Week 8: Gauge symmetries: the Abelian case

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Some remarks on $U(1)$ symmetries

Consider the case of two real scalar fields with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - \frac{1}{2} m^2 \phi_i \phi_i \qquad i=1,2$$

At the quantum level, the theory is solved exactly by

$$\phi_i(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\left[a_i(\vec{p})e^{-ip\cdot x} + a_i^\dagger(\vec{p})e^{ip\cdot x}\right] \qquad E_p = \sqrt{\vec{p}^2+m^2}$$

with commutation rules

$$[a_i(\vec{p}), a_j(\vec{p}\,')] = 0 \qquad [a_i(\vec{p}), a_j^\dagger(\vec{p}\,')] = \delta_{ij}(2\pi)^3 2E_p\, \delta^3(\vec{p}-\vec{p}\,')$$

The above Lagrangian is invariant under $SO(2) \simeq U(1)$ with conserved Noether current

$$J^\mu = (\partial^\mu \phi_1)\phi_2 - (\partial^\mu \phi_2)\phi_1$$

Consider the corresponding charge as an operator on the Fock space

$$Q = \int d^3x\, J^0(x) = i\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\left[a_1^\dagger(\vec{p})a_2(\vec{p}) - a_2^\dagger(\vec{p})a_1(\vec{p})\right]$$

For a given $\vec{p}$ we have

$$Q|\vec{p},1\rangle = -i|\vec{p},2\rangle \qquad Q|\vec{p},2\rangle = i|\vec{p},1\rangle \Rightarrow Q_{\vec{p}} = \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix} = -\sigma_2$$

For a given value of $\vec{p}$, $Q_{\vec{p}}$ has eigenvalues 1 and -1

Move to the basis of eigenvalues defining

$$|\vec{p},\pm\rangle = \frac{1}{\sqrt{2}}\left(|\vec{p},1\rangle \mp i|\vec{p},2\rangle\right) \Rightarrow Q|\vec{p},\pm\rangle = \pm|\vec{p},\pm\rangle$$

$$|\vec{p},+\rangle = a^\dagger(\vec{p})|0\rangle \qquad a(\vec{p}) = \frac{1}{\sqrt{2}}\left(a_1(\vec{p}) + i a_2(\vec{p})\right)$$

$$|\vec{p},-\rangle = b^\dagger(\vec{p})|0\rangle \qquad b(\vec{p}) = \frac{1}{\sqrt{2}}\left(a_1(\vec{p}) - i a_2(\vec{p})\right)$$

Instead of two real scalar fields we introduce a complex field

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\left(a(\vec{p})e^{-ipx} + b^\dagger(\vec{p})e^{ipx}\right)$$

This field describes particles with charge +1 (particles) created by $a^\dagger(\vec{p})$, and particles with charge −1 (anti-particles) created by $b^\dagger(\vec{p})$

Similarly, for a free Dirac field

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \sum_s \left(u_s(\vec{p})a(\vec{p})e^{-ipx} + v_s(\vec{p})b^\dagger(\vec{p})e^{ipx}\right)$$

With $u_s(\vec{p})$ and $v_s(\vec{p})$ such that $\psi(x)$ satisfies Dirac equation $(i\not\partial - m)\psi(x) = 0$

<u>Example:</u> symmetries of the pion-nucleon interaction term

$$\mathcal{L} \supset f_A \bar{\psi}\gamma^5 \frac{\tau^a}{2}\phi_a\psi \qquad \psi = \begin{pmatrix}\psi_p \\ \psi_n\end{pmatrix}$$

Baryon number $\psi \to e^{i\alpha_B}\psi$ symmetry

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s u_s(\vec{p}) \underbrace{\begin{pmatrix}a_p(\vec{p}) \\ a_n(\vec{p})\end{pmatrix}}_{\text{baryon}} e^{-ipx} + v_s(\vec{p}) \underbrace{\begin{pmatrix}b_p^\dagger(\vec{p}) \\ b_n^\dagger(\vec{p})\end{pmatrix}}_{\text{antibaryon}} e^{ipx}$$

$$Q_B = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \sum_{i=p,n}\left(a_i^\dagger(\vec{p})a_i(\vec{p}) - b_i^\dagger(\vec{p})b_i(\vec{p})\right)$$

