Spontaneous Symmetry Breaking in Maximal Supergravities

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Outline

Some history on $\mathcal{N} = 8, D = 4$ SUGRA

- Construction
- Potential Relevance
- Structure of the model
- Major Discoveries

Some history on $\mathcal{N} = 8, D = 4$ SUGRA: Construction

- 1915: Einstein, Hilbert: General Relativity
- 1941: Rarita and Schwinger: Theory of Spin-3/2 field
- 1973: Wess and Zumino: Lagrangian for a SUSY field theory
- 1976: Freedman, van Nieuwenhuizen, Ferrara: SUGRA
- 1976: $\mathcal{N} = 2$, $\mathcal{N} = 3$
- 1977: Freedman and Das: local $SO(2)$, $SO(3)$ invariance: Extra terms (spin-3/2 mass and cosmological constant)
- 1978: Cremmer and Julia: Model with 32 supercharges in D=11
- 1978: $\mathcal{N} = 1$ in $D = 11 \rightarrow \mathcal{N} = 8$ in $D = 4$
- 1980: de Wit, Nicolai: local $SO(8)$ symmetry

Some history on $\mathcal{N} = 8, D = 4$ SUGRA: Relevance

- SUGRA inevitable consequence of SUSY+GR. ($\delta_\epsilon^2 \rightsquigarrow \mathrm{diff}!$)
- SUSY $\sim$ Relations between particles and interactions $\sim$ constraining principle.
- Maximal SUGRA: Particle content and couplings completely determined by symmetry – rigid construction.
- *First completely unified model!*
- Phenomenological failure: No SM-type chiral gauge interactions for $\mathcal{N} \geq 2$.
- Low energy limit of Superstring Theory. Potentially relevant for GUT/Planck–scale physics?
- Unexpected other uses due to AdS/CFT correspondence

Some history on $\mathcal{N} = 8$, $D = 4$ SUGRA: Structure

$SO(8)$–gauged $\mathcal{N} = 8$ SUGRA in $D = 4$:

The Lagrangian

Some history on $\mathcal{N} = 8, D = 4$ SUGRA: Structure

- The full Lagrangian is fairly involved...

$$\begin{aligned}
\mathcal{L} \quad = \quad & -\tfrac{1}{2} eR(e,\omega) - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\bar\psi^i_\mu\gamma_\nu D_\rho\psi_{\sigma i} - \bar\psi^i_\mu \overleftarrow{D}_\rho\gamma_\nu\psi_{\sigma i}\right) \\
& -\tfrac{1}{12} e\left(\bar\chi^{ijk}\gamma^\mu D_\mu\chi_{ijk}\bar\chi^{ijk}\overleftarrow{D}_\nu\gamma^\nu\chi_{ijk}\right) - \tfrac{1}{96}\mathcal{A}^{ijk\ell}_\mu\mathcal{A}^\mu_{ijk\ell} \\
& -\tfrac{1}{8} e\left(F^+_{\mu\nu IJ}(2S^{IJ,KL} - \delta^{IJ}_{KL})F^{+\mu\nu}{}_{KL} + \text{h.c.}\right) \\
& -\tfrac{1}{2} e\left(F^+_{\mu\nu IJ}S^{IJ,KL}O^{+\mu\nu KL} + \text{h.c.}\right) \\
& -\tfrac{1}{4} e\left(O^+_{\mu\nu}{}^{IJ}(S^{IJ,KL} + u^{ij}{}_{IJ}v_{ijKL})O^{+\mu\nu KL} + \text{h.c.}\right) \\
& -\tfrac{1}{24} e\left(\bar\chi_{ijk}\gamma^\nu\gamma^\mu\psi_{\nu\ell}(\hat{\mathcal{A}}_\mu{}^{ijk\ell} + \mathcal{A}_\mu{}^{ijk\ell}) + \text{h.c.}\right) \\
& -\tfrac{1}{2} e\bar\psi^i_\mu\psi^j_\nu\bar\psi^\mu_k\psi^\nu_\ell\delta^{k\ell}_{ij} \\
& +\tfrac{\sqrt{2}}{4} e\left(\bar\psi^i_\lambda\sigma^{\mu\nu}\gamma^\lambda\chi_{ijk}\bar\psi^j_\mu\psi^k_\nu + \text{h.c.}\right) + \ldots
\end{aligned}$$

