A Generalisation of Closed Unbounded and Stationary Sets

Hazel Brickhill

University of Bristol

British Logic Colloquium
8th September 2017
A very vague question

What is a large/thick subset of an ordinal?
A very vague question

What is a large/thick subset of an ordinal?
- an unbounded set
A very vague question

What is a large/thick subset of an ordinal?

- an unbounded set
- a closed unbounded (club) set
What is a large/thick subset of an ordinal?

- an unbounded set
- a closed unbounded (club) set
  uncountable cofinality $\rightarrow$ clubs generate a filter (measure)
A very vague question

What is a large/thick subset of an ordinal?

- an unbounded set
- a closed unbounded (club) set
  uncountable cofinality $\rightarrow$ clubs generate a filter (measure)
- a stationary set
A very vague question

What is a large/thick subset of an ordinal?

- an unbounded set
- a closed unbounded (club) set
  uncountable cofinality $\rightarrow$ clubs generate a filter (measure)
- a stationary set
  defined at ordinals of uncountable cofinality only
A sketch

Definition

\( C \subseteq \kappa \) is stationary-closed if whenever \( \alpha < \kappa \) and \( C \cap \alpha \) is stationary in \( \alpha \) we have \( \alpha \in C \).

Definition

\( C \) is 1-club in \( \kappa \) iff \( C \) is stationary in \( \kappa \) and stationary-closed.
A sketch

**Definition**

$C \subseteq \kappa$ is *stationary-closed* if whenever $\alpha < \kappa$ and $C \cap \alpha$ is stationary in $\alpha$ we have $\alpha \in C$. 
A sketch

**Definition**

$C \subseteq \kappa$ is *stationary-closed* if whenever $\alpha < \kappa$ and $C \cap \alpha$ is stationary in $\alpha$ we have $\alpha \in C$.

**Definition**

$C$ is *1-club* in $\kappa$ iff $C$ is stationary in $\kappa$ and stationary-closed.
Definition 1: Generalised clubs

1. $S \subseteq On$ is 0-stationary in $\kappa$ if it is unbounded in $\kappa$. 

Notation $d_\gamma(A) := \{ \alpha : A \text{ is } \gamma\text{-stationary below } \alpha \}$.
Definition 1: Generalised clubs

Definition

1. $S \subseteq On$ is $0$-stationary in $\kappa$ if it is unbounded in $\kappa$.

2. $C \subseteq On$ is $\gamma$-stationary closed if for any $\alpha$ such that $C$ is $\gamma$-stationary in $\alpha$ we have $\alpha \in C$.

Notation

\[ d_{\gamma}(A) := \{ \alpha : A \text{ is } \gamma \text{-stationary below } \alpha \} \]
Definition 1: Generalised clubs

Definition

1. $S \subseteq \text{On}$ is 0-stationary in $\kappa$ if it is unbounded in $\kappa$.
2. $C \subseteq \text{On}$ is $\gamma$-stationary closed if for any $\alpha$ such that $C$ is $\gamma$-stationary in $\alpha$ we have $\alpha \in C$.
3. $C$ is $\gamma$-club in $\kappa$ if $C$ is $\gamma$-stationary closed and $\gamma$-stationary in $\kappa$. 

Notation

d$_{\gamma}(A) := \{\alpha : A$ is $\gamma$-stationary below $\alpha\}$
Definition 1: Generalised clubs

Definition

1. $S \subseteq \text{On}$ is 0-stationary in $\kappa$ if it is unbounded in $\kappa$.

2. $C \subseteq \text{On}$ is $\gamma$-stationary closed if for any $\alpha$ such that $C$ is $\gamma$-stationary in $\alpha$ we have $\alpha \in C$.

3. $C$ is $\gamma$-club in $\kappa$ if $C$ is $\gamma$-stationary closed and $\gamma$-stationary in $\kappa$.

4. $\kappa$ is $\gamma$-s-reflecting if for any $\gamma$-stationary $S$, $T \subseteq \kappa$ there is $\alpha < \kappa$ with $S$ and $T$ both $\gamma$-stationary below $\alpha$. 