Isospin

$$\psi \to U(\vec{\alpha})\psi \qquad \phi \to \Omega(\vec{\alpha})\phi$$

$$U(\vec{\alpha})\frac{\vec{\tau}}{2}\cdot\vec{\phi}\,U^{-1}(\vec{\alpha}) = \left(\Omega(\vec{\alpha})\vec{\phi}\right)\cdot\frac{\vec{\tau}}{2} \quad \text{symmetry}$$

Electric charge

$\psi_p \to e^{i\alpha}\psi_p \quad \psi_n \to \psi_n$ breaks isospin invariance

ψ_p and ψ_n belong to two different representations of $U_{em}(1)$ giving electric charge

The free Lagrangian is invariant with respect to $U_{em}(1)$

Now we expand the interaction term

$$\mathcal{L} \supset \frac{f_A}{2}\begin{pmatrix}\bar\psi_p\\ \bar\psi_n\end{pmatrix}^T(\tau_1\phi_1 + \tau_2\phi_2 + \tau_3\phi_3)\gamma^5\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix} =$$

$$= \frac{f_A}{2}\begin{pmatrix}\bar\psi_p\\ \bar\psi_n\end{pmatrix}^T\begin{pmatrix}\phi_3 & \phi_1 - i\phi_2\\ \phi_1 + i\phi_2 & -\phi_3\end{pmatrix}\gamma^5\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix} =$$

$$= \frac{f_A}{2}\Big[\bar\psi_p\phi_3\gamma^5\psi_p - \bar\psi_n\phi_3\gamma^5\psi_n +$$

$$+ \bar\psi_p(\phi_1 - i\phi_2)\gamma^5\psi_n + \bar\psi_n(\phi_1 + i\phi_2)\gamma^5\psi_p\Big]$$

We then define π_+, π_- and π_0 fields

$$\pi_\pm = \frac{1}{\sqrt2}(\phi_1 \mp i\phi_2) \qquad \pi_0 = \phi_3$$

$$\mathcal{L} \supset f_A\Big[\frac{1}{2}\left(\bar\psi_p\pi_0\gamma^5\psi_p - \bar\psi_n\pi_0\gamma^5\psi_n\right)$$
$$+ \frac{1}{\sqrt2}\left(\bar\psi_p\pi_+\gamma^5\psi_n + \bar\psi_n\pi_-\gamma^5\psi_p\right)\Big]$$

For this Lagrangian to be invariant under $U_{em}(1)$ we need to have

$$\pi_\pm \to e^{\pm i\alpha}\pi_\pm \qquad \pi_0 \to \pi_0$$

So that π_+, π_- and π_0 represent the two charged and the neutral pion

Neutrinos and lepton number

In the original formulation of the SM neutrinos were massless $\Rightarrow$ only ν_L with $m(\nu_L)=0$

Now, from neutrino oscillations, we know that neutrinos are massive $\Rightarrow$ ν_R should exist

Dirac mass term $\quad \mathcal{L} \supset -m_D \bar{\nu}_L \nu_R + h.c$

This Lagrangian, as well as that of other leptons, like the electron, possesses a $U(1)$ symmetry

$$\psi_{lepton} \to e^{i\alpha_L} \psi_{lepton}$$

As for all other $U(1)$ symmetries, the corresponding charge L counts the number of leptons minus the number of anti-leptons

$$L(\nu_i) = -L(\bar{\nu}_i)$$

The physical processes we observe all conserve lepton number, e.g. weak decay

$$n \to p\, e^- \,\bar{\nu}_e$$

$L=0 \qquad L = 1 + (-1) = 0$

Problem is that $m_\nu \sim eV \ll 100$ GeV, typical scale of EW interactions (see later SM lecture)

If ν_R does not have any charge, we can write a Majorana mass term for it

$$\mathcal{L} \supset -\frac{1}{2} m_R \bar{\nu}_R^c \nu_R + h.c. = -\frac{1}{2} m_R \nu_R^T C \nu_R + h.c.$$

How does this solve the problem of the small neutrino mass?