Some history on $\mathcal{N}=8, D=4$ SUGRA: Structure

- … but here we focus on the last line:

$$
\begin{aligned}
&\ldots\\
&+e\left(\tfrac{1}{144}\eta\epsilon_{ijk\ell mnpq}\bar{\chi}^{ijk}\sigma^{\mu\nu}\chi^{\ell mn}\bar{\psi}^{p}_{\mu}\psi^{q}_{\nu}\right.\\
&\qquad\left.+\tfrac{1}{8}\bar{\psi}^{i}_{\lambda}\sigma^{\mu\nu}\gamma^{\lambda}\chi_{ikl}\bar{\psi}_{\mu j}\gamma_{\nu}\chi^{jkl}+\text{h.c.}\right)\\
&+\tfrac{\sqrt{2}}{864}\eta e\left(\epsilon^{ijk\ell mnpq}\bar{\chi}_{ijk}\sigma^{\mu\nu}\chi_{\ell mn}\bar{\psi}^{r}_{\mu}\gamma_{\nu}\chi_{pqr}+\text{h.c.}\right)\\
&+\tfrac{1}{32}e\bar{\chi}^{ikl}\gamma^{\mu}\chi_{jkl}\bar{\chi}^{jmn}\gamma_{\mu}\chi_{imn}\\
&-\tfrac{1}{96}e\bar{\chi}^{ijk}\gamma^{\mu}\chi_{ijk}\bar{\chi}^{\ell mn}\gamma_{\mu}\chi_{\ell mn}\\
&+\sqrt{2}geA_{1\,ij}\bar{\psi}^{i}_{\mu}\sigma^{\mu\nu}\psi^{j}_{\nu}+\tfrac{1}{6}geA_{2i}^{jk\ell}\bar{\psi}^{i}_{\mu}\gamma^{\mu}\chi_{jk\ell}\\
&+\tfrac{\sqrt{2}}{144}g\eta e\epsilon^{ijkpq r\ell m}A^{n}_{2\,pqr}\bar{\chi}_{ijk}\chi_{\ell mn}+\text{h.c.}\\
&\\
&+g^{2}e\left(\tfrac{3}{4}A_{1}^{ij}A_{1\,ij}-\tfrac{1}{24}A_{2}{}^{i}{}_{jk\ell}A_{2\,i}{}^{jk\ell}\right)
\end{aligned}
$$

Some history on $\mathcal{N} = 8, D = 4$ SUGRA: Structure

- "Why is this so complicated if it has so much symmetry?" – Imagine listing the vertex coordinates of a dodecahedron in cartesian space.
- What are the key features?
  - Particle spectrum:
    - Graviton e ($\times$ 1),
    - Gravitini ψ^i ($\times$ 8),
    - Vectors F_{IJ} ($\times$ 28),
    - Fermions χ^{ijk} ($\times$ 56),
    - Scalars ϕ ($\times$ 70)
  - GR + Rarita–Schwinger + Dirac + YM + Higgses + var. Couplings + Scalar Potential

Some history on $\mathcal{N} = 8, D = 4$ SUGRA: Structure

- Generalized Electric/Magnetic duality invariance: 28 vector equations of motion + 28 Bianchi identities $\Rightarrow$ $\mathrm{Sp}(56)$–symmetry, *but* restricted by relation between F and $\delta\mathcal{L}/\delta F$ to subgroup G containing $SU(8)$.
- Another perspective: Dimensional reduction of $D = 11$ SUGRA ($R + F^2 + FFA$) gives 35 pseudo-scalars from 3–form A_{mnp} in $D = 11$, plus 28 scalars from g_{mn}, plus 7 scalars from dualizing $A_{\mu\nu i}$. Obvious $SO(7)$ symmetry can be enlarged to $SO(8)$; 35 scalars then live on $SL(8)/SO(8)$. *This can be extended to also include pseudo–scalars*: 35+35 (pseudo–)scalars live on $G/SU(8)$, with G a non–compact 70+63–dimensional Lie group with maximal compact subgroup $SU(8)$.
- $\mathcal{N} = 8$ SUGRA in $D = 4$ has hidden global $E_{7(+7)}$ symmetry!

Some history on $\mathcal{N} = 8$, $D = 4$ SUGRA: Structure

- General feature: After suitable dualizations, dimensionally reduced $D = 11$ SUGRA can be brought into form that exhibits global exceptional E-series symmetries: E_6 in $D = 5$, E_7 in $D = 6$, E_8 in $D = 3$: Generalized electric–magnetic dualities.
- Related to U–duality symmetry in Superstring Theory.
- Origin of these symmetries not yet fully clear, but tantalizing hints of strong relations to beautiful mathematics (del Pezzo surfaces, Octonions, hyperbolic Kac–Moody algebras, ...)