Notation

$d_{\gamma}(A) := \{ \alpha : A \text{ is } \gamma\text{-stationary below } \alpha \}$
Definition 1: Generalised clubs

Definition

1. $S \subseteq \text{On}$ is $0$-stationary in $\kappa$ if it is unbounded in $\kappa$.

2. $C \subseteq \text{On}$ is $\gamma$-stationary closed if for any $\alpha$ such that $C$ is $\gamma$-stationary in $\alpha$ we have $\alpha \in C$.

3. $C$ is $\gamma$-club in $\kappa$ if $C$ is $\gamma$-stationary closed and $\gamma$-stationary in $\kappa$.

4. $\kappa$ is $\gamma$-s-reflecting if for any $\gamma$-stationary $S$, $T \subseteq \kappa$ there is $\alpha < \kappa$ with $S$ and $T$ both $\gamma$-stationary below $\alpha$.

5. $S \subseteq \kappa$ is $\gamma$-stationary if for every $\gamma' < \gamma$ we have $\kappa$ is $\gamma'$-s-reflecting and for any $C \gamma'$-club in $\kappa$ we have $S \cap C \neq \emptyset$
### Definition 1: Generalised clubs

1. \( S \subseteq \text{On} \) is 0-stationary in \( \kappa \) if it is unbounded in \( \kappa \).
2. \( C \subseteq \text{On} \) is \( \gamma \)-stationary closed if for any \( \alpha \) such that \( C \) is \( \gamma \)-stationary in \( \alpha \) we have \( \alpha \in C \).
3. \( C \) is \( \gamma \)-club in \( \kappa \) if \( C \) is \( \gamma \)-stationary closed and \( \gamma \)-stationary in \( \kappa \).
4. \( \kappa \) is \( \gamma \)-s-reflecting if for any \( \gamma \)-stationary \( S, T \subseteq \kappa \) there is \( \alpha < \kappa \) with \( S \) and \( T \) both \( \gamma \)-stationary below \( \alpha \).
5. \( S \subseteq \kappa \) is \( \gamma \)-stationary if for every \( \gamma' < \gamma \) we have \( \kappa \) is \( \gamma' \)-s-reflecting and for any \( C \) \( \gamma' \)-club in \( \kappa \) we have \( S \cap C \neq \emptyset \).

### Notation

\( d_\gamma(A) := \{ \alpha : A \text{ is } \gamma \text{-stationary below } \alpha \} \)
Restating the Definitions in Terms of $d_\gamma$

**Notation**

$d_\gamma(A) := \{\alpha : A \text{ is } \gamma\text{-stationary below } \alpha\}$

**Definition (restated)**

1. $S \subseteq \text{On}$ is 0-stationary in $\kappa$ if it is unbounded in $\kappa$.
2. $C \subseteq \text{On}$ is $\gamma$-stationary closed if $d_\gamma(C) \subseteq C$.
3. $C$ is $\gamma$-club in $\kappa$ if $C$ is $\gamma$-stationary closed and $\gamma$-stationary below $\kappa$.
4. $\kappa$ is $\gamma$-reflecting if for any $\gamma$-stationary $S$, $T \subseteq \kappa$,
   \[d_\gamma(S) \cap d_\gamma(T) \cap \kappa \neq \emptyset.\]
5. $S \subseteq \kappa$ is $n+1$-stationary if $\kappa$ is $n$-reflecting and $S \cap C \neq \emptyset$ for every $C$ $n$-club in $\kappa$.
how large is a subset of $\kappa$?

If $\kappa$ is $n$-reflecting, then for a subset of $\kappa$ we have these implications:

\[
\begin{align*}
n\text{-club} & \quad \implies \quad n + 1\text{-stationary} \\
\uparrow & \\
n - 1\text{-club} & \quad \implies \quad n\text{-stationary} \\
\uparrow & \\
\vdots & \\
\uparrow & \\
0\text{-club (} \equiv \text{ club)} & \quad \implies \quad \text{stationary} \\
\uparrow & \\
\text{unbounded} & \quad \downarrow
\end{align*}
\]
Origins


Definition 2: Reflection

1. $S \subseteq \text{On}$ is 0-stationary in $\kappa$ if it is unbounded in $\kappa$. 
Definition 2: Reflection

1. $S \subseteq On$ is 0-stationary in $\kappa$ if it is unbounded in $\kappa$.
2. $S \subseteq \kappa$ is $\gamma$-stationary if for every $\eta < \gamma$ and for any $\eta$-stationary $T$, $T' \subseteq \kappa$ there is $\alpha \in S$ with $T$ and $T'$ both $\gamma$-stationary below $\alpha$. 
Definition 2: Reflection

1. $S \subseteq \text{On}$ is 0-stationary in $\kappa$ if it is unbounded in $\kappa$.

