Rotating quantum states

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Outline

1. Quantum field theory on curved space-time
   - General strategy
   - Quantum scalar field
   - Quantum fermion field
   - An example in Minkowski space
2. Rotating quantum states on Minkowski space-time
   - Quantum scalar field
   - Quantum fermion field
3. Rotating quantum states on anti-de Sitter space-time
   - Quantum scalar field
   - Quantum fermion field
4. Conclusions

Quantum field theory on curved space-time

Quantum field theory (QFT) on curved space-time

- Semi-classical limit of quantum gravity
- Background geometry fixed and classical
- Quantum field propagating on this background

- Semi-classical Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}$$

- Renormalized stress-energy tensor $\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}$

Quantum field theory (QFT) on curved space-time

- Semi-classical limit of quantum gravity
- Background geometry fixed and classical
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- Renormalized stress-energy tensor $\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}$

Steps in quantum field theory on curved space-time

1. Classical Lagrangian describing the field
2. Canonical quantization
3. Definition of quantum states
   (a) Orthonormal basis of field modes
   (b) Choice of positive frequency
4. Physical interpretation of states and observables

Quantum scalar field Φ

Step 1 - classical Lagrangian

- Classical Lagrangian density for a scalar field of mass M

$$\mathcal{L}_\Phi = \nabla_\mu \Phi \nabla^\mu \Phi + \left(M^2 + \xi \mathcal{R}\right) \Phi^2$$

- Klein-Gordon equation

$$\left[\Box - M^2 - \xi \mathcal{R}\right] \Phi = 0$$

- Klein-Gordon inner product

$$\left(\Phi_1, \Phi_2\right)_{KG} = i \int_S \left[\Phi_2^* \nabla_\mu \Phi_1 - \Phi_1 \nabla_\mu \Phi_2^*\right] dS^\mu$$

Involves *time derivative* of Φ

Quantum scalar field Φ

Step 2 - canonical quantization

- Canonical conjugate momentum

$$\Pi_\Phi = \frac{\delta \mathcal{L}_\Phi}{\delta \dot{\Phi}} \qquad \dot{\Phi} = \partial_t \Phi$$

- Impose equal-time *commutation* relations on $t =$ constant

$$\begin{aligned} \left[\hat{\Phi}(t, \boldsymbol{x}), \hat{\Pi}_\Phi(t, \boldsymbol{x}') \right] &= i\delta(\boldsymbol{x}, \boldsymbol{x}') \\ \left[\hat{\Phi}(t, \boldsymbol{x}), \hat{\Phi}(t, \boldsymbol{x}') \right] = & \; 0 = \left[\hat{\Pi}_\Phi(t, \boldsymbol{x}), \hat{\Pi}_\Phi(t, \boldsymbol{x}') \right] \end{aligned}$$

$$\int_S \delta(\boldsymbol{x}, \boldsymbol{x}') \, dS = 1$$

Quantum scalar field Φ

Step 3 - definition of quantum states

(a) *Choice of time t*

Pick a suitable time co-ordinate t

(b) *Choice of positive frequency modes ϕ_j*

Solutions of the Klein-Gordon equation such that

$$\phi_j \propto e^{-i\omega t} \qquad \omega > 0$$

(c) *Orthonormal basis of field modes*

- Positive frequency modes ϕ_j^+ have positive norm
- Negative frequency modes ϕ_j^- have negative norm
- Complete set of positive and negative frequency modes $\left\{ \phi_j^+, \phi_j^- \right\}$

Quantum scalar field Φ

Step 3 - definition of quantum states

- Expand classical field in terms of orthonormal basis

$$\Phi = \sum_j a_j \phi_j^+ + a_j^\dagger \phi_j^-$$

- Promote expansion coefficients to operators $\hat{a}_j$, $\hat{a}_j^\dagger$ with

$$\left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk} \qquad \left[\hat{a}_j, \hat{a}_k\right] = 0 \qquad \left[\hat{a}_j^\dagger, \hat{a}_k^\dagger\right] = 0$$

- $\hat{a}_j$ - particle annihilation operators
  $\hat{a}_j^\dagger$ - particle creation operators
- Define the *vacuum* state $|0\rangle$:

$$\hat{a}_j |0\rangle = 0$$

Choice of time

- Choice of ‘time’ co-ordinate important
- Definition of positive frequency
- Implications for definition of vacuum
- Not completely unrestricted - positive frequency modes *must* have positive norm

