j \rvert$ or m_j
1	1.0908	X_1 = 0.0908	m_1 = 0.0590
2	-0.8606	X_2 = -1.8606	m_2 = -1.8355
3	-0.0151 + 0.8634.i	$\lvert X_3 \rvert$ = 1.3326	$\lvert m_3 \rvert$ = 1.3161
4	-0.0151 - 0.8634.i	$\lvert X_4 \rvert$ = 1.3326	$\lvert m_4 \rvert$ = 1.3161
		From Eq. (8): $FC/C_0 = -(-X_1)(-X_2)\lvert X_3 \rvert \lvert X_4 \rvert$ = 0.30 = 30 cents	From Eq. (D5): $\pi/C_0 = -(-m_1)(-m_2)\lvert m_3 \rvert \lvert m_4 \rvert$ = 0.1877 = 18.8 cents

Notes to Table 4

See the notes to Table 2.

Table C1. The amortization schedule for a four-period loan based on Eq. (6)

Col. 1	Col. 2	Col. 3	Col. 4 (drawn from Col. 3)
Time	Cash flows	The number of \$ outstanding at each moment in time	The number of \$ marked up by $(1+X_1)$ during loan amortization
0	C_0	C_0	
1	$-D_1$	$C_0\,(1+X_1) - D_1$	C_0
2	$-D_2$	$C_0\,(1+X_1)^2 - D_1\,(1+X_1) - D_2$	$C_0\,(1+X_1) - D_1$
3	$-D_3$	$C_0\,(1+X_1)^3 - D_1\,(1+X_1)^2$ $- D_2\,(1+X_1) - D_3$	$C_0\,(1+X_1)^2 - D_1\,(1+X_1) - D_2$
4	$-D_4$	$C_0\,(1+X_1)^4 - D_1\,(1+X_1)^3$ $- D_2\,(1+X_1)^2 - D_3\,(1+X_1) - D_4 = 0$	$C_0\,(1+X_1)^3 - D_1\,(1+X_1)^2$ $- D_2\,(1+X_1) - D_3$
		The final element above is the loan equation, Eq. (6).	Y = sum of the elements above = the total number of dollars marked up during amortization of the loan.

Y/C_0 is the number of times a *borrowed* dollar is marked up by $(1+X_1)$ during amortization.