2. $S \subseteq \kappa$ is $\gamma$-stationary if for every $\eta < \gamma$ and for any $\eta$-stationary $T$, $T' \subseteq \kappa$ there is $\alpha \in S$ with $T$ and $T'$ both $\gamma$-stationary below $\alpha$.

- Defining $\gamma$-stationary sets in this way is equivalent to defining them in terms of generalised clubs.
Definition 2: Reflection

1. $S \subseteq \text{On}$ is 0-stationary in $\kappa$ if it is unbounded in $\kappa$.
2. $S \subseteq \kappa$ is $\gamma$-stationary if for every $\eta < \gamma$ and for any $\eta$-stationary $T$, $T' \subseteq \kappa$ there is $\alpha \in S$ with $T$ and $T'$ both $\gamma$-stationary below $\alpha$.

- Defining $\gamma$-stationary sets in this way is equivalent to defining them in terms of generalised clubs.
- This is easy to show by induction: the key is that if $\kappa$ is $\gamma$-stationary and $T \subseteq \kappa$ is $\eta$-stationary for $\eta < \gamma$, then $d_\eta(T)$ is $\eta$-club.
Definition 3: Topologies

Let $\Omega$ be an ordinal and $\mathcal{T}$ a topology on $\Omega$. For $A \subseteq \Omega$ we set

$$d_{\mathcal{T}}(A) = \text{the set of limit points of } A \text{ in the topology } \mathcal{T}.$$
Definition 3: Topologies

Let $\Omega$ be an ordinal and $\mathcal{T}$ a topology on $\Omega$. For $A \subseteq \Omega$ we set

$$d_{\mathcal{T}}(A) = \text{the set of limit points of } A \text{ in the topology } \mathcal{T}.$$ 

Set $\mathcal{T}_0$ to be the interval topology on $\Omega$. 

These topologies are closely related to $\gamma$-stationarity.
Definition 3: Topologies

Let $\Omega$ be an ordinal and $\mathcal{T}$ a topology on $\Omega$. For $A \subseteq \Omega$ we set

$$d_{\mathcal{T}}(A) = \text{the set of limit points of } A \text{ in the topology } \mathcal{T}.$$ 

Set $\mathcal{T}_0$ to be the interval topology on $\Omega$.

$\blacksquare$ $d_{\mathcal{T}_0}(A) = d_0(A)$
Definition 3: Topologies

Let $\Omega$ be an ordinal and $\mathcal{T}$ a topology on $\Omega$. For $A \subseteq \Omega$ we set

$$d_\mathcal{T}(A) = \text{the set of limit points of } A \text{ in the topology } \mathcal{T}.$$ 

Set $\mathcal{T}_0$ to be the interval topology on $\Omega$.

- $d_{\mathcal{T}_0}(A) = d_0(A)$
- $\alpha$ is non-isolated in $\mathcal{T}_0$ iff $\alpha$ is a limit ordinal.
Definition 3: Topologies

Let $\Omega$ be an ordinal and $\mathcal{T}$ a topology on $\Omega$. For $A \subseteq \Omega$ we set

$$d_{\mathcal{T}}(A) = \text{the set of limit points of } A \text{ in the topology } \mathcal{T}.$$ 

Set $\mathcal{T}_0$ to be the interval topology on $\Omega$.

- $d_{\mathcal{T}_0}(A) = d_0(A)$
- $\alpha$ is non-isolated in $\mathcal{T}_0$ iff $\alpha$ is a limit ordinal.

