The renormalization group flow of scalar-tensor theory

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Motivation and outline

- Search for a consistent combination of quantum theory and general relativity into quantum gravity
- Required to understand the physics at very high energies
- Well-defined ultraviolet behaviour
- Continuum, covariant quantum field theory methods
- Asymptotic safety
- Application of funtional renormalization group techniques to gravity
- Pure gravity and gravity coupled to matter
- Early-time cosmology

General Relativity as Effective Field Theory

- General Relativity used as an effective field theory with momenta in the loops cut off at some scale
  - Quantum corrections to the Newtonian potential
  - Independent of UV completion
- Beyond the Planck scale:
  - Higher order quantum corrections are no longer suppressed
  - New counterterms are required at each order of perturbation theory
  - Loss of predictivity

Scale dependent action

- Effective average action Γ_k includes all quantum fluctuations with momenta larger than the infrared cutoff scale k

$$\Gamma_k[\phi] = \sum_i g_i(k)\mathcal{O}^i[\phi] = \sum_i \tilde{g}_i k^{d_i}\mathcal{O}^i[\phi]$$

$$k\frac{\partial}{\partial k}\tilde{g}_i(k) = \beta_i; \qquad \beta_i(\tilde{g}_*) = 0$$

- Asymptotic safety
  - The renormalization group flow approaches a UV fixed point
  - The dimension of the UV critical surface formed by the trajectories attracted towards the fixed point is finite

UV Critical Surface

- Linearized flow at FP

$$\beta_i = M_{ij} \left(\tilde{g}_j(k) - \tilde{g}_j^* \right) \ , \ M_{ij} = \frac{\partial \beta_i(\tilde{g})}{\partial \tilde{g}_j} \|_{\tilde{g}=\tilde{g}^*}$$

- Can be solved to give

$$\tilde{g}_i(k) = \tilde{g}_i^* + C_k V_i^k k^{-\vartheta_k}$$

- $M_{ij} V_i^k = -\vartheta_k V_i^k$; C_k: initial constants
- If $C_k = 0$ for $\vartheta_k < 0$, the FP is reached for $k \to \infty$
- dim(UVCS) = number of positive ϑ_k

Scale dependent generating functionals

$$
\begin{aligned}
e^{-W_k[J]} &= \int D\phi \exp\left\{-S[\phi] - \Delta S_k[\phi] - \int dx\, J\phi\right\} \\
\Gamma_k[\phi] &= W_k[J] - \int dx\, J\phi - \Delta S_k[\phi]
\end{aligned}
$$

$$
\Delta S_k[\phi] = \frac{1}{2}\int d^4q\, \phi(q)\, \mathcal{R}_k(q^2)\, \phi(-q)
$$

- $\mathcal{R}_k(z) \to 0$ for $k^2 \ll z$, $\mathcal{R}_k(z) \simeq k^2$ for $z \ll k^2$
- Propagation of low-scale field modes is suppressed
- $\Gamma_k \to \Gamma$ for $k \to 0$

Exact RG equation

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr} \left(\frac{\delta^2 \Gamma_k}{\delta \Phi \delta \Phi} + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k$$

- RG time $t = \log(k/k_0)$
- Φ: all fields present in the theory
- STr: generalized functional trace (minus sign for fermionic variables, factor 2 for complex variables)
- Approximation schemes: truncations, large N limit, expansion in dimension

Heat-kernel expansion

- If the derivative terms combine to Laplacians

$$\mathrm{Tr}\, W(-\nabla^2) = Q_2\left(W\right) B_0(-\nabla^2) + Q_1\left(W\right) B_2(-\nabla^2) + Q_0\left(W\right) B_4(-\nabla^2) + \ldots$$

$$\begin{aligned} Q_n(W) &= \frac{1}{\Gamma\left(n\right)} \int_0^\infty dz z^{n-1} W(z) \,\mathrm{for\, n} > 0 \\ Q_n(W) &= (-1)^n W^{(-n)}(0) \,\mathrm{for\, n} \leq 0 \end{aligned}$$

