Week 9: Non-Abelian gauge theories

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Non-Abelian gauge symmetries

Old-fashioned nucleon model (Yang and Mills 1954)

$$\mathcal{L} = i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi \qquad \psi = \begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix}$$

$SU(2)$ is a global symmetry of $\mathcal{L}$ $\Rightarrow$ promote to local symmetry

$$\psi(x) \to U(x)\psi(x) \qquad U(x) = \exp\left[i\alpha_a(x)T^a\right]$$

The Lagrangian is not invariant under local $SU(2)$ transformation

$$i\bar\psi\gamma^\mu\partial_\mu\psi \to i\bar\psi U^\dagger(x)\gamma^\mu\left(U(x)\partial_\mu\psi + (\partial_\mu U(x))\psi\right) =$$

$$= i\bar\psi\gamma^\mu\partial_\mu\psi + \underbrace{i\bar\psi\gamma^\mu(U^\dagger\partial_\mu U)}_{\text{additional term}}\psi$$

We first show that $B_\mu = iU^\dagger\partial_\mu U \in su(2)$, the Lie algebra of $SU(2)$, i.e. $B_\mu^\dagger = B_\mu$ and $Tr(B_\mu) = 0$

i) $U^\dagger U = \mathbb{1} \Rightarrow 0 = \partial_\mu(U^\dagger U) = U^\dagger\partial_\mu U + \partial_\mu U^\dagger U = i(B_\mu^\dagger - B_\mu)$

ii) $iU^\dagger\partial_\mu U = -\partial_\mu\alpha_a U^\dagger T^a U$

$$Tr(U^\dagger T^a U) = Tr(T^a U U^\dagger) = Tr(T^a\mathbb{1}) = Tr(T^a) = 0$$

Then introduce three vector fields A_μ^a, so that $A_\mu \equiv A_\mu^a T^a \in su(2)$ and define a covariant derivative

$$D_\mu = \mathbb{1}\partial_\mu + iA_\mu$$

and impose the transformation law on A_μ such that

$$D_\mu'\psi'(x') = U(x)D_\mu\psi(x)$$

$$(\partial_\mu + iA_\mu')\psi' = (\partial_\mu + iA_\mu')U\psi =$$

$$= U(\partial_\mu + U^\dagger\partial_\mu U + iU^\dagger A_\mu' U)\psi \overset{!}{=} U(\partial_\mu + iA_\mu)\psi$$

$$A_\mu = U^\dagger A_\mu' U - iU^\dagger\partial_\mu U$$

$$A_\mu' = UA_\mu U^\dagger + i(\partial_\mu U)U^\dagger$$

The corresponding gauge-invariant Lagrangian is

$$\mathcal{L} = i\bar{\Psi}(\gamma^\mu D_\mu - m)\psi = i\bar{\Psi}(\gamma^\mu \partial_\mu - m)\psi - \bar{\Psi}\gamma^\mu A_\mu \psi$$

kinetic term   mass term   interaction term

$$\mathcal{L}_{int} = -\bar{\Psi}\gamma^\mu A_\mu \Psi = -A_\mu^a J_a^\mu$$

where $J_\mu^a = \bar{\Psi}_i \gamma_\mu T_{ij}^a \Psi_j$

<u>Kinetic term for A_μ</u>

Recall from QED that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -\frac{i}{e}(\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu) = -\frac{i}{e}(D_\mu, D_\nu)$$

Similarly, we define

$$F_{\mu\nu} = -i\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

In the non-Abelian case, $F_{\mu\nu}$ is no longer gauge-invariant, in fact

$$D_\mu' \psi' \to U D_\mu \psi \Rightarrow D_\mu' \to U D_\mu U^\dagger$$

Hence $F_{\mu\nu}' = -i[D_\mu', D_\nu'] = -i[U D_\mu U^\dagger, U D_\nu U^\dagger] =$

$$= -i\, U [D_\mu, D_\nu] U^\dagger = U F_{\mu\nu} U^\dagger$$

Therefore

$$F_{\mu\nu}F^{\mu\nu} \to U F_{\mu\nu} F^{\mu\nu} U^\dagger$$

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \to \mathrm{Tr}(U F_{\mu\nu} F^{\mu\nu} U^\dagger) = \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

