Field Theory in & out of Equilibrium and Cosmological Boltzmann Equations

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Motivation & Goals

- Dark Matter (its nature and origin) and the Baryon Asymmetry (BAU) of the Universe (its origin) are among the most important challenges for Particle Physics and Cosmology.
- Chief motivations for considering extensions of the Standard Model (SM) of Particle Physics.
- Key task for theorists: To compute the freeze-out abundances.
- Need reaction rates in the hot Early Universe.
- Freeze-out abundances are a non-equilibrium phenomenon: If the expansion were arbitrarily slow, the present Universe would only consist of particles with mass below T_{CMB}=2.73 K.

Formulate & solve equations that describe Particle Physics reactions in a hot non-equilibrium environment.

Outline

- Review standard approach to calculating freeze-out abundances in the Early Universe and its limitations.
- Introduce Closed-Time-Path method of calculating time-dependent expectation values (→ nonequilibrium abundances), compare with
  - approach based on scattering rates,
  - thermal field theory,
  - and what these correspond with in norelativistic QM.
- On the example of sterile right-handed neutrinos, discuss
  - the production of relativistic (massless) particles,
  - the production of massive particles,
  - Leptogenesis as an example involving *CP* violation.

Introduce & explain concepts, methods of Nonequilibrium Quantum Field Theory & some Early Universe applications.

Standard Approach

- Calculate reaction rates in terms of vacuum *S*-Matrix elements (quantum).

$$S = \mathbb{1} + iT$$

$$\langle \vec{k}_{B_1} \vec{k}_{B_2} \dots | iT | \vec{p}_{A_1} \vec{p}_{A_2} \dots \rangle = (2\pi)^4 \delta^4(k_{B_1} + k_{B_2} + \dots - p_{A_1} - p_{A_2} - \dots)\, i\mathcal{M}_{A_1 A_2 \dots \to B_1 B_2 \dots}$$

invariant matrix element

- Plug these in Boltzmann equations (classical statistics).

$$\nabla_\mu j_X^\mu = \partial_t n_X - \vec{\nabla} \cdot \vec{j}_X + 3Hq_X = \mathcal{C}_X$$

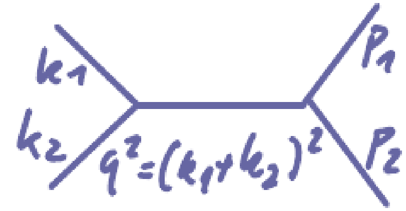
$$\mathcal{C}_X = \frac{1}{2E_X} \int \prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} \delta^4(p_X + p_{A_1} + \dots - p_{B_1} - \dots) \Big\{ (1 \pm f_X)(1 \pm f_{A_1}) \cdot \dots \cdot f_{B_1} \cdot \dots \cdot |\mathcal{M}_{B_1 B_2 \dots \to X A_1 A_2 \dots}|^2$$
$$- f_X f_{A_1} \cdot \dots \cdot (1 \pm f_{B_1}) \cdot \dots \cdot |\mathcal{M}_{X A_1 A_2 \dots \to B_1 B_2 \dots}|^2 \Big\}$$

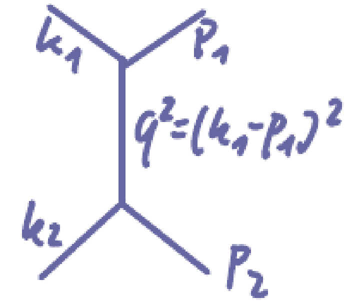
- Intuitive, heuristic approach, but very successfully applied to Cosmology (CMB, BBN). Why does it work so well and when we expect it to break down?

Standard Approach: Applicability & Limitations

<u>Applicability:</u>

- Vacuum *S*-Matrix elements appropriate for finite-temperature environment?
- Should be OK if the time an individual reaction takes is below the reaction rate.



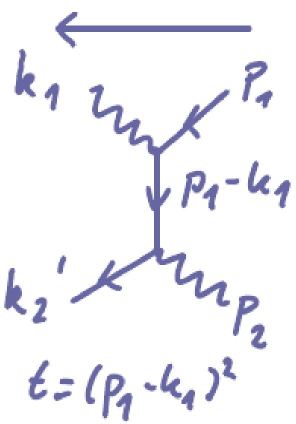


virtuality $\sqrt{|q^2|} \gg \Gamma$ reaction rate

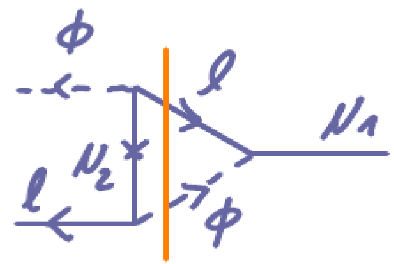
time for individual reaction << time between two reactions

- Valid approximation *e.g.* for freeze out of WIMPs at leading order (LO).

<u>Limitations:</u>

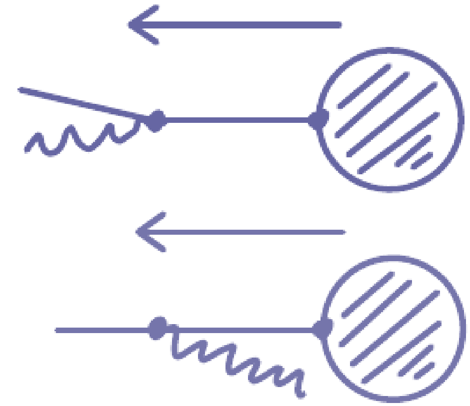


t-channel divergences, LO effect



CP-violating cuts: on-shell intermediate states, LO effect

strong coupling



Higher order corrections, *e.g.* soft & collinear emissions, NLO effect

- Reaction time large (virtuality small), or it is unclear how to define it.

How to Improve on Standard Approach

- Need to account for finite density effects $\longrightarrow$ Quantum Statistics.
- In many applications of Quantum Statistical mechanics, it is sufficient to consider equilibrium systems (time independent). Here want to generalize to non-equilibrium (time dependent).
- Briefly review how we treat scatterings and quantum statistics in nonrelativistic QM and in relativistic QFT. Then, introduce relativistic nonequilibrium approach: Closed Time Path (CTP) formalism.

Quantum Considerations, Nonrelativistic

- In standard Physics education, calculate time-independent expectation values and transition elements.

Examples for expectation values:

$$\langle\psi|A|\psi\rangle$$

standard QM

$$\varrho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$$

Quantum Statistical Mechanics

$$\mathrm{tr}\,\varrho A=\sum_n\langle n|\varrho A|n\rangle=\sum_{i,n}p_i\langle n|\psi_i\rangle\langle\psi_i|A|n\rangle=\sum_{i,n}p_i\langle\psi_i|A|n\rangle\langle n|\psi_i\rangle=\sum_i p_i\langle\psi_i|A|\psi_i\rangle=\langle A\rangle$$

Examples for transition elements:

$$\langle n|m,t\rangle=\langle n|e^{-\frac{i}{\hbar}T\int_{t_0}^{t}dt'H(t')}|m\rangle$$

$$=:\langle n|U(t;t_0)|m\rangle$$

$$=\delta_{nm}+\frac{1}{i\hbar}\int_{t_0}^{t}dt'\,e^{\frac{i}{\hbar}(E_n-E_m)}\langle n|V(t')|m\rangle+\dots$$

$$H=H_0+V$$
$$H_0|n\rangle=E_n|n\rangle$$
$$|m,t_0\rangle=|m\rangle$$

time-dependent perturbation theory, time-evolution operator *U*.

related methods

$$\langle n|U_I(t;t_0)|i\rangle=\delta_{ni}-\frac{i}{\hbar}T_{ni}\int_{t_0}^{t}dt'\,e^{\frac{i}{\hbar}(E_n-E_i)t'+\varepsilon t'}$$

scattering theory

$$S_{ni}=\lim_{t\to\infty}\lim_{\varepsilon\to 0}\langle n|U_I(t;-\infty)|i\rangle$$

- For particle-genesis in the Early Universe, need time-evolution of expectation values, *e.g.* particle number density → Nonequilibrium Theory.