If ν_R is very heavy, its field will be approximately constant, hence the EOM is

$$\frac{\partial \mathcal{L}}{\partial \nu_R} = 0 = -m_D \bar{\nu}_L - m_R \bar{\nu}_R^c \Rightarrow \nu_R = \frac{m_D}{m_R} \nu_L^c$$

Substituting into the Dirac mass term gives

$$\mathcal{L} \supset \frac{m_D^2}{m_R} \overline{\nu_L^c} \nu_L + h.c$$

This gives a Majorana mass term

If $m_R \gg m_D$, we have that

$$m_\nu \sim \frac{m_D^2}{m_R} \text{ small} \Leftarrow \text{see-saw mechanism}$$

A Majorana mass term for ν_L does not conserve lepton number $\Rightarrow$ expect lepton-number violating process

Experimental test: Neutrinoless double-beta decay ($0\nu\beta\beta$)

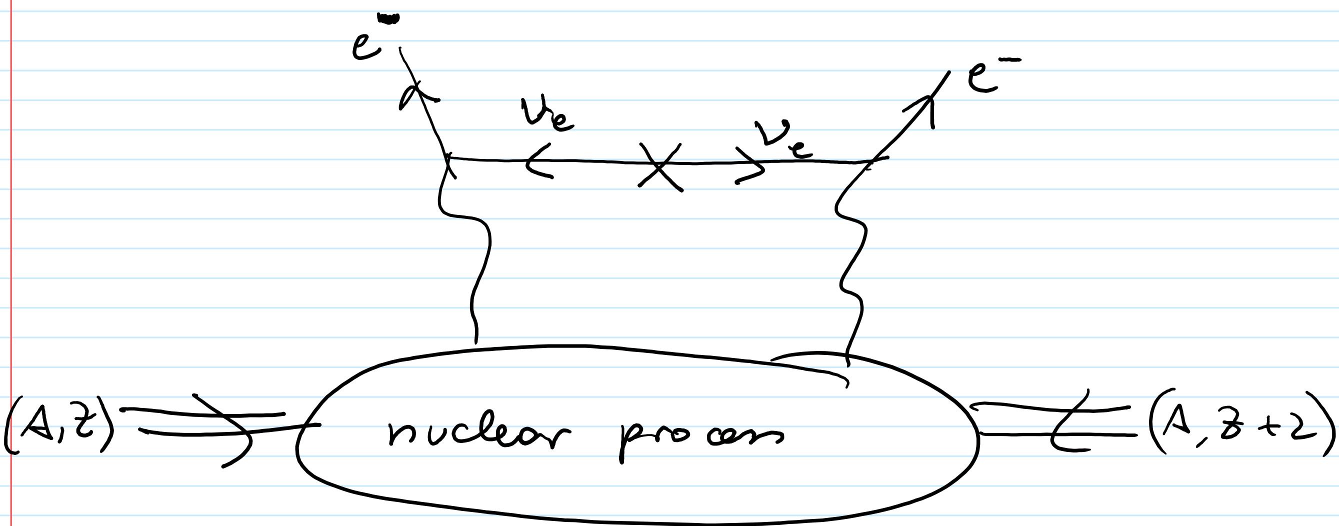
$$(A, Z) \rightarrow (A, Z+2) + 2e^-$$

nucleus protons electrons

The dominant process does not exist unless $\nu_i = \nu_i^c$

$$\nu = \begin{pmatrix} \nu_L \\ -i\sigma_2 \nu_L^* \end{pmatrix}$$

This is a Majorana spinor, the equivalent of a real scalar field for fermions



Local internal symmetries (gauge symmetries)

We consider the case in which the parameters of an internal transformation depend on the space-time point

Such transformations are called gauge transformations

Imposing that a classical Lagrangian is gauge invariant requires the introduction of a number of vector field equal to the dimension of the gauge group (gauge fields)