Quantum fermion field Ψ

Step 1 - classical Lagrangian

- Classical Lagrangian density for a fermion field of mass M

$$\mathcal{L}_{\Psi} = \overline{\Psi} \left[i\gamma^{\mu} \nabla_{\mu} - M \right] \Psi$$

- Dirac equation

$$\left[i\gamma^{\mu} \nabla_{\mu} - M \right] \Psi = 0$$

- Dirac inner product

$$\left(\Psi_1, \Psi_2 \right)_D = \int \overline{\Psi}_1 \gamma^{\mu} \Psi_2 \, dS_{\mu}$$

All modes have positive norm, regardless of the choice of positive frequency

Quantum fermion field Ψ

Step 2 - canonical quantization

- Canonical conjugate momentum

$$\Pi_\Psi = \frac{\delta \mathcal{L}_\Psi}{\delta \dot{\Psi}} \qquad \dot{\Psi} = \partial_t \Psi$$

- Impose equal-time *anti-commutation* relations on $t =$ constant

$$\begin{aligned} \left\{ \hat{\Psi}(t,\boldsymbol{x}), \hat{\Pi}_\Psi(t,\boldsymbol{x}') \right\} &= i\delta(\boldsymbol{x},\boldsymbol{x}') \\ \left\{ \hat{\Psi}(t,\boldsymbol{x}), \hat{\Psi}(t,\boldsymbol{x}') \right\} = &\ 0 = \left\{ \hat{\Pi}_\Psi(t,\boldsymbol{x}), \hat{\Pi}_\Psi(t,\boldsymbol{x}') \right\} \end{aligned}$$

$$\int_S \delta(\boldsymbol{x},\boldsymbol{x}')\, dS = 1$$

Quantum fermion field Ψ

Step 3 - definition of quantum states

(a) *Choice of time t*

Pick a suitable time co-ordinate t

(b) *Choice of positive frequency modes* ψ_j

Solutions of the Dirac equation such that

$$\psi_j \propto e^{-i\omega t} \qquad \omega > 0$$

(c) *Orthonormal basis of field modes*

- Both positive frequency modes ψ_j^+ and negative frequency modes ψ_j^- have positive norm
- Complete set of positive and negative frequency modes $\left\{ \psi_j^+, \psi_j^- \right\}$

Quantum fermion field Ψ

Step 3 - definition of quantum states

- Expand classical field in terms of orthonormal basis

$$\Psi = \sum_j b_j \psi_j^+ + c_j^\dagger \psi_j^-$$

- Promote expansion coefficients to operators $\hat{b}_j$, $\hat{c}_j$ with

$$\left\{ b_j, b_k^\dagger \right\} = \delta_{jk} = \left\{ c_j, c_k^\dagger \right\}$$

- $\hat{b}_j$, $\hat{c}_j$ - particle annihilation operators
  $\hat{b}_j^\dagger$, $\hat{c}_j^\dagger$ - particle creation operators
- Define the *vacuum* state $|0\rangle$:

$$\hat{b}_j \, |0\rangle = \hat{c}_j \, |0\rangle = 0$$

Observables for physical interpretation of states

Massive scalar field Φ

Stress-energy tensor

$$\hat{T}_{\mu\nu} = (1-2\xi)\nabla_\mu\hat{\Phi}\nabla_\nu\hat{\Phi} + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu}\nabla_\lambda\hat{\Phi}\nabla^\lambda\hat{\Phi} - 2\xi\hat{\Phi}\nabla_\mu\nabla_\nu\hat{\Phi}$$
$$+2\xi g_{\mu\nu}\hat{\Phi}\Box\hat{\Phi} + \xi\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}\right)\hat{\Phi}^2 - \frac{1}{2}g_{\mu\nu}M^2\hat{\Phi}^2$$

Massive fermion field Ψ

Stress-energy tensor

$$\hat{T}_{\mu\nu} = \frac{i}{8}\left\{\left[\hat{\overline{\Psi}}, \gamma_\mu\nabla_\nu\hat{\Psi}\right] + \left[\hat{\overline{\Psi}}, \gamma_\nu\nabla_\mu\hat{\Psi}\right] - \left[\nabla_\mu\hat{\overline{\Psi}}, \gamma_\nu\hat{\Psi}\right] - \left[\nabla_\nu\hat{\overline{\Psi}}, \gamma_\mu\hat{\Psi}\right]\right\}$$