The candidate for the kinetic term is then

$$\mathcal{L} \supset -c\, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

The normalisation constant c depends on the representation of $SU(2)$ according to which ψ transforms

In our case $T^a = \frac{\tau^a}{2} \Rightarrow A_\mu = \underset{\substack{\uparrow \\ \text{constant}}}{g} A_\mu^a \frac{\tau^a}{2}$

$$F_{\mu\nu} = g F_{\mu\nu}^a \frac{\tau^a}{2}$$

$$F_{\mu\nu} = g\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\frac{\tau^a}{2} + i g^2 \underbrace{\left[\frac{\tau^a}{2}, \frac{\tau^b}{2}\right]}_{= i\varepsilon_{abc}\frac{\tau^c}{2}} A_\mu^a A_\nu^b$$

$$\Rightarrow F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\,\varepsilon_{abc} A_\mu^b A_\nu^c$$

Canonical normalisation requires

$$-c\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu}$$

$$c\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = \frac{c}{4} g^2 F_{\mu\nu}^a F^{\mu\nu b} \underbrace{\mathrm{Tr}\left(\tau^a\tau^b\right)}_{= 2\delta_{ab}} = c\,\frac{g^2}{2} F_{\mu\nu}^a F_a^{\mu\nu}$$

$$c\,\frac{g^2}{2} = \frac{1}{4} \Rightarrow c = \frac{1}{2g^2}$$

To summarise, the full gauge-invariant Lagrangian is

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + i\bar{\psi}\left(i\not{D} - m\right)\psi$$

$$D_\mu = \partial_\mu + iA_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right]$$

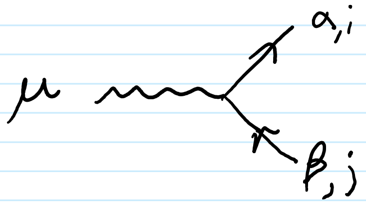
gauge transformations

$$\psi \to U\psi$$
$$A_\mu \to U A_\mu U^\dagger + i(\partial_\mu U)U^\dagger$$

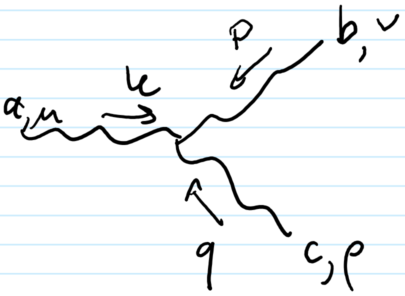
Since $F_{\mu\nu}$ transforms according to $SU(2)$, the gauge bosons A_μ^a interact among themselves. In fact

$$F_{\mu\nu}^a F^{\mu\nu}_a = (\partial_\mu A_\nu^a - \partial_\mu A_\nu^a)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) +$$
$$-4 g\, \varepsilon_{abc}\, \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} +$$
$$+ g^2 \varepsilon^{abc} \varepsilon_{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e}$$

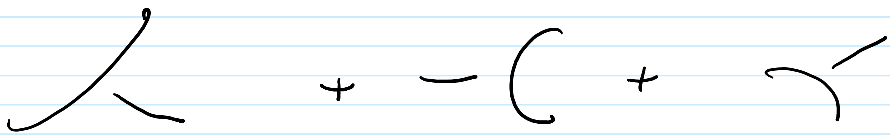
Feynman diagrams

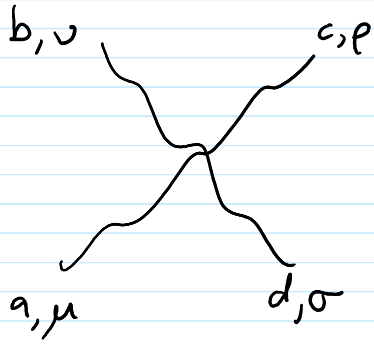


$$-ig \frac{\tau^a_{ij}}{2} \gamma^\mu_{\alpha\beta}$$



$$= -g\, \varepsilon_{abc} \left(\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\right)$$





$$= -ig^2 \Big[\varepsilon_{abe}\varepsilon_{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \varepsilon_{ace}\varepsilon_{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \varepsilon_{ade}\varepsilon_{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) \Big]$$