Relativistic Scattering Theory: *S*-Matrix

- The LSZ reduction formula relates *S*-Matrix elements to time-ordered Green functions. These quantities can be represented and calculated in terms of Feynman diagrams.
- Path integral representation of time-ordered Green-functions:

$$Z[J] = \mathcal{N}^{-1} \langle vac_{out} | vac_{in} \rangle_J = \int \mathcal{D}\phi \, e^{i \int d^4x \, (\mathcal{L} + J(x)\phi(x))}$$

$$\langle T[\phi(x)\phi(y)] \rangle = - \frac{\delta^2}{\delta J(x)\, \delta J(y)} \log Z[J] \Big|_{J=0}$$

- We call the partition function *Z* here the in-out generating functional.
- Alternatively, use explicit time-evolution and derive Feynman rules from Dyson series and Wick's theorem:

$$\langle T(\phi(x)\phi(y)) \rangle = \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle T(\phi_I(x)\phi_I(y) U(T,T')) \rangle}{\langle U(T,T') \rangle}$$

- This form makes once again the relation with time-dependent perturbation theory explicit.

Relativistic Quantum Statistical Mechanics: Thermal Field Theory

- Consider the transition amplitude

$$\langle \phi(\vec{x},t)|\, e^{-iHt}\, |\phi(\vec{x},0)\rangle \propto \int_{B.C.} D\phi \int D\pi \; e^{\,i\int_0^t dt' \int d^3x \left\{ \pi \frac{\partial \phi}{\partial t'} - \mathcal{H}[\phi,\pi] \right\}}$$

- Integrate in imaginary direction

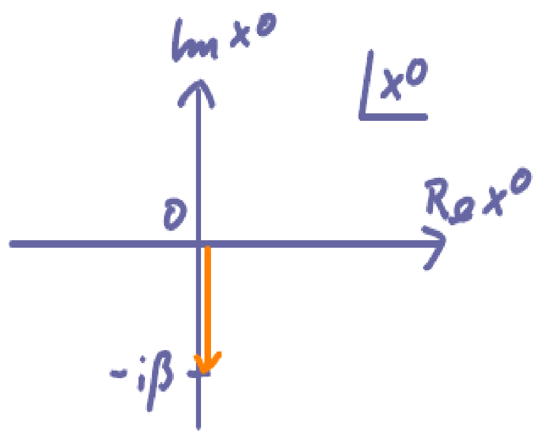
$$t = -i\beta = \frac{1}{iT}, \qquad \tau = it$$

$$\langle \phi(\vec{x},\tau)|\, e^{-\beta H}\, |\phi(\vec{x},0)\rangle \propto \int_{B.C.} D\phi \int D\pi \; e^{\,i\int_0^\beta dt' \int d^3x \left\{ i\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}[\phi,\pi] \right\}}$$

- Imposing periodic boundary conditions and integrating over complete set of states yields the canonical partition function:

$$\phi(\vec{x},0) = \phi(\vec{x},\tau)$$

$$Z = \mathrm{tr}\,\rho = \int_{\text{periodic}} D\phi \int D\pi \; e^{\,i\int_0^\beta dt' \int d^3x \left\{ i\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}[\phi,\pi] \right\}}$$



integration contour

Relativistic Thermal Field Theory – Remarks

- Describes finite-temperature effects, but no time evolution in first place.
- Can extract reaction rates as imaginary parts of equilibrium self-energies by *appealing* to optical theorem. Is this really applicable?
- What, if non-equilibrium particles run in the loop, as for Baryo-/Leptogenesis?

The Closed Time Path (CTP) Approach

- In-in generating functional:

$$\phi_{in}(x) = \phi(\vec{x}, \tau_0)$$

$$Z[J_+, J_-] = \int D\psi\, D\phi^-_{in}\, D\phi^+_{in}\, \langle \phi^-_{in} | \psi, \tau \rangle_{J_-} \langle \psi, \tau | \phi^+_{in} \rangle_{J_+} \langle \phi^-_{in} | \rho | \phi^+_{in} \rangle$$

$$= \int D\phi^+ D\phi^-\, e^{i \int d^4x \{ \mathcal{L}[\phi^+] + J_+ \phi^+ - \mathcal{L}[\phi^-] - J_- \phi^- \}} \langle \phi^-_{in} | \rho | \phi^+_{in} \rangle$$

- The Closed Time Path:

- Path-ordered Green functions:

$$i\Delta^{ab}_{\phi}(u,v) = -\frac{\delta^2}{\delta J_a(u)\, \delta J_b(v)} \log Z[J_+, J_-] \Big|_{J_\pm = 0} = i \langle \mathcal{C}[\phi^a(u)\, \phi^b(v)] \rangle$$

path ordering

- *Cf.* canonical time evolution:

$$\langle \phi(x) \phi(y) \rangle = \lim_{T \to \infty(1 - i\varepsilon)} \frac{\langle U(T, x^0)\, \phi_I(x)\, U(x^0, y^0)\, \phi_I(y)\, U(y^0, -T) \rangle}{\langle U(T, -T) \rangle}$$

Schwinger (1961);
Keldysh (1965);
Calzetta & Hu (1988).

- Thermal density matrix can be incorporated by attaching path into imaginary direction to the point τ_0^-.

Path-Ordered Green Functions

- Four propagators (two of which are linearly independent):

$$i\Delta_\phi^<(u,v) = i\Delta_\phi^{+-}(u,v) = \langle \phi(v)\phi(u)\rangle$$

$$i\Delta_\phi^>(u,v) = i\Delta_\phi^{-+}(u,v) = \langle \phi(u)\phi(v)\rangle$$

$$i\Delta_\phi^T(u,v) = i\Delta_\phi^{++}(u,v) = \langle T[\phi(u)\phi(v)]\rangle$$

$$i\Delta_\phi^{\bar{T}}(u,v) = i\Delta_\phi^{--}(u,v) = \langle \bar{T}[\phi(u)\phi(v)]\rangle$$

- Four propagators (two of which are linearly independent):

$$i\Delta_\phi^<(p) = 2\pi\,\delta(p^2-m^2)\left[\vartheta(p^0)f(\vec{p}) + \vartheta(-p^0)(1+\bar{f}(-\vec{p}))\right]$$

$$i\Delta_\phi^>(p) = 2\pi\,\delta(p^2-m^2)\left[\vartheta(p^0)(1+f(\vec{p})) + \vartheta(-p^0)\bar{f}(-\vec{p})\right]$$

$$i\Delta_\phi^T(p) = \frac{i}{p^2-m^2+i\varepsilon} + 2\pi\,\delta(p^2-m^2)\left[\vartheta(p^0)f(\vec{p}) + \vartheta(-p^0)\bar{f}(-\vec{p})\right]$$

$$i\Delta_\phi^{\bar{T}}(p) = -\frac{i}{p^2-m^2-i\varepsilon} + 2\pi\,\delta(p^2-m^2)\left[\vartheta(p^0)f(\vec{p}) + \vartheta(-p^0)\bar{f}(-\vec{p})\right]$$

particle & antiparticle distributions

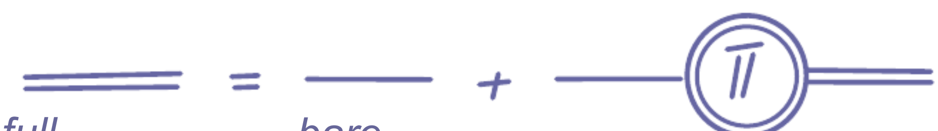
- Similarly for spin-1/2 fermions, gauge bosons, ...