At the classical level, gauge invariance gives rise to the interaction terms between the fields on which the symmetry group acts (matter fields) and the gauge fields

At quantum level, transition amplitudes resulting from gauge symmetries enjoy many constraint (Ward identities) that make the theory fully predictable ('t Hooft - Veltman 1972)

This is why gauge theories are the best way to introduce interactions between particles

Except for the interactions of the Higgs boson with fermions, all known interactions in nature follow from a gauge principle

Abelian gauge symmetry U(1)

Consider a spinor field ψ with a free Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

This Lagrangian has a global U(1) symmetry

$$\psi \to \psi' = e^{-i\alpha}\psi$$

Let us now consider a U(1) transformation whose parameters depend on the point

$$\psi(x) \to \psi'(x') = e^{-ie\alpha(x)} \qquad e = \text{constant}$$

The mass term is invariant under such transformations

$$m\bar{\psi}\psi \to m\bar{\psi}e^{ie\alpha(x)}e^{-ie\alpha(x)}\psi = m\bar{\psi}\psi$$

The kinetic term is not invariant

$$\bar{\psi}i\gamma^\mu\partial_\mu\psi \to \bar{\psi}(x)e^{ie\alpha(x)}i\gamma^\mu\partial_\mu\left(e^{-ie\alpha(x)}\psi(x)\right)$$

$$= \bar{\psi}i\gamma^\mu e^{ie\alpha(x)}\left[e^{-ie\alpha(x)}\psi(x) - i\partial_\mu\alpha\, e^{-ie\alpha(x)}\psi(x)\right] =$$

$$= \bar{\psi}i\gamma^\mu\partial_\mu\psi + e\bar{\psi}\gamma^\mu\partial_\mu\alpha\,\psi$$

The problem is that

$$\partial_\mu \psi \to e^{-ie\alpha}\partial_\mu \psi - ie\partial_\mu\alpha\, e^{-ie\alpha}\partial_\mu\psi + e^{-ie\alpha}\partial_\mu\psi$$

i.e. the ordinary derivative $\partial_\mu\psi$ does not transform as the field ψ, i.e. it is not "covariant"

Task: construct <u>covariant derivative</u> D_μ such that

$$D_\mu\psi \to D'_\mu\psi' = e^{-ie\alpha}D_\mu\psi$$

Ansatz: $D_\mu = \partial_\mu + ieA_\mu$ $\quad$ A_μ vector field

$$D_\mu'\psi' = (\partial_\mu + ieA_\mu')\, e^{-ie\alpha}\psi =$$

$$= e^{-i\alpha}(\partial_\mu + ieA_\mu' - ie\partial_\mu\alpha)\psi \overset{!}{=} e^{-ie\alpha}D_\mu\psi$$

This equality is possible only if

$$A_\mu' = A_\mu + \partial_\mu\alpha \qquad \text{(gauge transformation)}$$

Once we have found D_μ, any Lagrangian containg products of $\bar\psi$ with ψ or $D_\mu\psi$ will be automatically invariant under local $U(1)$ (a.k.a. "gauge") transformations

The new gauge-invariant Lagrangian is then

$$\mathcal{L} = \bar\psi(i\gamma^\mu D_\mu - m)\psi$$

The gauge field A_μ has <u>interactions</u> with ψ entirely fixed by the gauge symmetry

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = -e\bar\psi\gamma^\mu A_\mu\psi$$

<u>kinetic term for A_μ</u>

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant. In fact

$$F_{\mu\nu}' = \partial_\mu A_\nu' - \partial_\nu A_\mu' = F_{\mu\nu} + (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)A_\mu = F_{\mu\nu}$$

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

canonical normalization

Fermion electrodynamics

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} =$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - e\bar{\psi}\gamma^{\mu}A_{\mu}\psi$$

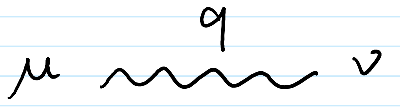
kinetic terms   mass   interaction

Note:

$$\mathcal{L}_{int} = -J^{\mu}A_{\mu} \qquad J^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$