An example in Minkowski space

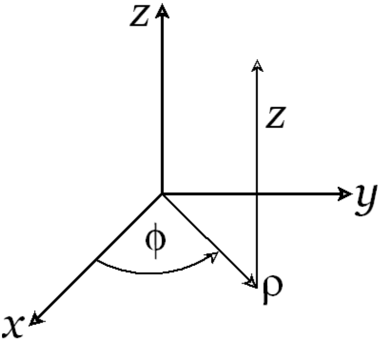
- Minkowski space in cylindrical co-ordinates

$$ds^2 = -dt^2 + d\rho^2 + \rho^2\, d\varphi^2 + dz^2$$

- Scalar field modes

$$\phi_j = \frac{1}{\sqrt{8\pi^2\,|\omega|}} e^{-i\omega t + im\varphi + ikz} J_m(q\rho)$$

- Norm of the modes is positive if $\omega > 0$
- Frequency of the modes is positive if $\omega > 0$



Rindler space

Rindler co-ordinates

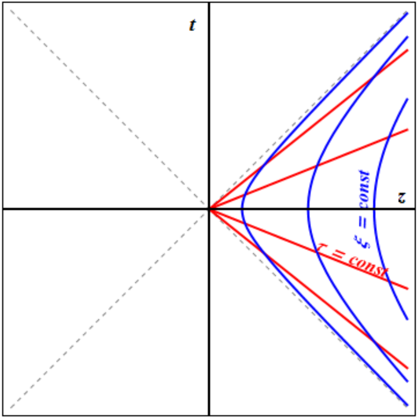
$$t = e^{g\zeta} \sinh(g\tau) \qquad z = e^{g\zeta} \cosh(g\tau)$$

Rindler scalar field modes

$$\phi_j = \frac{1}{2|\widetilde{\omega}|} e^{-i\widetilde{\omega}\tau} \widetilde{\phi}(x, y, \zeta)$$

Rindler vacuum

- $\widetilde{\omega} > 0$ - positive frequency
- $\widetilde{\omega} > 0$ - positive norm
- Rindler vacuum not equivalent to Minkowski vacuum



[Figure taken from Fulling and Matsas *Scholarpedia* **9** 31789 (2014)]

Rotating quantum states on Minkowski space-time

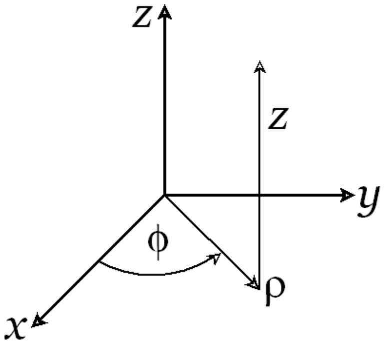
Minkowski space in rotating co-ordinates

Minkowski space in cylindrical co-ordinates

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 \, d\varphi^2 + dz^2$$

Rotating co-ordinates

$$t \to \tilde{t}, \qquad \varphi \to \widetilde{\varphi} = \varphi - \Omega t$$



Rotating metric

$$ds^2 = -\left(1 - \rho^2 \Omega^2\right) d\tilde{t}^2 + 2\rho^2 \Omega \, d\tilde{t} \, d\widetilde{\varphi} + d\rho^2 + \rho^2 \, d\widetilde{\varphi}^2 + dz^2$$

Speed of light surface (SOL) when $\rho = \Omega^{-1}$

Scalar field modes

$$\phi_j = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\omega t + im\varphi + ikz} J_m(q\rho) = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\tilde{\omega}\tilde{t} + im\tilde{\varphi} + ikz} J_m(q\rho)$$

$$\tilde{\omega} = \omega - m\Omega$$

Scalar field modes

$$\phi_j = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\omega t + im\varphi + ikz} J_m(q\rho) = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\tilde{\omega}\tilde{t} + im\tilde{\varphi} + ikz} J_m(q\rho)$$

$$\tilde{\omega} = \omega - m\Omega$$

Norm of these modes

$$\left(\phi_j, \phi_{j'}\right)_{KG} = \frac{\omega}{|\omega|} \delta\left(j, j'\right)$$