Definition

If $\mathcal{T}$ is a topology on $\Omega$ then the topology derived from $\mathcal{T}$ is the topology generated by

$$\mathcal{T} \cup \{d_{\mathcal{T}}(A) : A \subseteq \Omega\}.$$
Definition 3: Topologies

Let $\Omega$ be an ordinal and $\mathcal{T}$ a topology on $\Omega$. For $A \subseteq \Omega$ we set

$$d_{\mathcal{T}}(A) = \text{the set of limit points of } A \text{ in the topology } \mathcal{T}.$$ 

Set $\mathcal{T}_0$ to be the interval topology on $\Omega$.

- $d_{\mathcal{T}_0}(A) = d_0(A)$
- $\alpha$ is non-isolated in $\mathcal{T}_0$ iff $\alpha$ is a limit ordinal.

Definition

If $\mathcal{T}$ is a topology on $\Omega$ then the toplogy derived from $\mathcal{T}$ is the topology generated by

$$\mathcal{T} \cup \{ d_{\mathcal{T}}(A) : A \subseteq \Omega \}.$$ 

Set $\mathcal{T}_{\gamma+1}$ to be the topology derived from $\mathcal{T}_\gamma$ and for limit $\lambda$ set

$$\mathcal{T}_\lambda = \bigcup_{\gamma < \lambda} \mathcal{T}_\gamma.$$
Definition 3: Topologies

Let $\Omega$ be an ordinal and $\mathcal{I}$ a topology on $\Omega$. For $A \subseteq \Omega$ we set

$$d_{\mathcal{I}}(A) = \text{the set of limit points of } A \text{ in the topology } \mathcal{I}.$$  

Set $\mathcal{I}_0$ to be the interval topology on $\Omega$.

- $d_{\mathcal{I}_0}(A) = d_0(A)$
- $\alpha$ is non-isolated in $\mathcal{I}_0$ iff $\alpha$ is a limit ordinal.

Definition

If $\mathcal{I}$ is a topology on $\Omega$ then the \textit{topology derived from } $\mathcal{I}$ \textit{is the topology generated by}

$$\mathcal{I} \cup \{d_{\mathcal{I}}(A) : A \subseteq \Omega\}.$$  

Set $\mathcal{I}_{\gamma+1}$ to be the topology derived from $\mathcal{I}_\gamma$ and for limit $\lambda$ set

$$\mathcal{I}_\lambda = \bigcup_{\gamma < \lambda} \mathcal{I}_\gamma.$$  

- These topologies are closely related to $\gamma$-stationarity.
$\mathcal{T}_\gamma$ and $\gamma$-stationarity

What is $T_1$?
$\mathcal{T}_\gamma$ and $\gamma$-stationarity

What is $T_1$?
We have the following equivalences:

$$\alpha \in d_{\mathcal{T}_1}(A)$$

$$\iff \forall X, Y \subseteq \Omega \; \alpha \in d_0(X) \cap d_0(Y) \to d_0(X) \cap d_0(Y) \cap \alpha \cap A \neq \emptyset$$

$$\iff \forall C, D \text{ club in } \alpha \text{ we have } C \cap D \cap A \neq \emptyset$$

$$\iff A \text{ is stationary in } \alpha, \text{ i.e. } \alpha \in d_1(A)$$
What is $T_1$?
We have the following equivalences:

\[
\begin{align*}
\alpha \in d_{T_1}(A) & \iff \forall X, Y \subseteq \Omega \ \alpha \in d_0(X) \cap d_0(Y) \to d_0(X) \cap d_0(Y) \cap \alpha \cap A \neq \emptyset \\
& \iff \forall C, D \text{ club in } \alpha \text{ we have } C \cap D \cap A \neq \emptyset \\
& \iff A \text{ is stationary in } \alpha, \ i.e. \ \alpha \in d_1(A)
\end{align*}
\]

Thus:

\[
d_{T_1}(A) = d_1(A) = \{\alpha : A \text{ is stationary in } \alpha\}
\]
$\mathcal{T}_\gamma$ and $\gamma$-stationarity