Some history on $\mathcal{N} = 8$, $D = 4$ SUGRA: Major Discoveries

- Our first "Theory of Everything"!
  (But phenomenologically a failure.)
- Potentially renormalizable! (Question not settled yet.)
- Simplest interacting S–matrix in $D = 4$?
- KLT relations: $\mathcal{N} = 8$ SUGRA $\sim$ ($\mathcal{N} = 4$ SYM)^2 (Holds at tree level, evidence up to 4–loop level.)
- AdS/CFT application: RG Flow (CFT) $\leftrightarrow$ Geometry (AdS)
- AdS4/CFT3 application: Field theory on M2 branes
- Further AdS/CMT applications
- *Non–compact gauge groups possible*! (C. Hull, 1985); allow de Sitter solutions; Uses in cosmological inflation phenomenology: (Kallosh, Linde, Prokushkin, Shmakova, 2002)

The Scalar Sector and Vacua

- Situation for $\mathcal{N} = 8$ in $D = 4$ essentially very similar to situation in other dimensions.
- Stationary Lorentz–invariant solutions of field equations $\equiv$ critical points in scalar potential.
- 11-D perspective: Much of the complicated dynamics in $D = 11$ ends up in the scalar potential. By finding solutions & uplifting them to a full $D = 11$ embedding, we may learn something about D=11 SUGRA / $\mathcal{M}$–Theory.

The Scalar Sector and Vacua

- General form of scalar potential: $V(\phi) \sim \#A_1^2 - \#A_2^2$
  (A_{1IJ}: $\bar{\phi}^i{}_\mu A_{1\,ij}\sigma^{\mu\nu}\phi^j{}_\nu$ gravitino mass term;
  ${A_2}^i{}_{jkl}$: $\sim \bar{\psi}A_2\chi$ "gravitino–fermion mass term")
- A_1, A_2 are polynomial in *vielbein*
  $\mathcal{V} = \exp(\phi_A T^A) \in E_{11-d(11-d)}/K(E_{11-d(11-d)})$
  (D=3: quadratic, D=4,5: cubic)
- Computationally (D=4 case):
  1. Take values for 70 scalars
  2. Map to E_7 algebra generator in complex 56-dim. fundamental irrep.
  3. Exponentiate and form "T–tensor"
  4. Extract A_1, A_2
  5. Compute potential
  6. Check stationarity

The Scalar Sector and Vacua

- General Properties:
  - Potentials unbounded from below
  - For compact gauge groups: generally AdS solutions
  - All interesting stationary points are saddle points; can be stable nevertheless (BF-Bound).
  - SUSY stationary points stable; many non-SUSY s.p. unstable
- Solutions known before 2009:
  - $\mathrm{SO}(8)$ solution with $\mathcal{N}=8$ SUSY: $\phi=0$
  - $\mathrm{SO}(8)\to\mathrm{SO}(3)\times\mathrm{SO}(3)$
    (Oldest known solution; has $\mathcal{N}=5$ counterpart with $SO(5)\to SO(3)$)
  - $\mathrm{SO}(8)\to\mathrm{SO}(7)^{\pm}$ (2 inequivalent ways)
  - $\mathrm{SO}(8)\to\mathrm{G}_2$ with $\mathcal{N}=1$ SUSY
  - $\mathrm{SO}(8)\to\mathrm{SU}(3)\times\mathrm{U}(1)$ with $\mathcal{N}=2$ SUSY
  - $\mathrm{SO}(8)\to\mathrm{SU}(4)$

Methods: old and modern

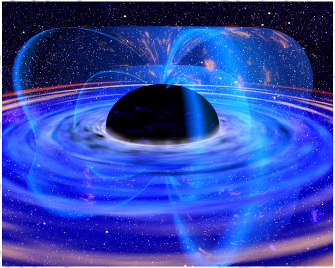
- *Conceptually*, the problem of finding critical points in the potential is very simple.
- Superficially involves just basic calculus and linear algebra.
- *Problem lies in the complexity of these potentials!*
- Example: $SO(8) \times SO(8)$ Gauged SUGRA with 32 supercharges in $D=3$: Scalar manifold is $E_{8(8)}/SO(16)$: 128 $(= 248 - 120)$–dimensional.
- "Euler angle" type parametrization of 128–dimensional submanifold of E_8 needed.