Full Lagrangian

$$\mathcal{L} = -i \bar\Psi \partial\!\!\!/ \psi - m \bar\Psi \psi - \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) +$$
$$- g A_\mu^a \bar\Psi \gamma^\mu \frac{\tau^a}{2} \psi + g\, \varepsilon_{abc} (\partial_\mu A_\nu^a) A^{\mu b} A^{\nu c} +$$
$$- \frac{1}{4} g^2 \varepsilon_{abc} \varepsilon_{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e}$$

Strong interactions

Theory of quarks and gluons $\Rightarrow SU(3)$ non-Abelian gauge theory

Matter content

Six Dirac spinors - the quarks - which also have electric charge

$$\text{quarks } q_f \begin{cases} \text{up} & u & \text{charm } c & \text{top } t & Q = +\frac{2}{3}e \\ \text{down} & d & \text{strange } s & \text{bottom } b & Q = -\frac{2}{3}e \end{cases}$$

Each quark can appear in three different states, called colour, red, green and blue

Each quark is a triplet under $SU(3)$

$$q_f = \begin{pmatrix} q_f^1 \\ q_f^2 \\ q_f^3 \end{pmatrix} \qquad i = u, c, t, d, s, f \quad (\text{flavour})$$

In general, we study $SU(N_c)$ with N_c the number of colours

$$U \in SU(N_c) \Rightarrow U = \exp[i\alpha_a t^a]$$

$$t^a = t^{a\dagger}$$

$$\mathrm{Tr}(t^a) = 0$$

$$\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab} \quad (\text{convention, orthogonality requirement})$$

$SU(N_c)$ has $N_c^2 - 1$ generators

$SU(2)$ has $2^2 - 1 = 3$ generators $\Rightarrow t^a = \frac{\tau^a}{2}$ (Pauli matrices)

$SU(3)$ has $3^2 - 1 = 8$ generators $\Rightarrow t^a = \frac{\lambda^a}{2}$ (Gell-mann matrices)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Structure constants

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}$$

Can be obtained by using

$$Tr([t^a, t^b] t^c) = i f^{abd} Tr(t^d t^c) = \frac{i}{2} f^{abc}$$

$$i f^{abc} = 2 Tr([t^a, t^b] t^c)$$

abc	123	147	156	246	257	345	367	458	678
f_{abc}	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

<u>Gauge-invariant Lagrangian</u>

Promote $SU(N_c)$ to a local symmetry

$$\psi \to U(x)\psi(x) \qquad U(x) = \exp[-i g_s \theta^a(x) t^a]$$

$$D_\mu = \partial_\mu + i g_s A_\mu \qquad A_\mu = A_\mu^a t^a \in SU(N_c)$$

$$A_\mu^a(x) \quad a = 1 \dots N_c^2 - 1 \quad (8 \text{ for } SU(3))$$

This gives $N_c^2 - 1$ gauge bosons, the <u>gluons</u>

$$D_\mu \psi \to D_\mu' \psi' = U(D_\mu \psi)$$

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g_s}(\partial_\mu U) U^\dagger$$

Check:

$$D_\mu' \psi' = (\partial_\mu + i g_s A_\mu') U \psi =$$

$$= (\cancel{\partial_\mu U} + U \partial_\mu + i g_s U A_\mu - \cancel{\partial_\mu U}) \psi =$$

$$= U(\partial_\mu + i g_s A_\mu) \psi = U D_\mu \psi$$

Field strength

$$F_{\mu\nu} = -\frac{i}{g_s}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_s [A_\mu, A_\nu] =$$

$$= G_{\mu\nu}^{\ a} t^a$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f^{abc} A_\mu^b A_\nu^c$$

$$F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$$

Full Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \sum_f \bar{q}_f (i\gamma^\mu D_\mu - m_f) q_f =$$

$$= -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} + \sum_f \bar{q}_f (i\gamma^\mu \partial_\mu - m_f) q_f +$$

$$- g_s A_\mu^a \sum_f \bar{q}_f \gamma^\mu t^a q_f$$

Infinitesimal gauge transformations

$$\psi_i \to \psi'_i = \left[\exp(-i g_s \theta^a(x) t^a)\right]_{ij} \psi_j \simeq$$

$$\simeq \psi_i - i g_s \theta^a(x) t^a_{ij} \psi_j \Rightarrow \delta\psi_i = -i g_s \theta^a t^a_{ij} \psi_j$$