Feynman Rules

- Vertices either $+$ or $-$
- Connect vertices $a=\pm$ and $b=\pm$ with $i\Delta^{ab}$
- Factor -1 for each $-$ vertex

Schwinger-Dyson Equations

$$i\Delta^{ab} = i\Delta^{(0)ab} + cd\, i\Delta^{(0)ac} \odot \Pi^{cd} \odot \Delta^{db}$$



full propagator

bare propagator

self energy from 2PI effective action

$$A(x,w) \odot B(w,y) = \int d^4w\, A(x,w)\, B(w,y)$$

- These describe in principle the real time evolution of the quantum system.
- Time evolution expressed in terms of Green functions. No reference to *S*-matrix elements for scattering processes in a vacuum background.

Toward Kinetic Equations: Wigner Transformation

- The <,> parts of the Schwinger-Dyson equations are the celebrated in Kadanoff-Baym equation:

$$(-\partial^2 - m^2)\Delta^{<,>} - \Pi^{H} \odot \Delta^{<,>} - \Pi^{<,>} \odot \Delta^{H} = \frac{1}{2}\left(\Pi^{>} \odot \Delta^{<} - \Pi^{<} \odot \Delta^{>}\right)$$

- Remaining linear combination gives pole-mass equation:

$$(-\partial^2 - m^2)\, i\Delta^{R,A} - \Pi^{R,A} \odot i\Delta^{R,A} = i\delta^4$$

Retarded/advanced propagators & self energies

- Wigner transformation:

$$A(k,x) = \int d^4 r\, e^{ikr} A\left(x+\frac{r}{2}, x-\frac{r}{2}\right)$$

Average coordinate – macroscopic evolution

Relative coodinate – microscopic (quantum) properties

- Suitable for describing particle number & charge densities.

Kinetic Equations

- For the convolutions, can show that

$$\int d^4r\, e^{ikr} \int d^4w\, A\left(x+\frac{r}{2}, w\right) B\left(w, x-\frac{r}{2}\right) = e^{-i\diamond}\{A(k,x)\}\{B(k,x)\}$$

$$\diamond\{\cdot\}\{\cdot\} = \frac{1}{2}\left(\partial_x^{(1)}\cdot\partial_k^{(2)} - \partial_k^{(1)}\cdot\partial_x^{(2)}\right)$$

- Gradient expansion: truncate derivatives w.r.t. *x*.
- The Wigner-transformed <,> Green functions are similar to phase-space densities – no such quantity can be defined in exact corresponence with classical phase-space densities due to the uncertainty relation.
- The Hermitian part of the Wigner-space Kadanoff Baym equations takes the form of kinetic equations:

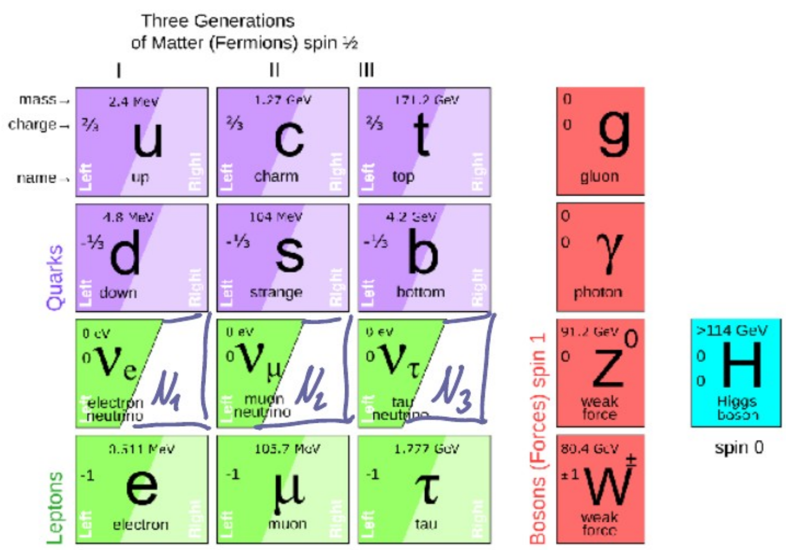
$$k_\mu \partial^\mu i\Delta^{<,>} = \underbrace{k^0\partial_t - \vec{k}\cdot\vec{\nabla}\, i\Delta^{<,>}}_{= k^0(\partial_t - \vec{v}\cdot\vec{\nabla})\, i\Delta^{<,>}} = \Pi^{>}\Delta^{<} - \Pi^{<}\Delta^{>}$$

collision term

- Have identified the collision term for relativistic non-equillibrium QFT. It is the *inclusive* rate that couples the four current. No over-/ undercounting issues.

Right-Handed Neutrinos

- Augment SM by singlet Majorana particles *N*, so-called right-handed neutrinos (RHNs). Plausible completion of the matter content, possible explanation of neutrino masses through the type-I seesaw mechanism.
- Wide allowed mass range. Exclusion window in keV to MeV range (depending on mixing with active ν s).
- Possible explanation of BAU through Leptogenesis.
- (Warm) Dark Matter candidate if mass in keV range → production through mixing. Light thermal RHNs are candidates for dark radiation.
- Light (GeV scale), approximately massless (compared to Electroweak scale) RHNs → "Leptogenesis via mixing" (ARS).
- Mass at time of decay of order *T* or heavier: standard Leptogenesis.

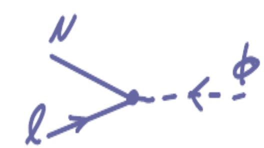


Production of (Nearly) Massless Right-Handed Neutrinos

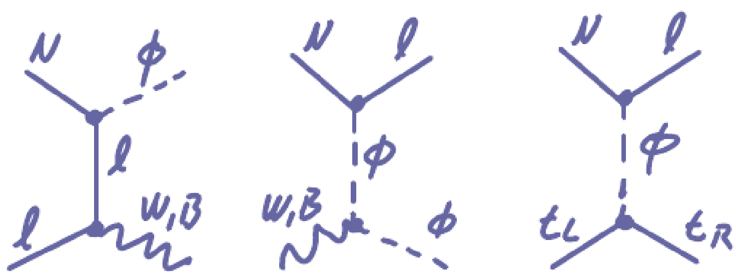
- Of interest for Leptogenesis from oscillations, lepton asymmetry can also influence subsequent DM abundance of RHNs.
- Tree-level production process: 

- Thermal masses suggest different process:

$$m_\phi^2 = \frac{1}{16}\left(3g_2^2+g_1^2+4h_t^2+8\lambda\right)T^2, \quad m_\ell^2 = \frac{1}{16}\left(3g_2^2+g_1^2\right)T^2, \quad m_N^2 \approx 0 \Longrightarrow m_\phi > m_\ell + m_N$$



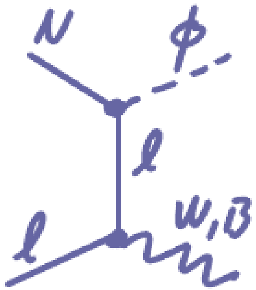
- Yet, the phase space is suppressed by the square of gauge and Yukawa couplings. At this order, there are also contributions from scattering amplitudes:



