Week 7: Global internal symmetries

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Symmetries and conservation laws (see QFT1)

Noether's theorem

For any continuous symmetry of the action, there exists a conserved current on the solutions of the equations of motion

For each parameter α_a of a continuous symmetry, the conserved current is

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \delta_0 \phi}{\partial \alpha_a} + \frac{\partial \delta x^\mu}{\partial \alpha_a} \mathcal{L} \qquad \partial_\mu J^\mu_a = 0$$

For each conserved current there exists a time-independent charge

$$Q_a = \int d^3\vec{x}\, J^0_a = \int d^3\vec{x} \left[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}}_{=\pi(x)} \frac{\partial \delta_0 \phi}{\partial \alpha_a} + \frac{\partial \delta x^0}{\partial \alpha_a} \mathcal{L} \right] =$$

$$= \int d^3\vec{x} \left[\pi(x) \frac{\partial \delta_0 \phi}{\partial \alpha_a} + \frac{\partial \delta x^0}{\partial \alpha_a} \mathcal{L} \right] \qquad \dot{Q}_a = 0$$

Example

Translations $\Rightarrow$ energy-momentum tensor

$$x^\mu \to x^\mu + a^\mu \Rightarrow \frac{\partial x^\mu}{\partial a_\nu} = \eta^{\mu\nu}$$

$$\delta_0 \phi = -a^\mu \partial_\mu \phi \Rightarrow \frac{\partial \delta_0 \phi}{\partial a_\nu} = -\partial^\nu \phi$$

Conserved noether current

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

charges

$$P^0 = H = \int d^3\vec{x} \left(\pi \partial^0 \phi - \mathcal{L}\right)$$

$$\vec{P} = \int d^3\vec{x}\, \pi \vec{\nabla} \phi$$

Global internal symmetries

The attribute "internal" refers to the fact that these are not symmetries of space-time, hence

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \delta \phi}{\partial \alpha_a}$$

The corresponding time-independent charges commute with all the generators of the Poincaré group, and then its irreducible representations on a Hilbert space give additional quantum numbers besides mass and spin/helicity

Some of this symmetries are not exact in nature, but are broken by small terms. So the results obtained by assuming the symmetry is exact are usually very good approximations

We will consider

i) Baryon number and nuclear isospin $U(2)$
ii) Electric charge $SO(2)$ vs $U(1)$
iii) Chirality $U(1)$
iv) Lepton number $U(1)$

Nuclear isospin

Properties of the nucleons (proton and neutron)

- They have approximately the same mass $m_p \simeq m_n$
- the strong force does not distinguish between them
- Elettromagnetic interactions are different $(Q_p = +1, Q_n = 0)$

Express this similarity as a symmetry transformation, that will be mildly broken by electromagnetic interactions

Construct a doublet of Dirac fermions

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and consider the Lagrangian

$$\mathcal{L} = i \bar{\psi}_i \not{\partial} \psi_i - m \bar{\psi}_i \psi_i \qquad \not{\partial} = \gamma^\mu \partial_\mu$$

Symmetry: $\psi \to \psi' = U \psi \qquad U \in U(2)$

Theorem: $U(2) = SU(2) \times U(1)$

Proof: $\forall\, U \in U(2),\ U^\dagger U = \mathbb{1} \Rightarrow |\det U| = 1$

$\Rightarrow \det U = e^{i\alpha}$

We can then write

$U = e^{i\alpha} V$ for some $V \in SU(2)$

Infinitesimal transformations

$$\psi \to \psi' = V\psi = \exp\left(i\alpha_a \frac{\tau_a}{2}\right)\psi \simeq \psi + \underbrace{i\alpha_a \frac{\tau_a}{2}\psi}_{\equiv \delta_0 \psi}$$

$a = 1,2,3$ $\quad \tau_a$ = Pauli matrices

$$\psi \to \psi' = e^{i\alpha}\psi \simeq \psi + \underbrace{i\alpha\psi}_{\equiv \delta_0\psi}$$