J^{μ} is the conserved current arising from global $U(1)$ symmetry

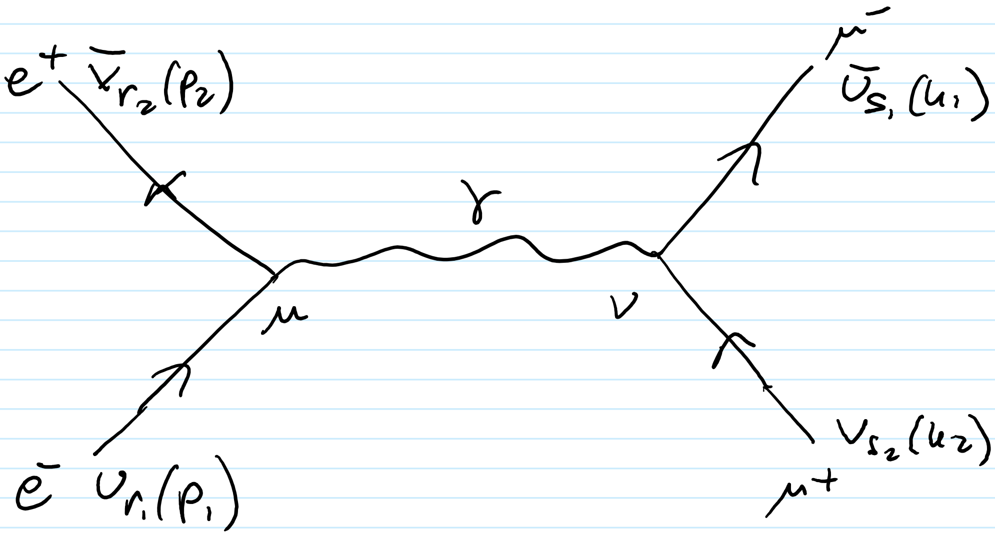
Feynman rules (for perturbative calculations)



$$-i\frac{\eta^{\mu\nu}}{q^2} + \text{gauge-dependent terms}$$

$$-ie(\gamma^{\mu})_{\alpha\beta}$$

$e^+e^- \to \mu^+\mu^-$



Scalar electrodynamics

Free Lagrangian for a complex field ϕ

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - V(\phi^* \phi)$$

This Lagrangian has the global $U(1)$ symmetry

$$\phi \to \phi' = e^{-i\alpha}\phi$$

Promote this symmetry to a local symmetry

$$\phi(x) \to \phi'(x') = e^{-ie\alpha(x)}\phi(x)$$

Promote the ordinary derivative to a covariant derivative

$$\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu$$

$$A_\mu' = A_\mu + \partial_\mu \alpha \Rightarrow D_\mu' \phi' \to e^{-ie\alpha} D_\mu \phi$$

New gauge-invariant Lagrangian, including the kinetic term for the gauge field A_μ

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu \phi)^*(D^\mu \phi) - V(\phi^*\phi) =$$

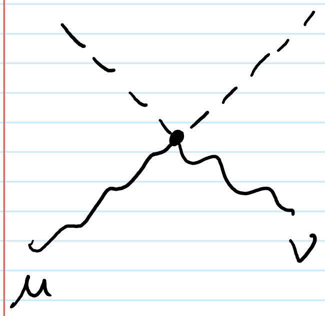
$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

$U(1)$ current J^μ

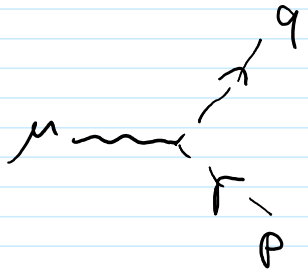
$$+ e^2 A_\mu A^\mu \phi^*\phi + ie\left(\phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi\right)A_\mu - \frac{\lambda}{2}(\phi^*\phi)^2 + \ldots$$

interactions

Feynman rules



$2ie^2\eta^{\mu\nu}$



$ie(p+q)^\mu$