Scalar field modes

$$\phi_j = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\omega t + im\varphi + ikz} J_m(q\rho) = \frac{1}{\sqrt{8\pi^2 |\omega|}} e^{-i\tilde{\omega}\tilde{t} + im\tilde{\varphi} + ikz} J_m(q\rho)$$

$$\tilde{\omega} = \omega - m\Omega$$

Norm of these modes

$$\left(\phi_j, \phi_{j'}\right)_{KG} = \frac{\omega}{|\omega|} \delta\left(j, j'\right)$$

Frequency of these modes

Corotating Hamiltonian $H = i\partial_{\tilde{t}} \neq i\partial_t$

Frequency in rotating co-ordinates $\tilde{\omega}$

Defining states

Rotating vacuum state

- Positive frequency modes ϕ_j^+ must have positive norm
- We must therefore choose positive frequency modes ϕ_j with $\omega > 0$
- Negative frequency modes are ϕ_j^*
- Expansion of the field

$$\Phi = \sum_m \int_M^\infty d\omega \int dk \left[a_j \phi_j + a_j^\dagger \phi_j^* \right]$$

- Vacuum state $|0\rangle$ is then identical to the non-rotating Minkowski vacuum

[Letaw and Pfautsch *PRD* **22** 1345 (1980)]

Defining states

Rotating thermal state

- Frequency in rotating co-ordinates $\widetilde{\omega} = \omega - m\Omega$
- Energy in rotating thermal expectation values at inverse temperature $\beta = T^{-1}$

$$\langle a_j^\dagger a_{j'} \rangle_\beta = \frac{\delta(j, j')}{\exp(\beta \widetilde{\omega}) - 1}$$

- Modes with $\omega > 0$ but $\widetilde{\omega} < 0$
    - Limit $\beta \to \infty$ is non-zero
    - Divergent when $\widetilde{\omega} \sim 0$
- Rotating thermal states are ill-defined *everywhere*

[Vilenkin *PRD* **21** 2260 (1980)]

[Duffy and Ottewill *PRD* **67** 044002 (2003)]

Defining states with a boundary present

- $\Phi = 0$ at reflecting boundary at $\rho = R$
- Field modes

$$\phi_j = \frac{1}{\sqrt{8\pi^2 \left| \omega \right|}} e^{-i\tilde{\omega}\tilde{t} + im\tilde{\varphi} + ikz} J_m(\eta_{m,n}\rho / R)$$

- Frequency

$$\omega = \pm \sqrt{k^2 + \eta_{m,n}^2 / R^2} \qquad \tilde{\omega} = \omega - m\Omega$$

- If $R < \Omega^{-1}$, by the properties of the zeros of the Bessel functions, $\tilde{\omega} > 0$ for all $\omega > 0$
- In this case rotating thermal states are well-defined

[Duffy and Ottewill *PRD* **67** 044002 (2003)]

Fermion field modes

$$\psi_j = \frac{1}{\sqrt{8\pi^2}} e^{-i\widetilde{E}\tilde{t}+ikz} \begin{pmatrix} \chi \\ \frac{2\lambda E}{|E|}\chi \end{pmatrix}$$

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\frac{2\lambda k}{p}} e^{im\widetilde{\varphi}} J_m(q\rho) \\ 2i\lambda\sqrt{1-\frac{2\lambda k}{p}} e^{i(m+1)\widetilde{\varphi}} J_{m+1}(q\rho) \end{pmatrix}$$

$$\widetilde{E} = E - \Omega\left(m+\frac{1}{2}\right)$$

[Ambrus and EW *PLB* **734** 296 (2014)]

Fermion field modes

$$\psi_j = \frac{1}{\sqrt{8\pi^2}} e^{-i\widetilde{E}\widetilde{t}+ikz} \begin{pmatrix} \chi \\ \frac{2\lambda E}{|E|}\chi \end{pmatrix}$$

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$$\widetilde{E} = E - \Omega\left(m + \frac{1}{2}\right)$$

Norm of these modes

$$\left(\psi_j, \psi_{j'}\right)_D = \delta\left(j, j'\right)$$

[Ambrus and EW *PLB* **734** 296 (2014)]