Conserved Noether currents

$SU(2)$ "iso-spin" $\qquad J^\mu_a = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\left(i \dfrac{\tau_a}{2}\psi\right) = -\bar{\psi}\gamma^\mu \dfrac{\tau_a}{2}\psi$

$U(1)$ "baryon number" $\qquad J^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(i\psi) = -\bar{\psi}\gamma^\mu\psi$

Interpretation: the "nucleon" is a particle of spin 1/2, of which the proton and the neutron are the two possible states

The charge Q associated to the $U(1)$ current counts the number of nucleons. This number is conserved in all physical processes in nature

Example: neutron decay $\quad n \to p\, e^- \bar{\nu}_e$

$Q_B = 1 \qquad Q_B = 1$

Isospin is conserved in strong interactions

$\Rightarrow$ isospin-orbit interactions in nuclear physics $\quad H \supset \vec{I}\cdot\vec{L}$

Electromagnetic interactions are different for the proton and the neutron $\Rightarrow \Delta m = m_n - m_p \neq 0$

$\partial_\mu J^\mu_I = f(\Delta m) \to 0$ for $\Delta m \to 0$

Pions and nuclear isospin

Three spin-0 particles $\pi^0, \pi^\pm$ $m_0 = 135$ MeV $\simeq m_\pm = 139.6$ MeV
Let assume that $m = m_0 \simeq m_\pm$ and write

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi_0)(\partial^\mu \pi_0) - \frac{1}{2} m_0^2 \pi_0^2 +$$
$$+ (\partial_\mu \pi^*)(\partial^\mu \pi) - m_\pm^2 \pi^* \pi$$

The second line is the Lagrangian for two real scalar fields having the same mass, in fact

$$\pi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$$

$$\pi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_3)$$

We also define

$$\pi_0 = \varphi_3$$

so that the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_i)(\partial^\mu \varphi_i) - \frac{1}{2} m^2 \varphi_i \varphi_i =$$
$$= \frac{1}{2}(\partial_\mu \phi^T)(\partial^\mu \phi) - \frac{1}{2}\phi^T m^2 \phi \qquad \phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

This Lagrangian in invariant under three-dimensional rotations

$$\phi \to O\phi \quad \text{with } O \in O(3), \text{ i.e. } O^T O = \mathbb{1}$$

Such symmetry is an isospin symmetry. Also, we know that $SO(3)$ and $SU(2)$ have the same Lie algebra

Is it possible to construct a Lagrangian for pions and nucleons with an interaction term that is both Lorentz and isospin invariant?

Consider $f_S \bar\psi \frac{\tau_a}{2} \psi \varphi^a \equiv f_S S$

$$f_A \bar\psi \frac{\tau_a}{2} \gamma^5 \psi \varphi^a \equiv f_A A$$

We observe that, for any $U(\vec\alpha) \in SU(2)$, and $O(\vec\alpha) \in SO(3)$

$U(\vec\alpha) = e^{i\vec\alpha\cdot\frac{\vec\tau}{2}}$ $\quad O(\vec\alpha) = e^{i\vec\alpha\cdot\vec J}$

we have

$$U(\vec\alpha) \frac{\tau_a}{2} \varphi_a U^\dagger(\vec\alpha) = \left(O(\vec\alpha) \phi \right)_b \frac{\tau_b}{2}$$

where J^a are the generators of $SO(3)$, which are also the generators of the adjoint representation of $SU(2)$

$$(J^a)_{bc} = -i\,\varepsilon_{abc}$$

Proof: a direct computation of $U(\vec\alpha)$ and $O(\vec\alpha)$ gives

$$U(\vec\alpha) = \cos\frac{\alpha}{2} + i(\hat\alpha\cdot\vec\tau)\sin\frac{\alpha}{2}$$