- $B_n = \int d^4x \sqrt{g}\, \mathrm{tr}\, \mathrm{b_n}$ include curvature invariants of n-th degree
- Obtain the beta functions by comparing the coefficients of the different curvature invariants

Einstein–Hilbert truncation

$$\Gamma_k[h, c, \bar{c}] = \frac{1}{16\pi G} \int d^4x \sqrt{\bar{g}}(2\Lambda - R) + S_{GF} + S_c; \; g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$\begin{aligned} \partial_t \left(\frac{2\Lambda}{16\pi G} \right) &= \frac{k^4}{16\pi}(A_1 + A_2\partial_t G + A_3\partial_t \Lambda) \\ -\partial_t \left(\frac{1}{16\pi G} \right) &= \frac{k^2}{16\pi}(A_4 + A_5\partial_t G + A_6\partial_t \Lambda) \end{aligned}$$

- Full solution: fixed point with two attractive directions

  M. Reuter (1996)

R^2 truncation

- Inclusion of R^2-coupling: fixed point with three attractive directions
- Occurence of anomalously large critical exponent
- Are there more and more relevant directions?

f(R)-gravity

$$\Gamma_k[h_{\mu\nu}, c_\mu, \bar{c}_\nu] = \int d^4x \, \sqrt{\bar{g}} f(R) + S_{GF} + S_c$$

$$f(R) = \sum_{m=0}^{n} g_m(k) R^m$$

- Expansion around spherical background

A. Codello, R. Percacci, C. R. (2007)

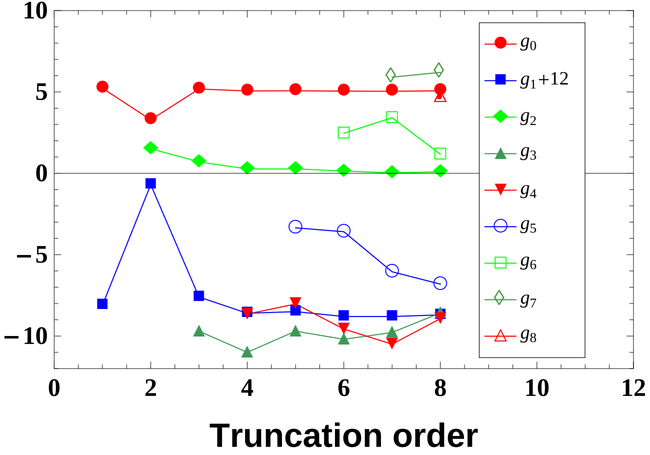
Position of the FP $\times$ 1000

n	$\tilde{g}_{0*}$	$\tilde{g}_{1*}$	$\tilde{g}_{2*}$	$\tilde{g}_{3*}$	$\tilde{g}_{4*}$	$\tilde{g}_{5*}$	$\tilde{g}_{6*}$	$\tilde{g}_{7*}$	$\tilde{g}_{8*}$
1	5.23	-20.1							
2	3.29	-12.7	1.51						
3	5.18	-19.6	0.70	-9.7					
4	5.06	-20.6	0.27	-11.0	-8.65				
5	5.07	-20.5	0.27	-9.7	-8.03	-3.35			
6	5.05	-20.8	0.14	-10.2	-9.57	-3.59	2.46		
7	5.04	-20.8	0.03	-9.78	-10.5	-6.05	3.42	5.91	
8	5.07	-20.7	0.09	-8.58	-8.93	-6.81	1.17	6.20	4.70