Methods: old and modern

- Reminder: Euler angles
- $SO(3)$ Euler angles in 3–d: 3×3 matrix; entries are polynomial in $\cos(\alpha_j)$, $\sin(\alpha_j)$.
- Rough estimate: Every matrix entry involves binary choice for each of the three angles: $3\times 3\times 2^3=72$ trigonometric factors (Actual matrix: 29).
- $E_{8(8)}/SO(16)$ Problem conceptually somewhat trickier here, as half–angles may alternatively appear, too: *more choice*!
- Generic matrix entry polynomial with at least binary choice (sinh / cosh) for each "angle" variable: $\sim 2^{128}$ summands.
- $D=3$ scalar potential 4–th order polynomial in matrix entries. Very crude estimate: $\sim 5^{128}\approx 10^{89}$ summands.
- Many of these potentials have more terms than there are electrons in the accessible Universe ($\sim 10^{80}$)!

Methods: old and modern

Douglas Adams inevitably comes to one's mind:



The rules to the game of Brockian Ultra-cricket, as played in the higher dimensions are strange and inexplicable. A full set of the rules is so massively complicated that the only time they were all bound together to form a single volume, they underwent gravitational collapse and became a black hole.

(D.N.A., Life, the Universe and Everything)

(Image source: `http://tinyurl.com/45xlvfb`)

. . . but these potentials are actually not *that* simple!

Methods: old and modern

- So, what to do?
- ... and by the way, how have the seven known solutions for $\mathcal{N} = 8, D = 4$ SUGRA been found?

Methods: old and modern

- One idea (N. Warner's approach, suggested by Tomaras):
  - Choose subgroup $H \subset G$.
  - Scalar potential V is a singlet w.r.t. gauge group G, and also w.r.t. H.
  - Let us expand $V(\phi_0 + \delta\phi)$ around a stationary point – we consider the action of G on $\delta\phi$:

$$V(\phi_0 + \delta\phi) = V(\phi_0) + \delta\phi(\partial V/\partial\phi) + \frac{1}{2}(\delta\phi)^2(\partial^2 V/\partial\phi^2) + \ldots$$

  - Stationarity $\equiv (\partial V/\partial\phi = 0)$.
  - If $\delta\phi$ transforms non–trivially under $H \subset G$, the linear term is absent (as one cannot form a G–singlet from a single H–nonsinglet).
  - Consequence: Stationary points on H–invariant submanifold lift to stationary points of the full potential!

Methods: old and modern

- Trade–off: Large subgroup $H \subset G$: low–dimensional H–invariant sub–manifold: Easy to analyze (few Euler angles, exponentiation feasible), but will not find solutions with residual unbroken gauge symmetry smaller than H.
- Useful choice: $H = SU(3)$ (N. Warner): 6–dimensional submanifold, analysis produced $SO(8), SO(7)^{\pm}, G_2, SU(4), SU(3) \times U(1)$ solutions.
- Another choice (T.F., 2003): $H = SO(3)$, Embedding: $\mathbf{8}_v \to \mathbf{3} + 5 \cdot \mathbf{1}$: 10 scalars, potential with 12 240 terms.

Methods: old and modern

- Since the 80's, no new solutions have been found for a long time...
- One article from 2009: "Are There Any New Vacua of Gauged N=8 Supergravity in Four Dimensions?" (C. Ahn, K. Woo – `arXiv:0904.2105`; none found).
- Another article from 2009: "Fourteen new stationary points in the scalar potential of SO(8)–gauged N=8, D=4 supergravity" (T.F. – `arXiv:0912.1636`)
- Among the new solutions: a SUSY vacuum with residual $U(1) \times U(1)$ symmetry, 8 critical points *without any* residual gauge symmetry: $SO(8)$ broken *completely*.

Methods: old and modern

- Breakthrough based on a completely new approach to the problem.
- Situation right now: there seem to be at least 160 different critical points, some with fairly interesting properties.
- Some calculations still running; first batch of 40 solutions was released recently (T.F., `arXiv:1109.1424`).

Methods: old and modern

- How does the new method work?
- Based on a combination of tricks:
  - Numerical validation of candidate solutions.
  - Complete intermediate avoidance of symbolic methods in search: numerical approach.
  - Re–phrasing stationarity condition as a (high–dimensional) numerical optimization problem: various choices for objective function, e.g. $|\nabla V|^2 = 0 = \min!$
  - Utilizing advanced algorithmic methods to obtain fast gradients.
  - ...