$$O(\vec\alpha) = \mathbb{1} + i(\hat\alpha\cdot\vec J)\sin\alpha + (\cos\alpha - 1)(\hat\alpha\cdot\vec J)^2$$

$$U(\vec\alpha)\frac{\tau_a}{2}\varphi_a U^\dagger(\vec\alpha) = \left(\cos\frac{\alpha}{2} + i\hat\alpha_b\tau_b\sin\frac{\alpha}{2}\right)\frac{\tau_a}{2}\varphi_a\left(\cos\frac{\alpha}{2} - i\hat\alpha_b\tau_b\sin\frac{\alpha}{2}\right) =$$

$$= \cos^2\frac{\alpha}{2}\,\frac{\tau_a}{2}\varphi_a + i\hat\alpha_b\varphi_a\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)\left[\frac{\tau_b}{2},\frac{\tau_a}{2}\right] +$$

$$+ \sin^2\frac{\alpha}{2}\,(\hat\alpha_b\tau_b)\frac{\tau_a}{2}(\hat\alpha_c\tau_c)\varphi_a$$

$$(\hat\alpha_b\tau_b)\frac{\tau_a}{2}(\hat\alpha_c\tau_c) = \underbrace{(\hat\alpha_b\tau_b)(\hat\alpha_c\tau_c)}_{=1}\frac{\tau_a}{2} +$$

$$+ (\hat\alpha_b\tau_b)(i\varepsilon_{acd}\tau_d)\hat\alpha_c =$$

$$= \frac{\tau_a}{2} + (i\varepsilon_{acd})(i\varepsilon_{bde})\tau_e\,\hat\alpha_b\hat\alpha_c =$$

$$= \frac{\tau_a}{2} - 2(\hat\alpha_c J^c)_{ad}(\hat\alpha_b J^b)_{de}\frac{\tau_e}{2}$$

$$U(\vec{\alpha})\frac{\tau_a}{2}\varphi_a U^\dagger(\vec{\alpha}) = \underbrace{\left(\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\right)}_{=1}\frac{\tau_a}{2}\varphi_a +$$

$$+ i\underbrace{\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)}_{=\sin\alpha}(\hat{\alpha}_b J^b)_{ac}\varphi^c\frac{\tau_a}{2} - \underbrace{2\sin^2\frac{\alpha}{2}}_{=\cos\alpha-1}\left[(\hat{\alpha}_b J^b)(\hat{\alpha}_c J^c)\phi\right]_a\frac{\tau_a}{2}$$

If we then perform the transformation

$$\psi \to \psi' = U(\vec{\alpha})\psi \qquad \phi \to \phi' = O(\vec{\alpha})\phi$$

we obtain, for $\Gamma = \mathbb{1}, \gamma^5$

$$\bar{\psi}\frac{\tau_a}{2}\Gamma\psi\varphi_a \to \bar{\psi}\Gamma U^\dagger(\vec{\alpha})\frac{\tau_a}{2}\varphi_a' U(\vec{\alpha})\psi =$$

$$= \bar{\psi}\Gamma\left[O^{-1}(\vec{\alpha})\phi'\right]_b\frac{\tau_b}{2}\psi =$$

$$= \bar{\psi}\Gamma\left[O^{-1}(\vec{\alpha})O(\vec{\alpha})\phi\right]_b\frac{\tau_b}{2}\psi =$$

$$= \bar{\psi}\Gamma\varphi_b\frac{\tau_b}{2}\psi$$

<u>Summary</u>

The free Lagrangian has SU(2) and O(3) symmetry. With interactions, only one SU(2) survives, and the trasformations of the nucleon and the pions are the same

Experimentally, $f_S = 0$

Also, $\bar{\psi}\gamma^5\psi$ is a pseudo-scalar under parity. Therefore, the pions have to be pseudoscalars for $\mathcal{L}$ to be Lorentz-invariant