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g_s}(\partial_\mu U) U^\dagger =$$

$$= e^{-i g_s \theta^a t^a} A_\mu^b t^b e^{+i g_s \theta^a t^a} + \frac{i}{g_s}(-i g_s \partial_\mu \theta^a t^a) \simeq$$

$$\simeq (A_\mu^a + \partial_\mu \theta^a) t^a - i g_s \theta^a A_\mu^b \underbrace{[t^a, t^b]}_{i f^{abc} t^c} =$$

$$= (A_\mu^a + \partial_\mu \theta^a + g_s f^{abc} \theta^b A_\mu^c) t^a$$

$$A'^a_\mu = A_\mu^a + \partial_\mu \theta^a + g_s f^{abc} \theta^b A_\mu^c$$

Remarks

i) The gauge transformations for A_μ^a depend explicitly on g_s $\Rightarrow$ all fields interacting with A_μ^a need to transform according to the same parameters

$$\phi(x) \to \phi'(x) = \exp[-ig_s\,\theta^a(x)T^a]$$

with T^a generating some representation of $SU(N_c)$

$$D_\mu = \partial_\mu \mathbb{1} + ig_s A_\mu \qquad A_\mu = A_\mu^a T^a$$

Although $A_\mu(x)$ depends on the representation, the gluon fields $A_\mu^a(x)$ are independent, and represent the physical degrees of freedom

ii) The infinitesimal transformation for A_μ^a can be written in a suggestive form using the covariant derivative for the adjoint representation

$$A_\mu'^a = A_\mu^a + \partial_\mu\theta^a + g_s f^{abc}\theta^b A_\mu^c =$$

$$= A_\mu^a + \left(\partial_\mu\delta^{ab} + ig_s\left(-if^{cab}A_\mu^c\right)\right]\theta^b =$$

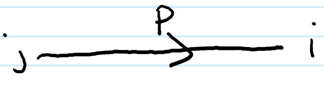
$$= A_\mu^a + \left[\partial_\mu\delta^{ab} + ig_s(T^c)^{ab}A_\mu^c\right]\theta^b =$$

$$= A_\mu^a + D_\mu^{ab}\,\theta^b$$

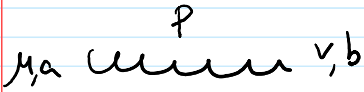
where D_μ^{ab} is the covariant derivative for the adjoint representation of $SU(N_c)$

$$D_\mu^{ab} = \partial_\mu\delta^{ab} + ig_s(T^c)^{ab}A_\mu^c$$

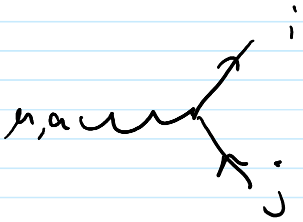
Feynman rules for QCD



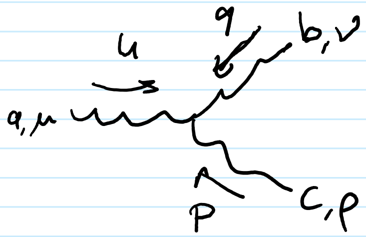
$$\frac{i}{\not{p}-m}\,\delta_{ij}$$



$$\frac{i}{p^2}\left(-\eta^{\mu\nu} + \text{gauge terms}\right)\delta^{ab}$$



$$-i g_s \gamma^\mu t^a_{ij}$$



$$-g_s \underbrace{f_{abc}}_{i(T^a)_{bc}}\left[\eta^{\mu\nu}(k-q)^\rho + \eta^{\nu\rho}(q-p)^\mu + \eta^{\rho\mu}(p-k)^\nu\right]$$

$$-i g_s^2\left[f^{abe}f^{cde}\left(\eta^{\mu\sigma}\eta^{\nu\rho}-\eta^{\mu\rho}\eta^{\nu\sigma}\right) + f^{ace}f^{bde}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}\right) + f^{ade}f^{cbe}\left(\eta^{\mu\sigma}\eta^{\nu\rho}-\eta^{\mu\nu}\eta^{\rho\sigma}\right)\right]$$

Add electromagnetic interactions

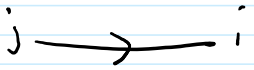
$$D_\mu = \partial_\mu + i g_s A^a_\mu t^a + i e Q_f A_\mu$$

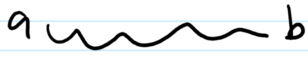
with $Q_f = \begin{cases} 2/3 & f = u,c,t \\ -\frac{1}{3} & f = d,s,b \end{cases}$