Methods: old and modern

- Based on a combination of tricks:
  - ...
  - 2nd "prettifying" optimization to bring numerical solution to nicely coordinate–aligned form.
  - Arbitrary–precision numerics for all optimizations to obtain numbers to 100+ valid digits.
  - *Inverse Symbolic Computation* (Integer relation algorithm – PSLQ) to map numerical values to "simple" polynomials with integer coefficients (which they are a root of).
- *Generally applicable to all models with many scalars and complicated potentials!*

Methods: old and modern

- Fast Gradients: Approach
  - Given a function $f : (\mathbb{R}^n \to \mathbb{R}) \in \mathcal{C}^2$ and a computer implementation of a program $\mathtt{P}$ calculating f that needs time T to calculate f at a given point.
  - Naive approach to calculate gradient needs at least $n+1$ function evaluations, gives half the available numerical accuracy.
  - Speelpenning's PhD thesis (1980): There is a (fully automatic) program transformation $\mathtt{P} \to \tilde{\mathtt{P}}$ such that $\tilde{\mathtt{P}}$ calculates both $f(\vec{x})$ and $\nabla f(\vec{x})$ in at most $5T$ – independent of n, and with full numerical accuracy!
  - Method widely used in engineering design optimization – e.g. to deal with many geometry parameters.

Methods: old and modern

- Fast Gradients: Approach
  - Price to be paid:
    - *Must evaluate f once and remember all intermediate numerical quantities...*
    - *...plus one extra number per intermediate value...*
    - *...and the complete execution path (i.e. all conditional branches/loop exits taken).*
  - Strategy: One first evaluates f, then goes once again through the calculation *backwards*, trying to answer for each intermediate value the question: "If, at the point this intermediate quantity y became first known, we replaced it by $y + \epsilon$, by how much (relative to ϵ) would the final result change (to 1st order in ϵ)?"
  - Can use chain rule to back–propagate these "sensitivities" through entire calculation.
  - Sensitivities on input data = gradient!

Methods: old and modern

- Inverse symbolic computation – Example:
  - Given the number such as 8.4721359549995793928183473374625524708812367192230514507 to 50 valid digits, is there a polynomial with integer coefficients carrying substantially less than $50\log_2 10$ bits of information of which this is a zero?
  - The PSLQ algorithm gives us a candidate analytic expression for this number: $\sqrt{36+16\sqrt{5}}$.
  - This analytic conjecture then can be checked to a very high number of digits (and often also analytically).
- In combination with other tricks, this allows us to completely avoid the intractable intermediate analytic stage – and nevertheless obtain analytic results!

Results and open problems

- Among the new solutions:
  - $U(1) \times U(1)$ $\mathcal{N} = 1$ vacuum.
  - *Supersymmetry not a pre–requisite for stability* – this was seen already in $D = 3$, but in $D = 4$ the *only* known example so far is the old $SO(3) \times SO(3)$ critical point!
  - Details: `arXiv:1010.4910` (T.F., K. Pilch, N. Warner)
  - Also: Novel solutions with many different symmetries (another $SO(3) \times SO(3)$, $SO(3) \times U(1) \times U(1)$, $SO(3) \times U(1)$ ($3\times$), $SO(3)$ ($4x$); many of type $U(1) \times U(1)$, $U(1)$ and $\emptyset$).

Open problems

- What about maximal SUGRA in $D = 5$ (and correspondingly, $\mathcal{N} = 4$ SYM)? $\rightarrow$ Currently investigated (T.F., Gereon Kaiping)
- Optimization–based method contains an element of chance: "fishing" for solutions. Some are found more often than others. This has been found to systematically miss some solutions (in particular, the stable $SO(3) \times SO(3)$ solution!) – are there ways to improve this?
- RG flows in $D = 3$ CFT that involve new critical points?
- 11–dimensional mother geometries?

Conclusion

- Detailed analysis of supergravity potentials in a (fairly large) class of models with many scalars has been considered intractable for a long time.
- This turned out to be wrong – there indeed is a general method to systematically analyze these potentials.
- Some of the methods employed probably can be used to great benefit for other problems in QFT as well. (Parameter fitting?)
- Powerful novel approaches produce many new results right now, but leave a number of blind spots. (Some solutions missed systematically.)
- Many options for creatively modifying/extending the method (e.g. tweaking the objective function, using conjectured properties of solutions, homotopy methods, etc.)