What is $T_1$?
We have the following equivalences:

$$\alpha \in d_{T_1}(A)$$

$$\iff \forall X, Y \subseteq \Omega \alpha \in d_0(X) \cap d_0(Y) \rightarrow d_0(X) \cap d_0(Y) \cap \alpha \cap A \neq \emptyset$$

$$\iff \forall C, D \text{ club in } \alpha \text{ we have } C \cap D \cap A \neq \emptyset$$

$$\iff A \text{ is stationary in } \alpha, \text{ i.e. } \alpha \in d_1(A)$$

Thus:

$$d_{T_1}(A) = d_1(A) = \{\alpha : A \text{ is stationary in } \alpha\}$$

In fact we can show that for any $\gamma$, $d_{T_\gamma} = d_{\gamma}$. 
$\mathcal{I}_\gamma$ and $\gamma$-stationarity

What is $T_1$?
We have the following equivalences:

$$\alpha \in d_{T_1}(A) \iff \forall X, Y \subseteq \Omega \ \alpha \in d_0(X) \cap d_0(Y) \rightarrow d_0(X) \cap d_0(Y) \cap \alpha \cap A \neq \emptyset$$

$$\iff \forall C, D \text{ club in } \alpha \text{ we have } C \cap D \cap A \neq \emptyset$$

$$\iff A \text{ is stationary in } \alpha, \text{ i.e. } \alpha \in d_1(A)$$

Thus:

$$d_{T_1}(A) = d_1(A) = \{\alpha : A \text{ is stationary in } \alpha\}$$

In fact we can show that for any $\gamma$, $d_{T_\gamma} = d_\gamma$. Thus a point $\alpha$ is non-isolated in $\mathcal{I}_\gamma$ iff for every $\gamma' < \gamma$, $\alpha$ is $\gamma'$-s-reflecting (i.e. $\alpha$ is $\gamma$-stationary), and $\mathcal{I}_\gamma$ is non-discrete iff there is an ordinal $\alpha < \Omega$ that is $\gamma$ stationary.
Consistency Strength?

Upper bound:

- An easy argument shows that $\Pi^1_n$-indescribable cardinals are $n$-stationary reflecting.
Consistency Strength?

Upper bound:

- An easy argument shows that $\Pi^1_n$-indescribable cardinals are $n$-stationary reflecting.

- We can define a notion of $\Pi^1_\gamma$-indescribability for ordinals $\gamma$ such that $\Pi^1_\gamma$-indescribable cardinals are $\gamma$-stationary reflecting.
Consistency Strength?

Upper bound:

- An easy argument show that $\Pi^1_n$-indescribable cardinals are $n$-stationary reflecting.
- We can define a notion of $\Pi^1_\gamma$-indescribability for ordinals $\gamma$ such that $\Pi^1_\gamma$-indescribable cardinals are $\gamma$-stationary reflecting.

Under $V = L$:

Theorem (Jensen)
In $L$ a regular cardinal reflects stationary sets iff it is $\Pi^1_1$-indescribable (=weakly compact).
Consistency Strength?

Upper bound:

- An easy argument show that $\Pi_1^n$-indescribable cardinals are $n$-stationary reflecting.
- We can define a notion of $\Pi_1^\gamma$-indescribability for ordinals $\gamma$ such that $\Pi_1^\gamma$-indescribable cardinals are $\gamma$-stationary reflecting.

Under $V = L$:

Theorem (Jensen)

In $L$ a regular cardinal reflects stationary sets iff it is $\Pi_1^1$-indescribable (=weakly compact).

Theorem (Bagaria, Magidor, Sakai) (1 < $n$ < $\omega$)

In $L$ a regular cardinal reflects $n$-stationary sets iff it is $\Pi_1^n$-indescribable.
Consistency Strength?

Upper bound:

- An easy argument shows that $\Pi^1_n$-indescribable cardinals are $n$-stationary reflecting.
- We can define a notion of $\Pi^1_\gamma$-indescribability for ordinals $\gamma$ such that $\Pi^1_\gamma$-indescribable cardinals are $\gamma$-stationary reflecting.

Under $V = L$:

**Theorem (Jensen)**

In $L$ a regular cardinal reflects stationary sets iff it is $\Pi^1_1$-indescribable (=weakly compact).