+ crossings

Massless Right-Handed Neutrinos: *t*-Channel Divergence

- The *t*-channel divergence arises when integrating over thermal phase-space:



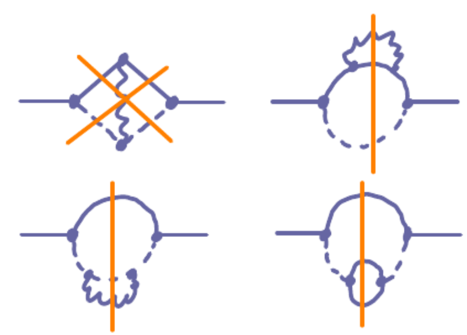
$$\mathcal{M} \sim \frac{1}{(p_N - p_\phi)^2}$$

The following integral is divergent:

$$\int \frac{d^4p_\ell \, d^4p_N \, d^4p_\phi \, d^4p_W}{(2\pi)^{16} \, 2E_\ell \, 2E_N \, 2E_\phi \, 2E_W} (2\pi)^4 \delta^4(p_N + p_\ell - p_\phi - p_W) \sum_{\text{pol.}} |\mathcal{M}|^2 (1 - f_N(p_N))(1 - f_\ell(p_\ell)) f_\phi(p_\phi) f_W(p_W)$$

- Physically, expect screening of the exchange of soft leptons.
- Readily implemented in 2PI CTP approach. RHN production rate is given by:

Double lines indicate resummed propagators. To LO, the two-loop diagram can be expressed in terms of tree-level propagators.



Kinematically accessible cut contributions.

Massless Right-Handed Neutrinos: Screening of *t*-Channel Divergences

- For the resummed propagators (spectral functions), take:

$$S_\ell^{A}(k) = P_L \frac{2(k-\not\Sigma^{H})\cdot\Sigma^{A}(k-\Sigma^{H}) - \not\Sigma^{A}(k-\not\Sigma^{H})^2 + \not\Sigma^{A3}}{[(k-\not\Sigma^{H})^2 - \not\Sigma^{A2}]^2 + 4[\Sigma^{A}\cdot(k-\Sigma^{H})]^2} \qquad \not\Sigma = \gamma_\mu \Sigma^\mu$$

$$\Delta_\phi^{A} = \frac{\Pi_\phi^{A}}{(k^2-\Pi^{H})^2 + \Pi^{A2}} \qquad G^{A} = \frac{i}{2}(G^{>} - G^{<}) \qquad G^{H} = \frac{1}{2}(G^{T} - G^{\bar T})$$

cf. $e^+e^- \to Z^0 \to X\bar{X}$

thermal mass

width

Note reduction to delta-functions when the widths goes to zero.

- Result [BG, Glowna, Schwaller (2013); see also Bödeker & Besak (2012)]:

$$\frac{\Gamma^N}{y^2 V} = \left(3{,}1 * 10^{-3} - 3{,}7 * 10^{-4} \log\left(\tfrac{3}{2}g_2^2 + \tfrac{1}{2}g_1^2\right)\right) * \left(\tfrac{3}{2}g_2^2 + \tfrac{1}{2}g_1^2\right) T^4 + 5{,}2 * 10^{-4} h_t^2 T^4$$

- Logarithms indicate the resummation.
- These are the full LO results, *i.e.* due to the *t*-channel divergence, LO production of massless particles is already an involved calculation.

Massless Particle Production: Remarks

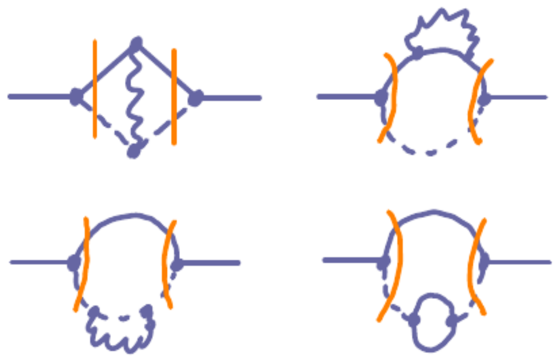
- Method can also be applied to the rate of lepton flavour violation, that is of interest for flavoured Leptogenesis. We find that [BG, Glowna, Schwaller (2013)]:

$$\frac{\Gamma^{\delta\ell}}{h^2_{e,\mu,\tau}} = \left(5{,}4 * 10^{-3} - 8{,}3 * 10^{-4} * \log\left(\frac{3}{2} g_2^2 + \frac{1}{2} g_1^2\right)\right) * \left(\frac{3}{2} g_2^2 + \frac{1}{2} g_1^2\right) T + \left(4{,}7 * 10^{-3} - 1{,}7 * 10^{-3} \log g_1\right) g_1 T + 1{,}3 * 10^{-3} h_t^2 T$$

- Similar techniques have been developed long ago [see e.g. Braaten and Yuan (1991) on axion production], matching thermal field theory with *S*-matrix techniques. Here, we have formulated and resolved the problem within a single formalism (CTP). → Can be crucial for Leptogenesis.
- Inclusiveness: In the CTP approach, all crossings & interferences are included by construction and do not need to be tracked individually.
- Other applications: Diffusion constants for Electroweak Baryogenesis (w. D. Güter/TUM), automatisation of these calculations (w. J. Klaric/TUM).

Massive Right-Handed Neutrinos: Soft and Collinear Divergences

- When the RHN mass is of the same order or above the temperature, the 1 ↔ 2 contributions to RHN production become dominant/relevant.
- Due to the RHN mass, new cuts are possible, that may be interpreted as virtual corrections. The loops involved suffer from IR divergences:



- Soft and collinear divergences occur, when the intermediate propagator in the (inverse) decay diagrams is on-shell:

- At zero temperature, it is well known that the IR, soft & collinear divergences cancel when added (inclusive rates/cross sections → KLN theorem).

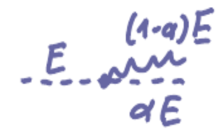
Regularisation of Soft and Collinear Divergences

- Strategy for regularisation & cancellation of divergences:
  - introduce a fictitious gauge boson mass λ,
  - calculate the individual real and virtual contributions and isolate the divergences in terms ~$\log \lambda$,
  - add the diagrams and observe the cancellation of the $\log \lambda$ terms in the *inclusive* rate, i.e. when summing $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$ processes.
- In the vacuum, all integrals (for the present example) can be calculated analytically and cancellations be seen explicitly.
- Fully analytical result is not available at finite temperature, due to the Fermi-Dirac and Bose-Einstein distribution functions.
- Possible approaches [BG, Glowna, Herranen (2013)]:
  - Separate integrals into simple contributions that can be calculated analytically and that exhibit the cancellations, and more complicated terms, that are finite and should be evaluated numerically (our tactic for wave-function contributions).
  - Show cancellation at integrand level (our tactic for vertex contributions).