Rotating vacuum state

Vilenkin quantization [Vilenkin *PRD* **21** 2260 (1980)]

- Positive frequency $E > 0$
- Positive frequency modes ψ_j
- Negative frequency modes $\widetilde{\psi}_j = i\gamma^2 \psi_j^*$
- Expansion of the field

$$\Psi_V = \sum_m \int_M^\infty dE \int dk \left[b_{j:V} \psi_j + c_{j:V}^\dagger \widetilde{\psi}_j \right]$$

- Vacuum state $|0_V\rangle$ is then identical to the non-rotating Minkowski vacuum

Rotating vacuum state

Iyer quantization [Iyer *PRD* **26** 1900 (1982)]

- Positive frequency $\widetilde{E} > 0$
- Expansion of the field

$$\Psi_I = \sum_m \int_{\widetilde{E}>0, |E|>M}^{\infty} dE \int dk \left[b_{j:I} \psi_j + c_{j:I}^{\dagger} \widetilde{\psi}_j \right]$$

- Vacuum state $|0_I\rangle$ is then *not* the non-rotating Minkowski vacuum

$$b_{j:I} = \begin{cases} b_{j:V} & E > 0 \\ i^{2m+1} c_{j':V}^{\dagger} & E < 0 \end{cases}$$

Rotating thermal state

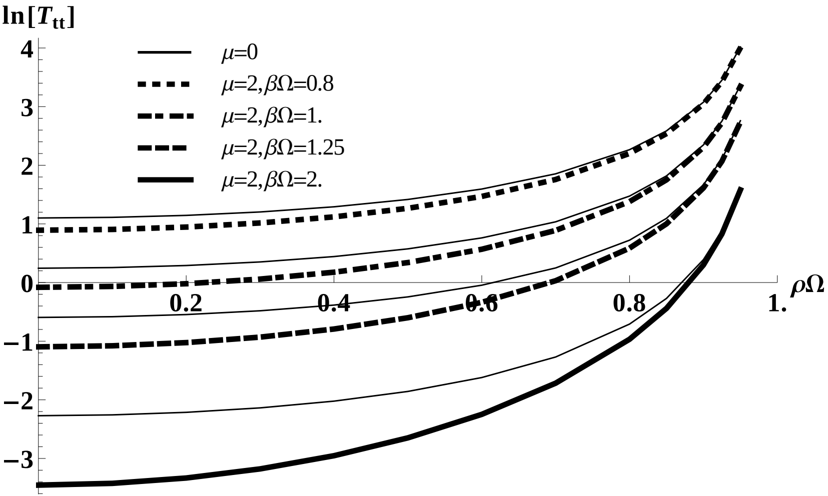
- Frequency in rotating co-ordinates $\widetilde{E} = \omega - \Omega \left(m + \frac{1}{2} \right)$
- Energy in rotating thermal expectation values at inverse temperature $\beta = T^{-1}$

$$\langle b_j^\dagger b_{j'} \rangle_\beta = \langle c_j^\dagger c_{j'} \rangle_\beta = \frac{\delta\left(j, j'\right)}{\exp\left(\beta \widetilde{E}\right) + 1}$$

- Limit $\beta \to \infty$ is non-zero for modes with $\widetilde{E} < 0$
- Rule out such modes by using Iyer quantization
- Fermi-Dirac density of states factor finite for all $\widetilde{E}$
- Rotating thermal state can be defined on the unbounded space-time for fermions but not bosons

[Ambrus and EW *PLB* **734** 296 (2014)]

Rotating thermal state



[Ambrus and EW *PLB* **734** 296 (2014)]

Rotating quantum states on anti-de Sitter space-time

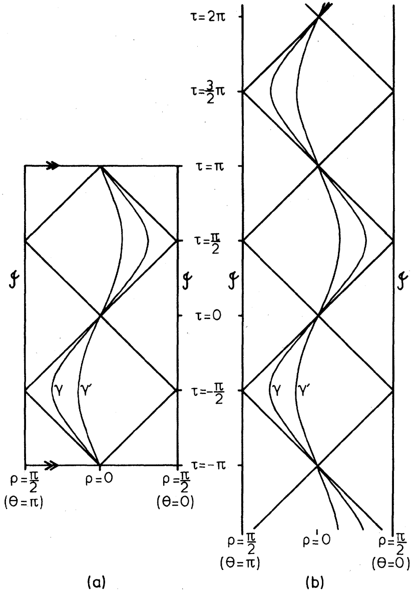
Anti-de Sitter (adS) space-time

Metric

$$ds^2 = a^2 \sec^2\rho \left[-d\tau^2 + d\rho^2 + \sin^2\rho \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right]$$

a - radius of curvature of adS

[Figure taken from Avis, Isham and Storey, *PRD* **18** 3565 (1978)]



