We consider a model of directed last passage growth model in two dimensions, where each lattice site (i, j) of $\mathbb{Z}_+^2$ is given a random weight $\tau_{i,j}$ according to some background measure $\mathbb{P}$.

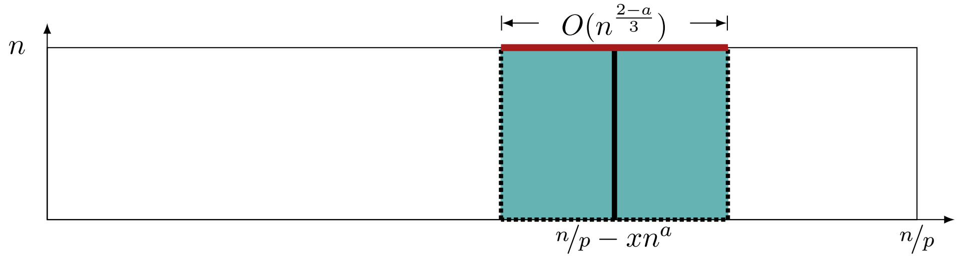

Figure 2. Tracy-Widom fluctuations to the last passage time of the independent model depend on position of the endpoint in the thickset red line. When $a \in (1/2, 2/3)$ the Tracy-Widom reveals itself just by centering according to the first and second order macroscopic approximation of the LLN for G. However when $a \in (3/2, 5/7)$, a third order approximation to the LLN, cn^{3a-2} is necessary in order to see the Tracy-Widom fluctuations.

Given lattice points $(a, b), (u, v) \in \mathbb{Z}_+^2$, $\Pi_{(a,b),(u,v)}$ is the set of lattice paths $\pi = \{(a, b) = (i_0, j_0), (i_1, j_1), \ldots, (i_p, j_p) = (u, v)\}$ whose admissible steps satisfy

$$(i_\ell, j_\ell) - (i_{\ell-1}, j_{\ell-1}) \in \{(1, 0), (0, 1)\}. \tag{3.1}$$

If $(a, b) = (0, 0)$ we simply denote this set by $\Pi_{u,v}$.

For $(u, v) \in \mathbb{Z}_+^2$ and $n \in \mathbb{N}$ denote the *last passage time*

$$G_{(a,b),(u,v)} = \max_{\pi \in \Pi_{(a,b),(u,v)}} \left\{ \sum_{(i,j)\in\pi} \tau_{i,j} \right\}. \tag{3.2}$$

Again, if $(a, b) = (0, 0)$ and no confusion arises, we simply denote $G_{(0,0),(u,v)}$ with $G_{u,v}$. In the homogeneous setting, $\{\tau_{i,j}\}_{(i,j)\in\mathbb{Z}_2^+}$ are i.i.d. under $\mathbb{P}$ and standard subadditivity arguments give the existence of a point-to-point scaling limit

$$\lim_{n\to\infty} \frac{G_{\lfloor nx\rfloor,\lfloor ny\rfloor}}{n} = g_{pp}(x, y).$$

When the environment $\tau_{i,j} \sim \text{Exp}(1)$, the last passage model is one of the exactly solvable models of the KPZ class. The strong law of large numbers in the exponential model is explicitly computed by Rost in 1981

$$\lim_{n\to\infty} \frac{G_{\lfloor nx\rfloor,\lfloor ny\rfloor}}{n} = \gamma(x, y) = (\sqrt{x} + \sqrt{y})^2, \quad \mathbb{P}\text{-a.s.} \tag{3.3}$$