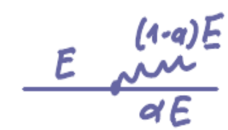
Cancellation of Soft and Collinear Divergences

- The cancellation from direct computation of the integrals (wave-function) is rather technical.
- For the cancellation at the integrand-level, need to combine integrals from virtual and real cuts in a pertinent manner. It was thus very useful to observe a relation with detailed balance for collinear splitting processes:

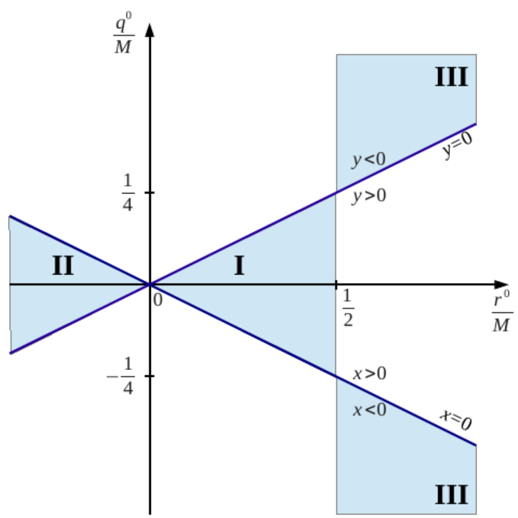
$$f_\phi(E)\left(1+f_\phi(\alpha E)\right)\left(1+f_A((1-\alpha)E)\right) = \left(1+f_\phi(E)\right) f_\phi(\alpha E)\, f_A((1-\alpha)E)$$



$$f_\ell(E)\left(1-f_\ell(\alpha E)\right)\left(1+f_A((1-\alpha)E)\right) = \left(1-f_\ell(E)\right) f_\ell(\alpha E)\, f_A((1-\alpha)E)$$



Can also cross these relations. Note that always one of the lines is virtual, no matter which cut we consider.



- Kinematics in real and virtual diagrams agrees whenever momentum square of a propagator variables vanishes. → Also where soft & collinear divergences occur.
- Can demonstrate cancellation at integrand level, when deforming integrals to figure on the left.

Region I: 1↔3 processes,
Region II: interference of t-channel amplitudes,
Region III: interference of *s*- and *t*-channel amplitudes.

Nonrelativistic Limit

- Due to the presence of two scales, RHN mass *M* and temperature *T*, general results need to be obtained numerically and are somewhat intransparent.
- When *M>>T*, result can be presented in a simple form that can even be expanded analytically in powers of *T/M*:

  Can use the following approximations:

  - RHNs are at rest and
  - replace quantum-statistical distributions by Maxwell distributions.

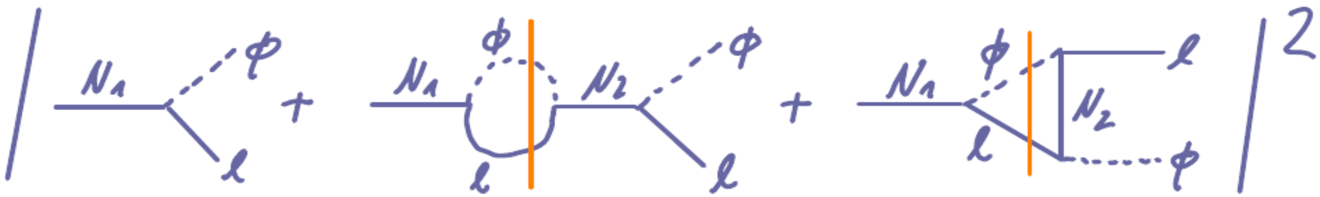
  [Lodone, Salvio, Strumia (2011); Laine, Schröder (2011); Biondini,Brambilla, Vairo (2013)].

$$\Gamma_N = \frac{y^2 M}{8\pi}\left[1 + \frac{29}{2^7\pi^2}\left(3g_2^2 + g_1^2\right) - \frac{21}{2^5\pi^2} h_t^2 - \lambda \frac{T^2}{M^2}\right]$$

- NLO result, corrections are perturbative and at the few percent level.
- Similar calculation could be applied to the production of nonrelativistic Dark Matter.

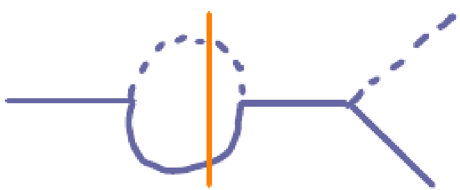
Leptogenesis – Standard Approach

- Interference of tree and loop amplitudes → *CP* violation.

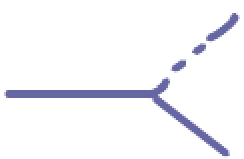
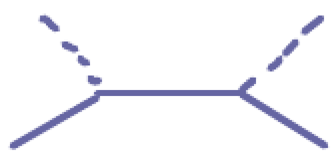


(*)

- *CP* violating contributions from discontinuities → loop momentum where cut particles are on shell (Cutksosky rules).
- Is

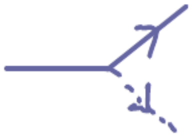


an extra process or is it already accounted for by

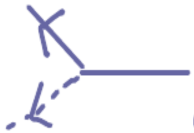
-  and  ?
- Including (*) only → *CP* violation generated in equilibrium.

(Inverse) Decays & *CP* Asymmetry

- Consider squared matrix elements, ε the parameter that quantifies the CP asymmetry.

 $|\mathcal{M}_{N\to\ell\phi}|^2 \sim 1+\varepsilon$  $|\mathcal{M}_{N\to\bar{\ell}\phi^*}|^2 \sim 1-\varepsilon$

$\overset{CPT}{=}$ $\overset{CPT}{=}$

 $|\mathcal{M}_{\bar{\ell}\phi^*\to N}|^2 \sim 1+\varepsilon$  $|\mathcal{M}_{\ell\phi\to N}|^2 \sim 1-\varepsilon$

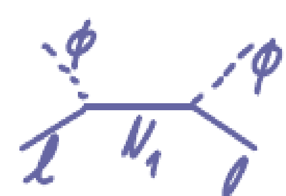
- Naive multiplication* suggests that an asymmetry is generatred even in equilibrium:

$$\Gamma_{\bar{\ell}\phi^*\to\ell\phi} \sim 1+2\varepsilon$$

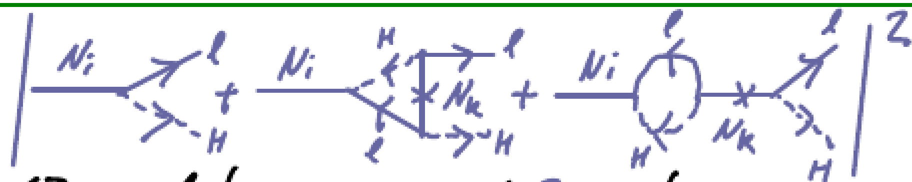
*Do not try this at home: the unstable *N* are not asymptotic states of a unitary *S* matrix → conflict with *CPT* theorem.

Usual Fix: RIS Subtraction

- Generation of *CP*-asymmetry in equilibrium is at odds with *CPT* theorem.
- Usual fix: subtract Real Intermediate States (RIS) from [Kolb & Wolfram (1980)].