Rotating adS

Rotating co-ordinates

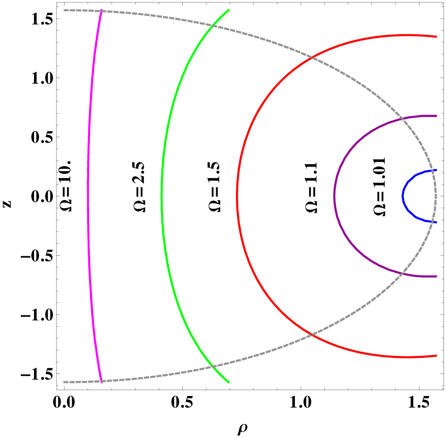
$$\tau \to \tilde{\tau}, \quad \varphi \to \tilde{\varphi} = \varphi - \Omega\tau$$

$$\begin{aligned} ds^2 &= a^2 \sec^2 \rho \left[-\varepsilon \, d\tilde{\tau}^2 + d\rho^2 \right. \\ &\quad + 2\Omega \sin^2 \rho \sin^2 \theta \, d\tilde{\tau} \, d\tilde{\varphi} \\ &\quad \left. + \sin^2 \rho \left(d\theta^2 + \sin^2 \theta \, d\tilde{\varphi}^2 \right) \right] \end{aligned}$$

where

$$\varepsilon = \left(1 - \Omega^2 \sin^2 \rho \sin^2 \theta \right)$$

Speed-of-light surface (SOL) where $\varepsilon = 0$



[Ambrus and EW arXiv:1405.2215 [gr-qc]]

Scalar field modes

$$\phi_{n\ell m} = N_{n\ell} e^{-i\omega\tau + im\varphi} R_{n\ell}(\rho)\Theta_{\ell m}(\theta) = N_{n\ell} e^{-i\widetilde{\omega}\widetilde{\tau} + im\widetilde{\varphi}} R_{n\ell}(\rho)\Theta_{\ell m}(\theta)$$

Scalar field modes

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Quantum numbers

$$\tilde{\omega} = \omega - m\Omega \qquad \omega = 2n + \ell + \kappa$$

$$\kappa = \frac{3}{2} + \sqrt{M^2a^2 + \xi\mathcal{R}a^2 + \frac{9}{4}} > 0$$

$$n \geq 0 \qquad \ell \geq |m| \geq 0$$

Scalar field modes

$$\phi_{n\ell m} = N_{n\ell} e^{-i\omega\tau + im\varphi} R_{n\ell}(\rho)\Theta_{\ell m}(\theta) = N_{n\ell} e^{-i\widetilde{\omega}\widetilde{\tau} + im\widetilde{\varphi}} R_{n\ell}(\rho)\Theta_{\ell m}(\theta)$$

Quantum numbers

$$\widetilde{\omega} = \omega - m\Omega \qquad \omega = 2n + \ell + \kappa$$

$$\kappa = \frac{3}{2} + \sqrt{M^2a^2 + \xi\mathcal{R}a^2 + \frac{9}{4}} > 0$$

$$n \geq 0 \qquad \ell \geq |m| \geq 0$$

Norm of these modes

$$(\phi_{n\ell m}, \phi_{n'\ell'm'})_{KG} = \frac{\omega}{|\omega|}\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}$$

Defining states

Rotating vacuum state

- Positive frequency modes $\phi^{+}_{n\ell m}$ must have positive norm
- We must therefore choose positive frequency modes $\phi_{n\ell m}$ with $\omega > 0$
- Negative frequency modes are $\phi^{*}_{n\ell m}$
- Expansion of the field

$$\Phi = \sum_{n}\sum_{\ell}\sum_{m}\left[a_{n\ell m}\phi_{n\ell m} + a^{\dagger}_{n\ell m}\phi^{*}_{n\ell m}\right]$$