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\substack{\text{e.m.}\\ \gamma}} \underbrace{-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a}}_{\substack{\text{strong}\\ 8\text{ gluons}}} + \sum_f \bar{q}_f(i\not{D}_f - m_f)q_f$$

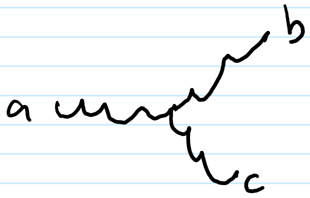
Colour algebra

It is useful to cast $SU(N_c)$ identities in terms of colour Feynman diagrams, by just keeping the colour part of the corresponding Feynman diagrams

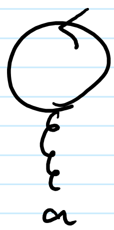
$\delta_{ij} \rightarrow$ 

$\delta_{ab} \rightarrow$ 

$t^a_{ij} \rightarrow$ 

$if_{abc} \rightarrow$ 

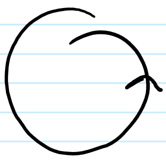
- Colour identities

$\mathrm{Tr}(t^a) = 0 \longrightarrow$ 

$= 0$

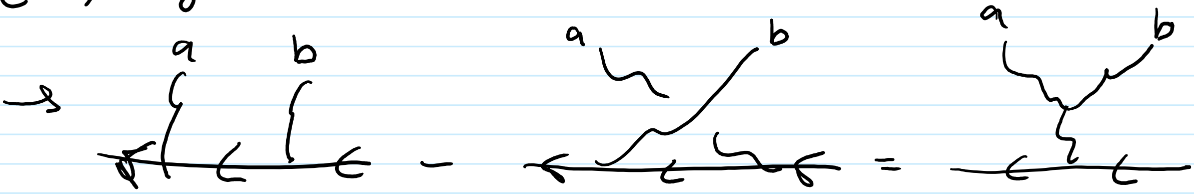
$\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab} \rightarrow$ 

$= \frac{1}{2}$ 

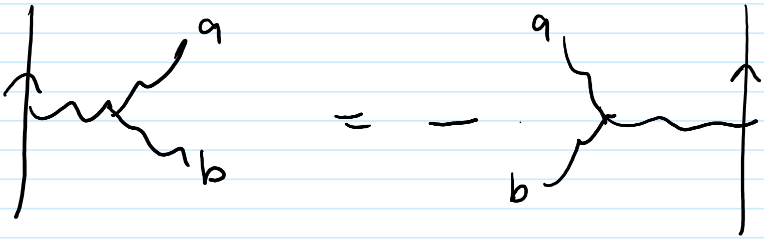
$\delta_{ii} = N_c \longrightarrow$  $= N_c$

$\delta_{aa} = N_c^2 - 1 \longrightarrow$  $= N_c^2 - 1$

$[t^a, t^b] = i f^{abc} t^c \rightarrow$



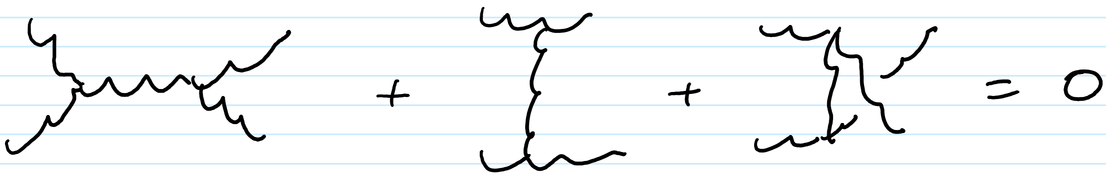
Note:



("pedestrian" rule)

• Jacobi identity

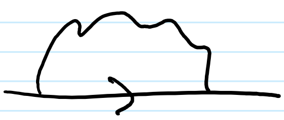
$f_{ade} f_{bcd} + f_{cde} f_{abd} + f_{bde} f_{cad} = 0$