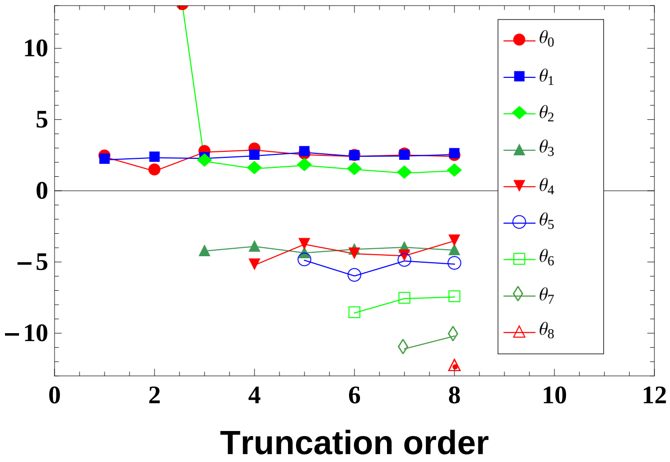
Critical exponents, first two critical exponents are a complex conjugate pair of the form $\vartheta' \pm \vartheta'' i$.

n	ϑ'	ϑ''	ϑ_2	ϑ_3	ϑ_4	ϑ_5	ϑ_6	ϑ_7	ϑ_8
1	2.38	2.17							
2	1.38	2.32	26.9						
3	2.71	2.27	2.07	-4.23					
4	2.86	2.45	1.55	-3.91	-5.22				
5	2.53	2.69	1.78	-4.36	-3.76	-4.88			
6	2.41	2.42	1.50	-4.11	-4.42	-5.98	-8.58		
7	2.51	2.44	1.24	-3.97	-4.57	-4.93	-7.57	-11.1	
8	2.41	2.54	1.40	-4.17	-3.52	-5.15	-7.46	-10.2	-12.3

Fixed point values × 1000



Critical exponents



UV critical surface

- Determine all couplings from a subset of three couplings, e. g. $\tilde{g}_0,\ \tilde{g}_1,\ \tilde{g}_2$ from

$$\tilde{g}_i\,(k) = \tilde{g}_i^* + C_k\, V_i^k\, k^{-\vartheta_k}$$

$$\begin{aligned}
\tilde{g}_3 &= +0.0006 + 0.0682\,\tilde{g}_0 + 0.4635\,\tilde{g}_1 + 0.8950\,\tilde{g}_2 \\
\tilde{g}_4 &= -0.0092 - 0.8365\,\tilde{g}_0 - 0.2089\,\tilde{g}_1 + 1.6208\,\tilde{g}_2 \\
\tilde{g}_5 &= -0.0157 - 1.2349\,\tilde{g}_0 - 0.7254\,\tilde{g}_1 + 1.0175\,\tilde{g}_2 \\
\tilde{g}_6 &= -0.0127 - 0.6226\,\tilde{g}_0 - 0.8240\,\tilde{g}_1 - 0.6468\,\tilde{g}_2 \\
\tilde{g}_7 &= -0.0008 + 0.8139\,\tilde{g}_0 - 0.1484\,\tilde{g}_1 - 2.0181\,\tilde{g}_2 \\
\tilde{g}_8 &= +0.0091 + 1.2543\,\tilde{g}_0 + 0.5085\,\tilde{g}_1 - 1.9012\,\tilde{g}_2
\end{aligned}$$

Results in pure gravity

- $f(R) = \sum_{i=0}^{n} g_i R^i \Rightarrow$ three relevant directions up to $n = 8$
- Series seems to converge
- Parametrization of UV critical surface at the fixed point
- Relied on heat-kernel expansion on a spherical background
  $\Rightarrow$ different couplings of higher curvature invariants are combined

Scalar-Tensor theory

$$\Gamma_k[h_{\mu\nu}, c_\mu, \bar{c}_\nu, \phi] = \int d^4x\sqrt{g}\left\{F(\phi^2, R) + \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi\right\} + S_{GF} + S_{gh}$$

$$F(\phi^2, R) = V_0(\phi^2) + V_1(\phi^2)\,R + \cdots + V_p(\phi^2)\,R^p = \sum_{a=0}^{p} V_a(\phi^2)\,R^a$$

- Expand around spherical background
- Arbitrary potential terms, analytic and Taylor expandable around $\phi^2 = 0$