Decay Asymmetries in the Standard Approach



CP-violating squared S-matrix elements (quantum)

$\mathcal{L}[f] = \mathcal{C}[f]$

Boltzmann equation (classical)

Lepton Asymmetry

- Decay asymmetries:
  [Fukugita, Yanagida (1986), Covi, Roulet, Vissani (1996)]:

$$\varepsilon_{N_i \to \ell_a} = \frac{\Gamma_{N_i \to \ell_a H} - \Gamma_{N_i \to \bar{\ell}_a \bar{H}}}{\Gamma_{N_i \to \ell_a H} + \Gamma_{N_i \to \bar{\ell}_a \bar{H}}}$$

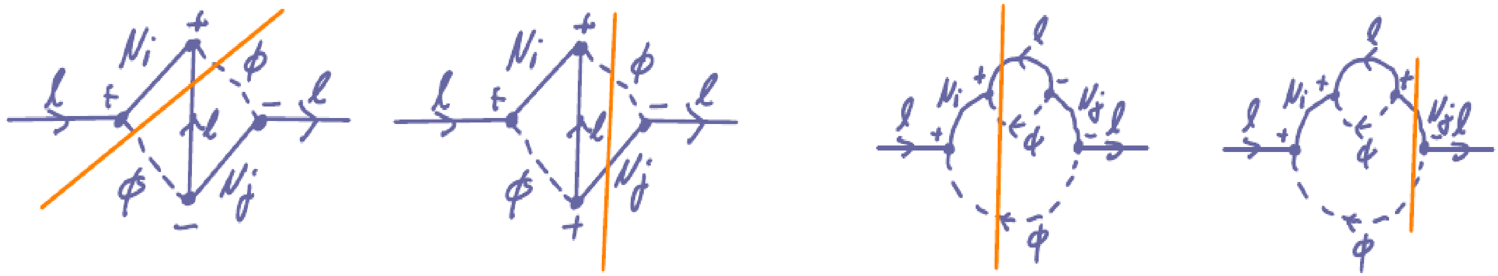
$$\varepsilon^{wf}_{N_i \to \ell_a} = \frac{1}{8\pi} \sum_{j \neq i} \frac{M_i}{M_i^2 - M_j^2} \frac{\mathrm{Im}\left[(Y^\dagger Y^* Y^t)_{aj} M_j Y_{ja} + Y^\dagger Y Y^\dagger M_j Y_{ja} \right]}{[Y^\dagger Y]_{aa}}$$

$$\varepsilon^{vertex}_{N_i \to \ell_a} = \frac{1}{8\pi} \sum_{j \neq i} \sqrt{\frac{M_j}{M_i}} \left[1 - \left(1 + \frac{M_j}{M_i}\right) \log\left(1 + \frac{M_i}{M_j}\right) \right] \frac{\mathrm{Im}\left[(Y^\dagger Y^* Y^t)_{aj} M_j Y_{ja} \right]}{[Y^\dagger Y]_{aa}}$$

- Note the resonant enhancement of the wave-function contribution as $M_1 \to M_2$.

Unitarity in CTP Approach to Leptogenesis

- The inclusiveness of the CTP approach guarantees that no over-/undercounting occurs.



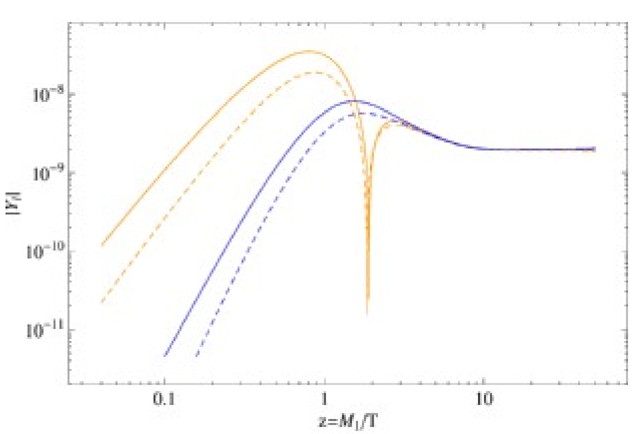
- As byproduct, obtain corrections due to quantum statistical distributions:

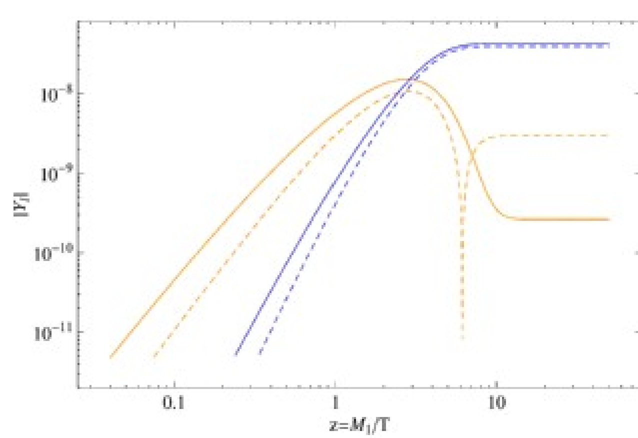
$$\Sigma^{\mathcal{A}}(k) = -\frac{1}{2}\int \frac{d^3p\, d^3q}{(2\pi)^6\, 4|\vec{p}||\vec{q}|}(2\pi)^4\delta^4(k-p-q)\,\mathrm{sign}(p^0)\left[Y_iY_j^*\, \not{p}\, P_R + Y_i^*Y_j\, \not{p}\, P_L\right] * \left[1 - f_\ell(\vec{p}) + f_\phi(\vec{q})\right]$$

$=$ 

$$S = 3\,\mathrm{Im}\left[Y_1^2 Y_2^{*2}\right]\left(-\frac{M_1}{M_2}\right)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{\vec{k}^2+M^2}}\left[f_N(\vec{k}) - f_N^{eq}(\vec{k})\right]\Sigma^{\mathcal{A}}_{\mu}(k)\,\Sigma^{\mathcal{A}\mu}(k)$$

$M_2 >> M_1$





Beneke, BG, Herranen, Schwaller (2010); Garny, Hohenegger, Kartavtsev, Lindner (2009,2010); Anisimov, Buchmüller, Drewes, Mendizabal (2010).

Resonant Mixing

- We can derive equations for the distribution functions that resemble equations for density-matrices:

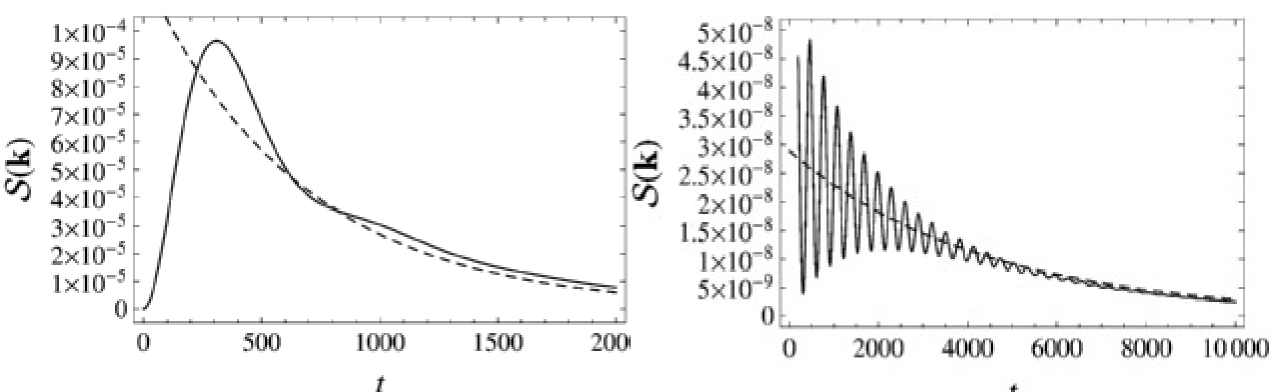
$$2 i k^0 \partial_t \left[\delta f_N(k) + \delta f_N^{eq}(k)\right] - \left[M^2, \delta f_N(k)\right] = -i \left\{\Pi^{\mathcal{A}}(k), \delta f_N(k)\right\}$$

- Mass commutators induce flavour oscillations:

$$M^2 = \begin{pmatrix} M_1^2 & 0 \\ 0 & M_2^2 \end{pmatrix} \Rightarrow \left[M^2, \delta f_N(k)\right] = (M_1^2 - M_2^2)\begin{pmatrix} 0 & \delta f_{N12}(k) \\ -\delta f_{N21}(k) & 0 \end{pmatrix}$$

- Note that a sign change of k^0 reverses the oscillation frequency – irrelevant for Majorana neutrinos, but will be of importance when we can distinguish between particles and anti-particles.
- Unless we are close to the resonance, we can average over the oscillations → recover standard decay asymmetry:

$$M_1^2 - M_2^2 \gg \Pi^{\mathcal{A}} \Rightarrow \delta f_{N12} \approx \frac{\Pi^{\mathcal{A}}_{11}(k)}{M_1^2 - M_2^2}\left(\delta f_{N11} + \delta f_{N22}\right)$$



[BG, Herranen (2011)]

- Otherwise, must solve above equations for flavour oscillations in order to obtain lepton asymmetry.