**Theorem (Bagaria, Magidor, Sakai) ($1 < n < \omega$)**

In $L$ a regular cardinal reflects $n$-stationary sets iff it is $\Pi^1_n$-indescribable.

**Theorem (B., Bagaria)**

In $L$ a regular cardinal reflects $\gamma$-stationary sets iff it is $\Pi^1_\gamma$-indescribable.
Consistency Strength?

Lower bound?

Theorem (Magidor)
A regular cardinal that is 1-s-reflecting is $\Pi^1_1$-indescribable in $L$. Thus the existence of a 1-s-reflecting cardinal is equiconsistent with the existence of a $\Pi^1_1$-indescribable.

Conjecture:
For $\gamma > 1$ the consistency strength of a $\gamma$-s-reflecting cardinal is below that of a $\Pi^1_\gamma$-indescribable.
Consistency Strength?

Lower bound?

Theorem (Magidor)
A regular cardinal that is 1-s-reflecting is $\Pi_1^1$-indescribable in $L$. Thus the existence of a 1-s-reflecting cardinal is equiconsistent with the existence of a $\Pi_1^1$-indescribable.

Theorem (B.)
Let $\kappa$ be a regular cardinal that is $\gamma$-s-reflecting such that the $\gamma$-club filter on $\kappa$ is normal, and for “many” cardinals $\lambda$ below $\kappa$ we have $\lambda$ is $\eta$-s-reflecting implies the $\eta$-club filter on $\lambda$ is normal. Then $\kappa$ is $\Pi_1^\gamma$-indescribable in $L$. 
Consistency Strength?

Lower bound?

Theorem (Magidor)
A regular cardinal that is 1-s-reflecting is $\Pi_1^1$-indescribable in $L$. Thus the existence of a 1-s-reflecting cardinal is equiconsistent with the existence of a $\Pi_1^1$-indescribable.

Theorem (B.)
Let $\kappa$ be a regular cardinal that is $\gamma$-s-reflecting such that the $\gamma$-club filter on $\kappa$ is normal, and for “many” cardinals $\lambda$ below $\kappa$ we have $\lambda$ is $\eta$-s-reflecting implies the $\eta$-club filter on $\lambda$ is normal. Then $\kappa$ is $\Pi_{\gamma}^1$-indescribable in $L$.

Conjecture:
For $\gamma > 1$ the consistency strength of a $\gamma$-s-reflecting cardinal is below that of a $\Pi_{\gamma}^1$-indescribable.
Generalised □ sequences

Definition

A □γ sequence on κ is a sequence \( \langle C_\alpha : \alpha \in d_\gamma (\kappa) \rangle \) such that for each \( \alpha \):

1. \( C_\alpha \) is an \( \gamma \)-club subset of \( \alpha \)
2. (Coherence) for every \( \beta \in d_\gamma (C_\alpha) \) we have \( C_\beta = C_\alpha \cap \beta \)
Generalised □ sequences

Definition

A □γ sequence on κ is a sequence ⟨Cα : α ∈ dγ(κ)⟩ such that for each α:

1. Cα is an γ-club subset of α
2. (Coherence) for every β ∈ dγ(Cα) we have Cβ = Cα ∩ β

- We need to add an extra condition for □ sequences to be non-trivial (and useful)
- There are many ways to do this for the standard case
- Standard □ sequences are useful for a variety of constructions
Definition

A $\square^\gamma$ sequence on $\kappa$ is a sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa) \rangle$ such that for each $\alpha$:

1. $C_\alpha$ is an $\gamma$-club subset of $\alpha$
2. (Coherence) for every $\beta \in d_\gamma(C_\alpha)$ we have $C_\beta = C_\alpha \cap \beta$

- We need to add an extra condition for $\square$ sequences to be non-trivial (and useful)
- There are many ways to do this for the standard case
- Standard $\square$ sequences are useful for a variety of constructions

Theorem (B.)($V = L$)

If $\kappa$ is $\Pi^1_\gamma$- but not $\Pi^1_{\gamma+1}$-indescribable then there is an $\gamma + 1$-stationary set $E \subseteq \kappa$ and $\square^\gamma$ sequence avoiding $E$. Thus $\kappa$ is not $\gamma + 1$-reflecting.