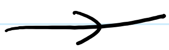
• Casimirs

$\sum_a t^a_{ik} t^a_{kj} = C_F \delta_{ij}$ (Analogous to $J_1^2 + J_2^2 + J_3^2 = j(j+1)\mathbb{1}$)

fundamental



$= C_F$



$\sum_a f^{abd} f^{acd} = C_A \delta^{ac}$

adjoint

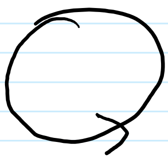
$= C_A$

Calculation of the Casimir C_F

 $= C_F$ 

 $=$  $= \frac{1}{2}$  $= \frac{N_c^2-1}{2}$

$\parallel$

C_F  $= C_F N_c$

Hence $C_F = \frac{N_c^2-1}{2N_c}$ $\quad \left(C_F = \frac{4}{3} \text{ for } N_c = 3\right)$

Fierz identity

Completeness relation: any $N_c \times N_c$ matrix can be written as a linear combination of the identity and of the $SU(N_c)$ generators t^a

$$A_{ij} = A_0 \delta_{ij} + A_a t^a_{ij}$$

$$\mathrm{Tr}(A) = A_0 \mathrm{Tr}(\mathbb{1}) = N_c A_0 \Rightarrow A_0 = \frac{\mathrm{Tr} A}{N_c}$$

$$\mathrm{Tr}(A t^a) = A_b \mathrm{Tr}(t^b t^a) = \frac{1}{2} A_b \delta^{ab} = \frac{1}{2} A_a \Rightarrow A_a = 2\mathrm{Tr}(A t^a)$$

$$A_{ij} = \frac{\mathrm{Tr} A}{N_c} \delta_{ij} + 2\mathrm{Tr}(A t^a) t^a_{ij}$$

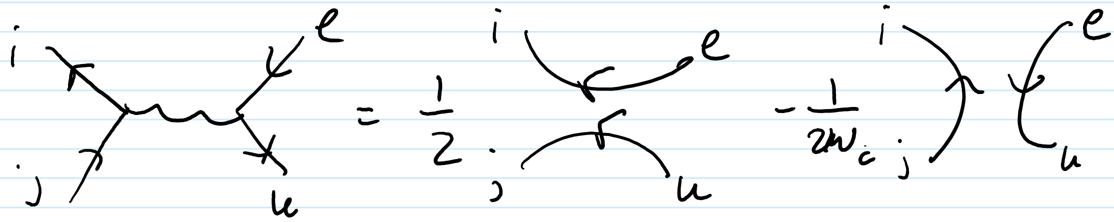
with all the indices displayed

$$A_{kl} \delta_{ik} \delta_{jl} = A_{kl} \left[\frac{\delta_{kl} \delta_{ij}}{N_c} + 2\left(t^a_{lk} t^a_{ij}\right)\right]$$

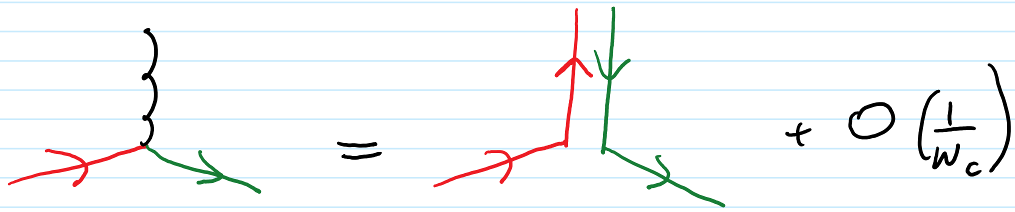
This gives

$$t^a_{ij} t^a_{kl} = \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{2N_c} \delta_{ij} \delta_{kl}$$

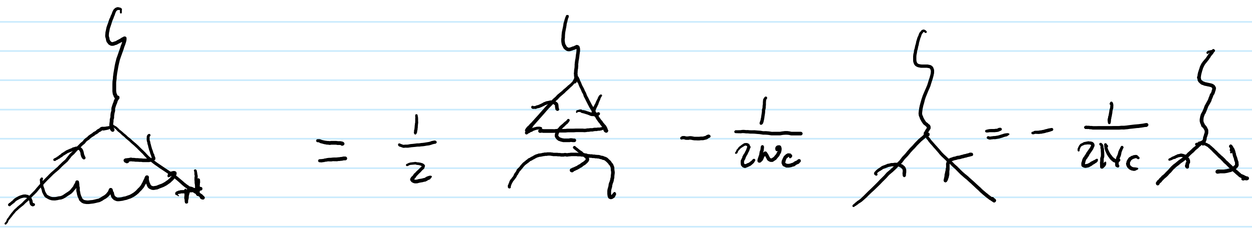
Pictorial representation of the Fierz identity