Flavoured Leptogenesis

- RHN Yukawa couplings including flavour: $Y_{ia} N_i \phi \ell_a$
- When insensitive to SM flavour a, can transform SM lepton basis and bring Y to triangular form:

$$(N_1 \;\; N_2 \;\; N_3)\begin{pmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{pmatrix}\begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}$$

only ℓ_1 couples to N_1 $\longrightarrow$ single flavour leptogenesis, but $\ell_1 = \alpha_e \ell_e + \alpha_\mu \ell_\mu + \alpha_\tau \ell_\tau$ in general

- When T falls below 10^{12} GeV (10^9 GeV, 10^4 GeV), $h_{\tau\tau}$ ($h_{\mu\mu}$, h_{ee}) come into equilibrium (reactions faster than expansion rate H).

[Abada, Davidson, Josse-Michaux, Losada, Riotto (2006); Nardi, Nir, Roulet, Racker (2006)]

$$\begin{pmatrix} q_{ee} & q_{e\mu} & q_{e\tau} \\ q_{\mu e} & q_{\mu\mu} & q_{\mu\tau} \\ q_{\tau e} & q_{\tau\mu} & q_{\tau\tau} \end{pmatrix} \xrightarrow[\text{decoherence}]{\text{complete flavour}} \begin{pmatrix} q_{ee} & 0 & 0 \\ 0 & q_{\mu\mu} & 0 \\ 0 & 0 & q_{\tau\tau} \end{pmatrix}$$

- Between 10^9 GeV and 10^{12} GeV, only the off-diagonal components involving τ evaporate and we may effectively distinguish two flavours.
- Goal: Describe intermediate regime between full flavour coherence and decoherence.

Flavoured Leptogenesis

- Kinetic equations for (anti-) leptons:

$$\frac{\partial n^{\pm}_{\ell ab}}{\partial t} = \pm S_{ab} \mp i\Delta\omega^{th}_{ab}\,\delta n^{\pm}_{\ell ab} - [W, \delta n_{\ell}]_{ab} - \gamma^{\ell\ell}(\delta n^{+}_{\ell ab} + \delta n^{-}_{\ell ab}) - \Gamma^{\ell\ell}_{ab}$$

CP-violating source — *oscillations induced by thermal masses* — *washout* — *flavour-blind pair creation & annihilation* — *flavour-sensitive interactions*

$$\Gamma^{\ell\ell} \propto h_\tau^2 \left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \delta n^{\pm}_{\ell}\right] - 2h_\tau^2\,\delta n^{\pm}_{R}$$

thermal masses like to induce flavour oscillations in opposite directions

$\delta n^{+}_{\ell ab}$

$\delta n^{-}_{\ell ab}$

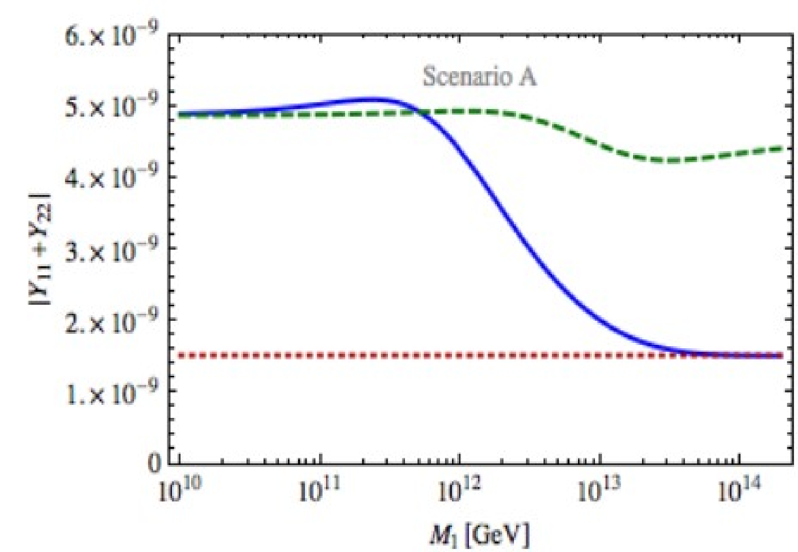
oscillations overdamped

$W^{0,\pm}, B$ like to keep these aligned

$\Gamma^{direct} \sim (h_{aa}^2 + h_{bb}^2)\,T$ where $a = e, \mu, \tau$

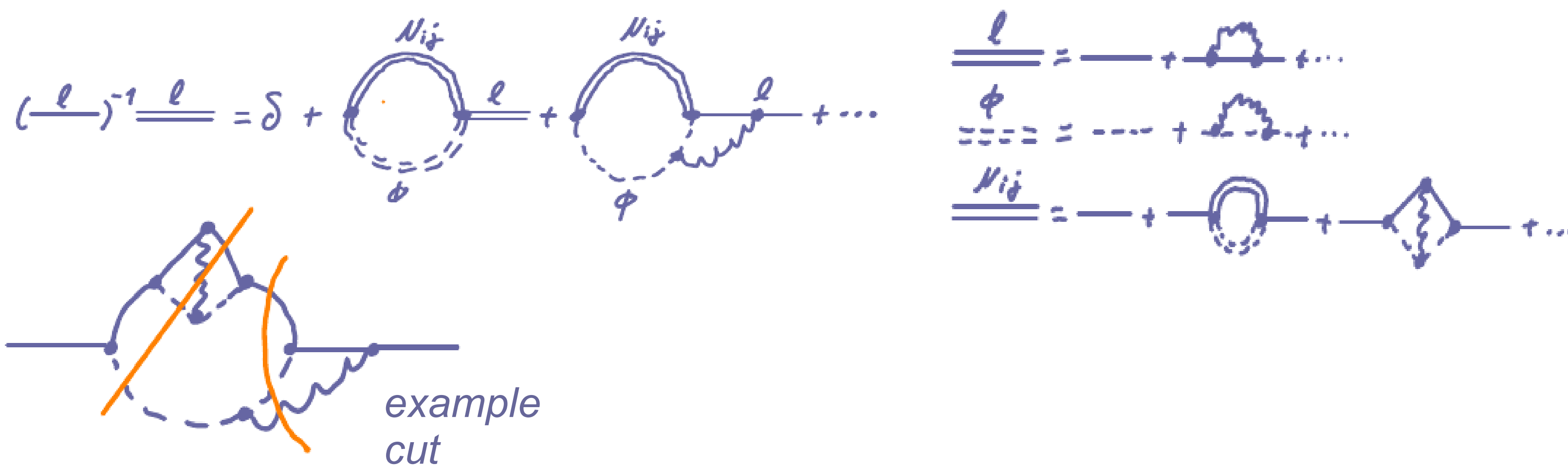
$$\Gamma^{overdamped} \sim \frac{(h_{aa}^2 - h_{bb}^2)^2}{g_2^4}\,T$$

- Can interpolate between fully flavoured and unflavoured regimes:
[Beneke, BG, Fidler, Herranen, Schwaller (2010)]