G. Narain, C.R. (2009)

Gaussian matter fixed point

- Matter interactions can become weak in UV
- At Gaussian matter fixed point $\tilde{V}_a$ are $\tilde{\phi}^2$-independent

$$\tilde{V}_a^{(i)}(0) = 0 \text{ for } i \geq 1$$

- Can be shown to exist for arbitrary potential
- For specific form

$$V_a(\phi^2) R^a = \sum_{i=0}^{q} \lambda_{2i}^a(k) \phi^{2i} R^a \Rightarrow \tilde{\lambda}_{2i}^a(k) \to 0 \text{ for } i \geq 1$$

Linearized flow

$$M = \begin{pmatrix} M_{00} & M_{01} & 0 & 0 & \cdots \\ 0 & M_{11} & M_{12} & 0 & \ddots \\ 0 & 0 & M_{22} & M_{23} & \ddots \\ 0 & 0 & 0 & M_{33} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ with } M_{ij} = \begin{pmatrix} \frac{\partial \beta_{2i}^{(0)}}{\partial \tilde{\lambda}_{2j}^{(0)}} & \cdots & \frac{\partial \beta_{2i}^{(0)}}{\partial \tilde{\lambda}_{2j}^{(p)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \beta_{2i}^{(p)}}{\partial \tilde{\lambda}_{2j}^{(0)}} & \cdots & \frac{\partial \beta_{2i}^{(p)}}{\partial \tilde{\lambda}_{2j}^{(p)}} \end{pmatrix}$$

$$M_{ii} = 2i\,\mathbf{1} + M_{00} \;; \qquad M_{i,i+1} = (i+1)(2i+1)M_{01}$$

$$\rho_{2i}^{(a)} = \rho_0^{(a)} - 2\,i$$

- Gravitational corrections to the critical exponents are always the same

Example stability matrix

$$F(\phi^2, R) = \lambda_0^0 + \lambda_0^1 R + (\lambda_2^0 + \lambda_2^1 R)\phi^2 + (\lambda_4^0 + \lambda_4^1 R)\phi^4$$

$$(\tilde{\lambda}_0^0 \to 0.0065, \tilde{\lambda}_0^1 \to -0.022)$$

$$M|_{\mathrm{GMFP}} = \begin{pmatrix} -0.5804 & 1.5694 & -0.0058 & 0.0017 & 0 & 0 \\ -5.9041 & -4.4052 & -0.0028 & -0.0083 & 0 & 0 \\ 0 & 0 & 1.4196 & 1.5694 & -0.0346 & 0.0103 \\ 0 & 0 & -5.9041 & -2.4052 & -0.0166 & -0.0499 \\ 0 & 0 & 0 & 0 & 3.4196 & 1.5694 \\ 0 & 0 & 0 & 0 & -5.9041 & -0.4052 \end{pmatrix}$$

$$(2.49 \pm 2.37i, 0.49 \pm 2.37i, -1.51 \pm 2.37i)$$

$$M_{12} = 2 \times 3 \times \; M_{01}$$

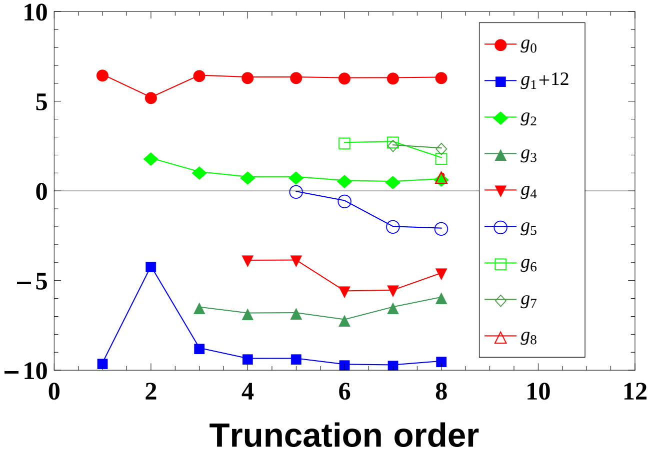
Position of the fixed point $\times$ 1000