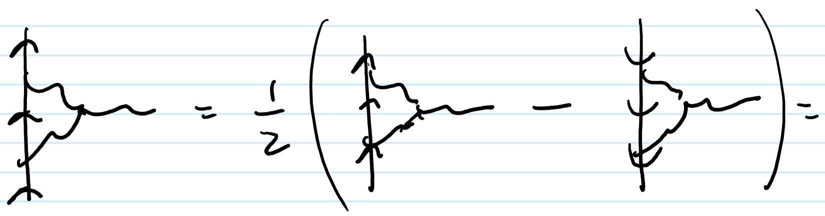
In the limit of large N_c, a gluon can be represented as a quark-antiquark line!

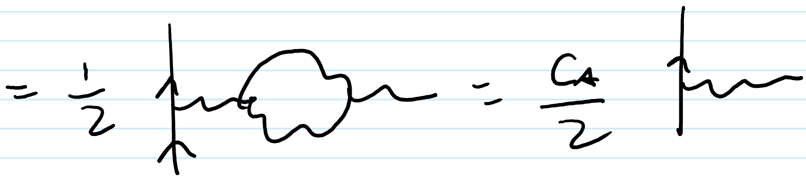


Quark-gluon vertex



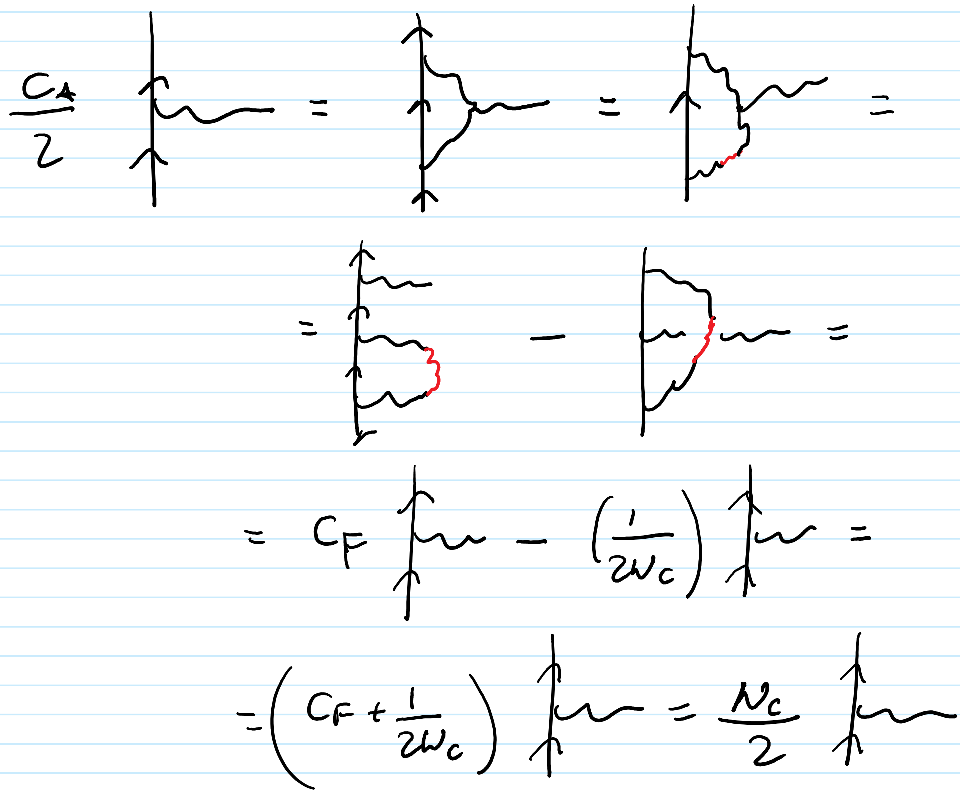
$$t^a t^b t^a = -\frac{1}{2N_c} t^b$$





$$-i f_{abc} t^b t^c = \frac{C_A}{2} t^a$$

Calculation of the Casimir C_A



From this we read $C_A = N_c$

Representation of f_{abc}

$$if^{abc} = 2\,\mathrm{Tr}\left([t^a, t^b], t^c\right)$$