- Vacuum state $|0\rangle$ is then identical to the non-rotating anti-de Sitter vacuum

[Kent and EW `arXiv:1410.3215 [gr-qc]`]

Defining states

Rotating thermal state

- Frequency in rotating co-ordinates $\widetilde{\omega} = \omega - m\Omega$
- Energy in rotating thermal expectation values at inverse temperature $\beta = T^{-1}$

$$\langle a_j^\dagger a_{j'} \rangle_\beta = \frac{\delta(j, j')}{\exp(\beta\widetilde{\omega}) - 1}$$

- From the properties of the quantum numbers

$$\omega = \kappa + 2n + \ell \geq \kappa + 2n + |m| > |m|$$

$$\Rightarrow \qquad \widetilde{\omega} \geq |m| - m\Omega > 0 \qquad \text{if } \Omega < 1$$

- If $\Omega < 1$ rotating thermal states will be well-defined
- If $\Omega < 1$ there is no speed-of-light surface

Fermion field modes

$$\psi_{E\ell m} = e^{-i\widetilde{E}\widetilde{\tau}} \chi_{E\ell m}(\rho, \theta, \widetilde{\varphi})$$

[Cotaescu *PRD* **60** 124006 (1999)]
[Ambrus and EW `arXiv:1405.2215 [gr-qc]`]

Fermion field modes

$$\psi_{E\ell m} = e^{-i\widetilde{E}\widetilde{\tau}} \chi_{E\ell m}(\rho, \theta, \widetilde{\varphi})$$

Quantum numbers

$$\widetilde{E} = E - m\Omega \qquad E = Ma + 2n + \ell + 2$$

$$n \geq 0 \qquad \ell \geq |m| \geq 0$$

[Cotaescu *PRD* **60** 124006 (1999)]
[Ambrus and EW `arXiv:1405.2215 [gr-qc]`]

Fermion field modes

$$\psi_{E\ell m} = e^{-i\widetilde{E}\widetilde{\tau}} \chi_{E\ell m}(\rho, \theta, \widetilde{\varphi})$$

Quantum numbers

$$\widetilde{E} = E - m\Omega \qquad E = Ma + 2n + \ell + 2$$

$$n \geq 0 \qquad \ell \geq |m| \geq 0$$

Norm of these modes

$$\left(\psi_{E\ell m}, \psi_{E'\ell' m'}\right)_D = \delta_{\ell\ell'}\delta_{mm'}\delta\left(E, E'\right)$$

[Cotaescu *PRD* **60** 124006 (1999)]

[Ambrus and EW `arXiv:1405.2215 [gr-qc]`]

Defining states

Rotating vacuum state

- All fermion modes have positive norm
- Non-rotating vacuum state arises from choosing $E > 0$ to be positive frequency
- Rotating vacuum state arises from choosing $\widetilde{E} > 0$ to be positive frequency
- From the properties of the quantum numbers

$$E = Ma + 2n + \ell + 2 \geq Ma + 2n + |m| + 2 > |m|$$

$$\Rightarrow \qquad \widetilde{E} > |m| - m\Omega \geq 0 \qquad \text{if } \Omega \leq 1$$

- If $\Omega \leq 1$, the rotating vacuum coincides with the non-rotating vacuum
- If $\Omega > 1$ the rotating and non-rotating vacua are different

Rotating thermal state

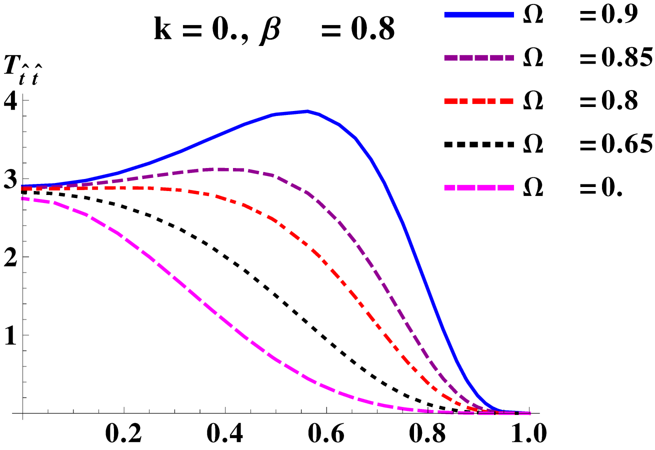
- Frequency in rotating co-ordinates $\widetilde{E} = E - m\Omega$
- Energy in rotating thermal expectation values at inverse temperature $\beta = T^{-1}$

$$\langle b_j^\dagger b_{j'} \rangle_\beta = \langle c_j^\dagger c_{j'} \rangle_\beta = \frac{\delta(j, j')}{\exp\left(\beta \widetilde{E}\right) + 1}$$