p	$\tilde{\lambda}_{0*}^{(0)}$	$\tilde{\lambda}_{0*}^{(1)}$	$\tilde{\lambda}_{0*}^{(2)}$	$\tilde{\lambda}_{0*}^{(3)}$	$\tilde{\lambda}_{0*}^{(4)}$	$\tilde{\lambda}_{0*}^{(5)}$	$\tilde{\lambda}_{0*}^{(6)}$	$\tilde{\lambda}_{0*}^{(7)}$	$\tilde{\lambda}_{0*}^{(8)}$
1	6.50	-21.58							
2	5.22	-16.20	1.83						
3	6.45	-20.76	1.07	-6.47					
4	6.35	-21.34	0.79	-6.81	-3.87				
5	6.36	-21.34	0.79	-6.79	-3.85	-0.02			
6	6.31	-21.67	0.59	-7.17	-5.58	-0.54	2.70		
7	6.32	-21.70	0.53	-6.47	-5.53	-1.98	2.76	2.57	
8	6.34	-21.49	0.68	-5.92	-4.57	-2.07	1.86	2.39	0.83

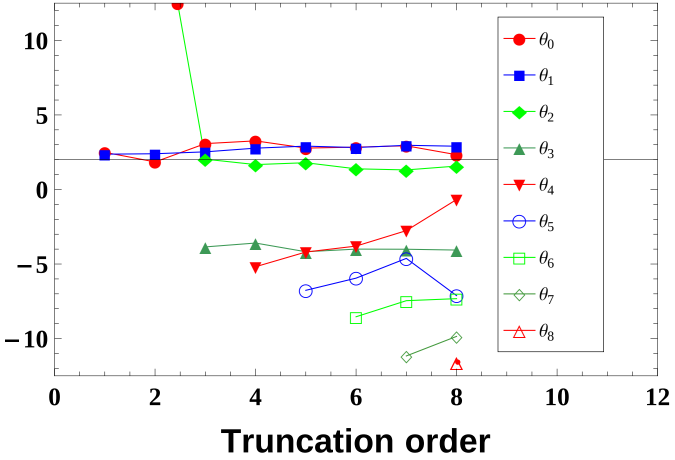
Critical exponents

p	ϑ_0'	ϑ_0''	$\vartheta_0^{(2)}$	$\vartheta_0^{(3)}$	$\vartheta_0^{(4)}$	$\vartheta_0^{(5)}$	$\vartheta_0^{(6)}$	$\vartheta_0^{(7)}$	$\vartheta_0^{(8)}$
1	2.49	2.39							
2	1.85	2.40	21.03						
3	3.08	2.52	2.03	-3.85					
4	3.26	2.77	1.67	-3.59	-5.18				
5	2.78	2.91	1.80	-4.18	-4.20	-6.76			
6	2.84	2.81	1.39	-4.00	-3.80	-5.95	-8.54		
7	2.93	2.96	1.31	-4.01	-2.76	-4.62	-7.46	-11.17	
8	2.33	2.90	1.57	-4.06	-0.67	-7.12	-7.32	-9.85	-11.61

Fixed point values × 1000



Critical exponents



Curvature squared truncation

$$\Gamma_k = \int d^4x \sqrt{g} \left[\frac{1}{16\pi G}(2\Lambda - R) + \frac{1}{2\sigma} C^2 - \frac{\omega}{3\sigma} R^2 + \frac{\theta}{\sigma} E \right] + S_{GF} + S_c$$

- Expansion on Einstein-space background
- Can distinguish R^2 and C^2
- Fully non-Gaussian fixed point with three attractive directions

  D. Benedetti, P. Machado, F. Saueressig (2009)

- Non-Gaussian fixed point with vanishing C^2-coupling and four attractive directions