In this article we derive the limiting constant for a sequence of scaled last passage times on the lattice. The passage times themselves are coupled through a common realization of exponential random variables. However, the rates of these random variables will be chosen according to a discrete approximation of a macroscopic function

$$c : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+.$$

Consider the lattice corner $\mathbb{Z}_+^2$. The environment $\tau = \{\tau_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ is a collection of i.i.d. exponential random variables of rate 1. For any $n \in \mathbb{N}$ we alter the rate of each of these random variables by a scalar multiplication using the macroscopic speed function $c(x, y)$. Namely, define

$$r_{i,j}^{(n)} = c\Big(\frac{i}{n}, \frac{j}{n}\Big)^{-1}, \qquad (i,j) \in \mathbb{Z}_+^2, \tag{3.4}$$

and define n-scaled, inhomogeneous environment by

$$\tau_{i,j}^{(n)} = r_{i,j}^{(n)} \tau_{i,j}^{n}. \tag{3.5}$$

The rate of the exponential random variable $\tau_{i,j}^{(n)}$ is now determined by the scalar $c\big(\frac{i}{n}, \frac{j}{n}\big)$. On each site the rate is completely determined by the speed function $c(\cdot,\cdot)$. We indicate the corresponding exponential 1 random variable as $\tau_{i,j}^{n}$.

For $(u, v) \in \mathbb{Z}_+^2$ and $n \in \mathbb{N}$ denote the last passage time

$$G_{u,v}^{(n)} = \max_{\pi \in \Pi_{u,v}} \Big\{ \sum_{(i,j)\in\pi} r_{i,j}^{(n)} \tau_{i,j}^{n} \Big\} = \max_{\pi \in \Pi_{u,v}} \Big\{ \sum_{(i,j)\in\pi} \tau_{i,j}^{(n)} \Big\}. \tag{3.6}$$

We impose several conditions on the function $c(x, y)$ but one important one is that for any compact set $K \subseteq \mathbb{R}_+^2$ there exist finite constant m_K and M_K such that

$$m_K \le c(x, y) \le M_K \quad \text{ for all } (x, y) \in K$$

and there are a finite number (that depends on K) of discontinuity curves of the function $c(x, y)$. These are to avoid degeneracies: If $c(x, y)$ can take the value 0 then the environment could take the value ∞ which leads to trivial passage times. If $c(x, y)$ can be infinity, that region of space will never be explored by a path. If the discontinuities have an accumulation point, then no finite descritisation of $c(x, y)$ can capture that.

We prove a strong law of large numbers for $n^{-1}G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor}$. The limiting last passage constant $\Gamma_c(x, y)$ has a variational characterisation that naturally leads to a continuous version of a last passage time model,

$$\lim_{n\to\infty} n^{-1}G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor} = \Gamma_c(x, y) = \sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \Big\{ \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s), x_2(s))} ds \Big\}, \quad \mathbb{P} - \text{a.s.}$$

We study the variational formula above, discuss properties of the shape $\Gamma_c(x, y)$ and obtain explicit minimizers in two cases of interest in the article " Last passage percolation in an exponential environment with discontinuous rates " that can be found here.

Keywords: last passage time, corner growth model, flat edge, shape theorem, discontinuous percolation, discontinuous environment, inhomogeneous corner, two-phase models, variational formula.

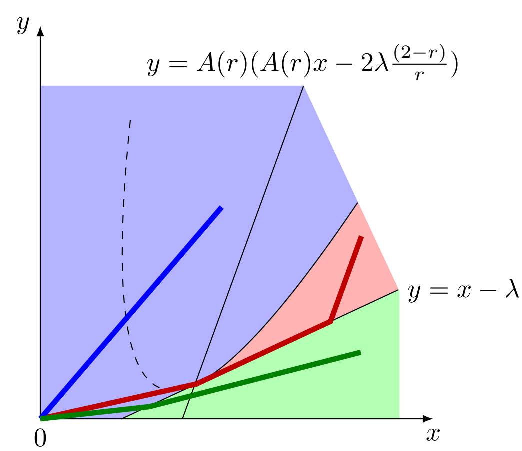

FIGURE 3. Maximal macroscopic paths for the shifted two-phase corner growth model. In the blue region we have a straight line path, in the red region we have a three piecewise linear path and in the green region we have a two piecewise linear path.