Leptogenesis from RHN Oscillations: Finite Temperature Cuts

- When *T>>M*, the standard cuts are strongly suppressed.
- However, the rates including extra radiation that we discussed above lead to new kinematically viable cuts:



- Efficient way of calculating Asymmetry from RHN oscillations (ARS scenario) [Akhmedov, Rubakov & Smirnov (1998)].

$$S_{ab} = \sum_{\substack{cij \\ i\neq j}} \frac{64}{M_i^2 - M_j^2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + M_i^2}} \left(k_\mu \hat{\Sigma}_N^\mu\right)^2 \mathrm{Im}\left[Y_{ai}^{t} Y_{ic}^{*} Y_{cj}^{t} Y_{jb}^{*} \right]$$

Leptogenesis from GeV Scale RHNs Requires no Mass Degeneracy

- Source of flavoured asymmetries, including thermal cuts:

$$S_{ab} = \sum_{\substack{i,j \\ i\neq j}} \frac{32}{M_{jj}^2 - M_{ii}^2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + M_{ii}^2}}$$

$$* \Big\{ \mathrm{Im}\left[Y_{ai}^\dagger (YY^\dagger)_{ij} Y_{jb}\right] * \left[(M_{ii}^2 + 2\vec{k}^2)\left(\hat{\Sigma}_N^{\mu 0 2} + \hat{\Sigma}_N^{\mu i 2}\right) - 4|\vec{k}|\sqrt{\vec{k}^2 + M_{ii}^2}\, \hat{\Sigma}_N^{\mu 0}\, \hat{k}^i\, \hat{\Sigma}_N^{\mu i}\right]$$

$$+ \mathrm{Im}\left[Y_{ai}^\dagger (Y^* Y^t)_{ij} Y_{jb}\right] M_{ii} M_{jj}\, \hat{\Sigma}_{N\mu}^{\mu}\, \hat{\Sigma}_N^{\mu} \Big\}\, \delta f_{0h\,ii}(\vec{k})$$

(Standard) lepton number violating contribution proportional to $M^2/\Delta M^2$ → need $\Delta M^2 << M^2$ for large enhancement.

Lepton number conserving (but flavour violating) contribution proportional to $T^2/\Delta M^2$ → large enhancement for $\Delta M^2 << T^2$ → no mass degeneracy required.

- Leptogenesis is viable with non-degenerate (in mass) RHNs of GeV-scale mass. [Drewes, BG (2012)].

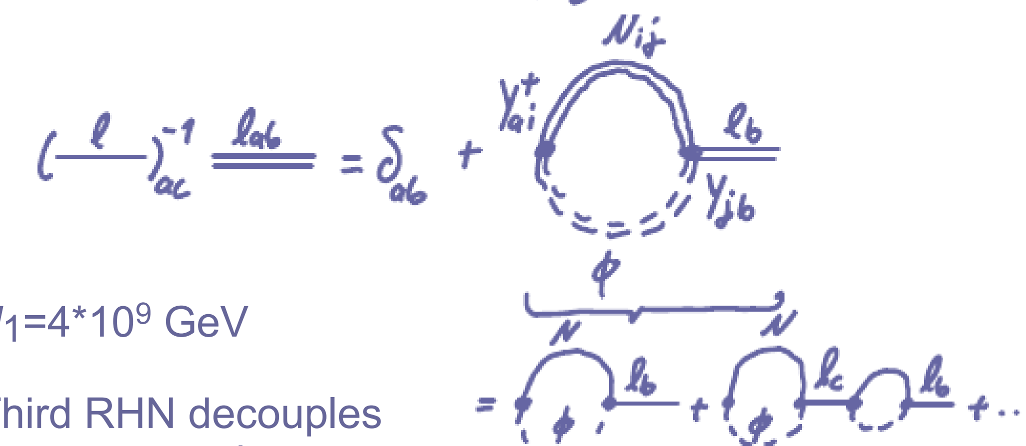
Leptogenesis: Asymmetry from Mixing Leptons

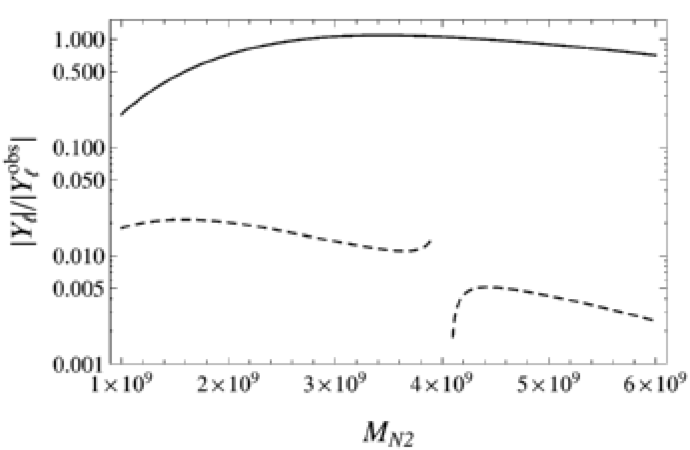
- Wave-function corrections lead to off-diagonal correlations, that in turn induce asymmetries in diagonal charge densities.
- Off-diagonal correlations are a non-equilibrium effect → can occur for SM leptons from loop that propagate nonequilibrium RHNs:

$$q_{\ell ab} = \delta n^+_{\ell ab} - \delta n^-_{\ell ab} = i\,\frac{(h_{aa}^2 - h_{bb}^2)\frac{T^2}{8}\sum_i Y^\dagger_{ai} Y_{ib} B_i^Y \delta n_N}{\left[(h_{aa}^2 - h_{bb}^2)\frac{T^2}{16}\right]^2 + (h_{aa}^2 + h_{bb}^2) B^{\ell\ell}\left[2B^g + (h_{aa}^2 + h_{bb}^2) B^{\ell\ell}\right]}$$

$B^{\ell\ell} \approx 10^{-2}\, T^2 \quad B_g \approx 1{,}7 * 10^{-3}\, T^2 \longrightarrow$ max. enhancement: $\frac{T^2}{8 B^g} \quad \left(h_\tau^2 \frac{T^4}{256} \ll 2 B_{\ell\ell} B_g\right)$

[BG (2012)]





solid: asymmetry from lepton mixing
dashed: standard asymmetry

$M_1 = 4 * 10^9$ GeV

Third RHN decouples
→ strong washout
→ freezeout occurs around $z=10$
→ reheat temperature as low as $5 * 10^8$ GeV possible.

with I. Izaguirre (preliminary)

Summary & Conclusions

- Have introduced CTP approach for calculating the time dependence of operator expectation values ($\rightarrow$ number/charge densities) and compared with
  - quantum statistical mechanics $\leftrightarrow$ thermal field theory,
  - time-dependent perturbation/scattering theory $\leftrightarrow$ relativistic S-matrix elements.
- Applied to production of relativistic/massive RHNs and Leptogenesis. Improved calculations of flavour/resonance effects, identified qualitatively new source from mixing of SM leptons.
- Criterion for applicability of S-matrix/Boltzmann approach:

  virtuality>>reaction rate $\longleftrightarrow$ time of a reaction << time between reactions.

  $\rightarrow$ OK, e.g., for WIMP freeze out at LO, in general not OK for freeze out/freeze in of relativistic particles.
- CTP approach appears to be required when non-equilibrium particles run in Feynman diagrams. Otherwise, calculations can be synthesized as a of hybrid thermal field theory and Boltzmann equations. $\rightarrow$ A matter of taste.
- CTP approach however most systematic.