  D. Litim, C. R. (in preparation)

Gravity and matter

- Nonminimally coupled scalar fields (scalar-tensor theory)
- Fixed point with vanishing matter couplings
- UV critical surface maintains low dimensionality
- Scalar, Dirac, and Maxwell matter fields minimally coupled
- Gravitational corrections to Yukawa and Yang-Mills couplings (on flat background)

Scalar-tensor theory in the early universe

- Possible applications for scalar-tensor theory in early universe cosmology
- Need for accelerated expansion to explain the short-comings of big-bang theory (flatness, horizon, entropy problem ...)
- Many different models
- Could be driven by a scalar field, the inflaton
- One of the most plausible $V = \frac{1}{2} m^2 \phi^2$

Renormalization group effects

- Different in our approach: extends to time-scales shorter than the Planck time
- Renormalization group scale should increase with decreasing time
- Strong renormalization group effects at short time scales
- Correct the equations of motion by inserting scale-dependent couplings

Cosmological evolution equations

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{g}(2\Lambda - R + \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + V(\phi))$$

$$\begin{aligned}
\dot{H} &= -4\pi G\left((1+w)\rho + \dot{\phi}^2\right)\\
\dot{\rho} &= -3(1+w)H\,\rho\\
\ddot{\phi} &= -3H\dot{\phi} - \frac{\mathrm{dV}}{\mathrm{d}\phi}\\
H^2 &= \frac{8\pi G}{3}\left(\rho + \frac{\dot{\phi}^2}{2} + V(\phi)\right)
\end{aligned}$$

Dimensionless variables

$$x = \sqrt{\frac{4\pi G}{3}}\frac{\dot{\phi}}{H}\ ;\ \ y = \sqrt{\frac{8\pi G}{3}}\frac{\sqrt{V}}{H}\ ; \lambda = -\frac{V'(\phi)}{8\pi G V}\ ;\ \ \Gamma = \frac{V(\phi)V''(\phi)}{V'(\phi)^2}$$

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{dN}} &= 3x - \sqrt{\frac{3}{2}}\lambda y^2 - \frac{3}{2}x\left((1-w)x^2 + (1+w)(1-y^2)\right) \\
\frac{\mathrm{d}y}{\mathrm{dN}} &= \sqrt{\frac{3}{2}}\lambda xy - \frac{3}{2}y\left((1-w)x^2 + (1+w)(1-y^2)\right) \\
\frac{\mathrm{d}\lambda}{\mathrm{dN}} &= \sqrt{6}\lambda^2(\Gamma - 1)x \\
1 &= \frac{8\pi G\rho}{3H^2} + x^2 + y^2 \qquad\qquad N = -\ln a
\end{aligned}$$

Fixed point behaviour

$$V = \frac{1}{2} m^2 \phi^2 \Rightarrow \Gamma = 1/2 \; ; \; \lambda = -1/(4\pi G \phi)$$

λ	x	y	EVs
∞	0	0	$\left(0, \frac{3(1+w)}{2}, -\frac{3(1-w)}{2}\right)$
0	± 1	0	$(-3, 0, -3(1+w))$
0	0	± 1	$(3, 0, 6-3(1+w))$

- Gives behaviour dominated by cosmological constant, kinetic term or potential
- Are these universal features modified by RG effects?

Fixed point behaviour

- Assume e.g. that RG scale depends explicitly on time $k = k(t)$
- Fulfilling the Bianchi requires modifying the equation for energy-momentum conservation

$$\dot{\rho} + 3(1 + w)H\rho + \dot{G}\rho + \frac{1}{8\pi G}\dot{\Lambda} = 0$$

- Requiring no energy transfer to matter gives relation between k, H, and the couplings
- Gives $x^2 + y^2 = \frac{1}{2}$

D. Litim, M. Hindmarsh, C.R., D. Yokoyama (in preparation)

Summary

- Nongaussian fixed point exists in pure gravity and in scalar-tensor theory
- The UV critical surface is parametrized by a small number of couplings
- Serious hints of possibility of asymptotic safety