<u>Electric charge</u>

Electric charge

Consider N real scalar fields ϕ_a, $a = 1, \dots N$, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - V(\phi^a)$$

If $V(\phi^a) = F(\phi^a \phi^a) \Rightarrow \mathcal{L}$ invariant under $SO(N)$

Example: $N=2$

symmetry $\phi_1 \to \phi_1' = \phi_1 \cos\theta + \phi_2 \sin\theta$

$$\phi_2 = -\phi_1 \sin\theta + \phi_2 \cos\theta$$

$$\phi' = \Omega \phi \qquad \Omega = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)$$

infinitesimal transformation

$$\delta_0 \phi = \theta \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \phi$$

$$\delta_0 \phi_a = \theta \varepsilon_{ab} \phi_b \qquad \varepsilon_{ab} = \begin{cases} 1 & ab = 12 \\ -1 & ab = 21 \\ 0 & a = b \end{cases}$$

conserved current

$$J^\mu = (\partial^\mu \phi_a) \varepsilon_{ab} \phi_b = (\partial^\mu \phi_1)\phi_2 - (\partial^\mu \phi_2)\phi_1$$

Complex field

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - V(\phi^* \phi)$$

symmetry: $\phi \to \phi' = e^{-i\theta} \phi \qquad U(1)$

with $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ we recover $SO(2)$ symmetry

Infinitesimal transformations

$$\delta \phi = -i\theta \phi \qquad \delta \phi^* = i\theta \phi^*$$

conserved current

$$J^\mu = -i(\partial^\mu \phi^*)\phi + i(\partial^\mu \phi)\phi^* = i\phi^* \overleftrightarrow{\partial^\mu} \phi$$

Spinor field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Symmetry : $U(1)$

$$\psi \to e^{-i\theta}\psi \qquad \delta\psi = -i\theta\psi$$

$$\bar{\psi} \to e^{i\theta}\bar{\psi} \qquad \delta\bar{\psi} = i\theta\bar{\psi}$$

current:

$$J^\mu = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}}_{= i\bar{\psi}} \frac{\partial \delta\psi}{\partial\theta} + \frac{\partial \delta\bar{\psi}}{\partial\theta} \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}}_{=0} = \bar{\psi}\gamma^\mu\psi$$

Both in the scalar and in the spinor case, the theory describes two particles having the same mass
The time-independent $U(1)$ charge counts the number of particles of one kind (particles) minus the number of particles of the other kind (anti-particles) $\Rightarrow$ electric charge

Chiral symmetry

Consider a massless spinor field with a Lagrangian

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$$

This Lagrangian acquires an additional symmetry

$$\psi \to e^{-i\gamma^5\theta}\psi \qquad \delta\psi = -i\theta\gamma^5\psi$$

$$\bar{\psi} \to \bar{\psi} e^{-i\gamma^5\theta} \qquad \delta\bar{\psi} = -i\theta\bar{\psi}\gamma^5$$

conserved (axial) current

$$J_5^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)} \frac{\partial\delta\psi}{\partial\theta} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

This symmetry exists because we can split spinors into left- and right-handed parts according to the eigenvalues of γ^5 (chirality)

$$\psi_{R/L} = \frac{1}{2}(1 \pm \gamma^5)\psi \Rightarrow \begin{matrix} \psi_L \to e^{+i\theta}\psi_L \\ \psi_R \to e^{-i\theta}\psi_R \end{matrix}$$

A Dirac mass term mixes left- and right-handed spinors and is therefore not invariant under chiral transformations, e.g.

$$\psi_R^\dagger \psi_L \to e^{2i\theta}\psi_R^\dagger\psi_L$$

A Majorana mass term contains two left- or right-handed spinor fields, so it is also not invariant under chiral transformations, e.g.

$$\psi_L^T \psi_L \to e^{2i\theta}\psi_L^T\psi_L$$

The corresponding charge counts the number of left-handed minus the number of right handed fermions