- Limit $\beta \to \infty$ is non-zero for modes with $\widetilde{E} < 0$
- Rule out such modes by using rotating vacuum
- Fermi-Dirac density of states factor finite for all $\widetilde{E}$
- Rotating thermal state can be defined on the unbounded space-time for all Ω

[Ambrus and EW, `arXiv:1405.2215 [gr-qc]`]

Rotating thermal state



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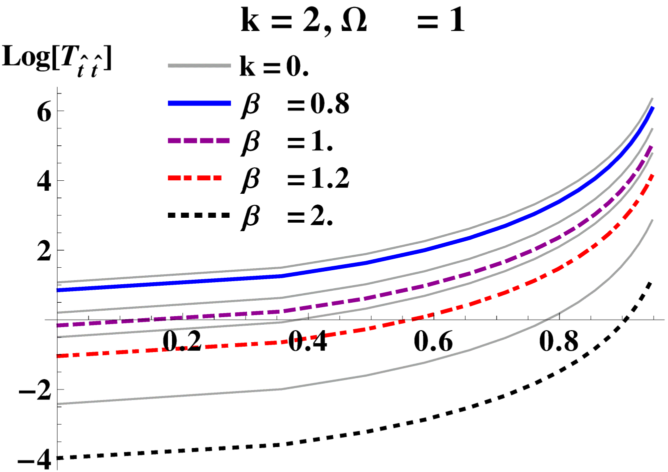
Rotating thermal state



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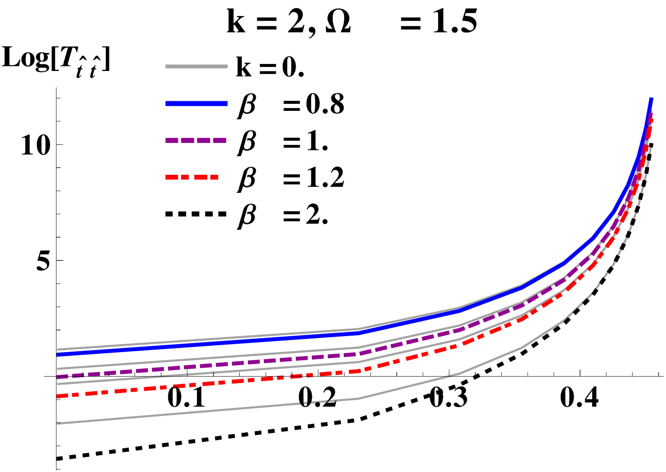

[Ambrus and EW `arXiv:1405.2215 [gr-qc]`]

Conclusions

Minkowski space

Scalars

- Positive frequency modes must have positive norm
- Rotating vacuum is the same as the Minkowski vacuum
- Rotating thermal states cannot be defined unless the system is enclosed in a boundary sufficiently close to the axis of rotation

Fermions

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- Two possible rotating vacua: Vilenkin and Iyer
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Anti-de Sitter space

Scalars

- Rotating vacuum is identical to the non-rotating vacuum
- If $\Omega < 1$, positive norm modes have positive frequency as seen by the rotating observer
- Rotating thermal states well-defined in this case

Fermions

- If $\Omega \leq 1$, rotating vacuum is identical to the non-rotating vacuum
- If $\Omega > 1$, rotating vacuum is distinct from the non-rotating vacuum
- For all Ω rotating thermal states can be defined

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Conclusions

Key points

- Fermions and bosons are different when it comes to defining quantum states
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Implications

- Considerations apply to general curved space-times
- No rotating thermal state exists on a Kerr black hole for bosonic fields
- A rotating thermal state can be defined for fermions on a Kerr black hole

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