Week 6: Fermionic fields

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Spinor representations of the Lorentz group

Recall the Lie algebra of the Lorentz group

$$S_i = \frac{1}{2}(J_i + iK_i) \qquad [S_i, S_j] = i\epsilon_{ijk} S_k$$

$$[S_i, A_j] = 0$$

$$A_i = \frac{1}{2}(J_i - iK_i) \qquad [A_i, A_j] = i\epsilon_{ijk} A_k$$

Consider now a generic Lorentz transformation

$$\Omega = \exp\left(\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}\right) = \exp\left[i(\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K})\right]$$

Consider now the two-dimensional "spinor" representations

$$\left(\tfrac{1}{2}, 0\right) \qquad S_i = \frac{\sigma_i}{2} \qquad A_i = 0 \qquad \text{"Left"}$$

(σ_i = Pauli matrices)

$$\left(0, \tfrac{1}{2}\right) \qquad S_i = 0 \qquad A_i = \frac{\sigma_i}{2} \qquad \text{"Right"}$$

This gives

$$\vec{J} = \frac{1}{2}\vec{\sigma} \qquad \vec{K} = \mp \frac{i}{2}\vec{\sigma}$$

left ($-$), right ($+$)

Let us consider an $SU(2)$ group element generated by the left algebra

$$\Omega_L(\vec{\alpha}) = \exp\left(i\frac{\vec{\alpha}}{2}\cdot(\vec{J} + i\vec{K})\right) = \exp\left(i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}\right) = \exp(i\vec{\alpha}\cdot\vec{J})$$

But such transformations give us only the rotations. To obtain the boosts as well we need to consider the complex Lie groups generated by $\vec{S}$ and $\vec{A}$

$$\Omega_L(\vec{\alpha}, \vec{\beta}) = \exp\left(i(\vec{\alpha} - i\vec{\beta})\cdot\frac{\vec{\sigma}}{2}\right) = \exp\left(i\left(\vec{\alpha}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\left(-\frac{i\vec{\sigma}}{2}\right)\right)\right) =$$

$$= \exp\left[i(\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K})\right]$$

$$\Omega_R(\vec{\alpha}, \vec{\beta}) = \exp\left(i(\vec{\alpha} + i\vec{\beta})\cdot\frac{\vec{\sigma}}{2}\right)$$

Which group have we generated?

$$\ln\det\Omega = \operatorname{Tr}\ln\Omega = \frac{i}{2}\operatorname{Tr}(\vec{\sigma}\cdot(\vec{\alpha} \pm i\vec{\beta})) = \frac{i}{2}(\vec{\alpha} \pm i\vec{\beta})\cdot\operatorname{Tr}(\vec{\sigma}) = 0$$

This implies $\det\Omega = 1 \Rightarrow \Omega \in SL(2, \mathbb{C})$

- Properties

$$\Omega_L^{-1} = \Omega_R^\dagger$$

Pauli matrices obey the relation

$$\sigma_2 \sigma_i \sigma_2 = -\sigma_i^*$$

Therefore

$$\sigma_2 \Omega_L \sigma_2 = \Omega_R^* \qquad \text{(exercise: show explicitly)}$$

Consider now

$$\Omega_L^{-1} = \exp\left[-\frac{i}{2}(\vec{\alpha} - i\vec{\beta})\cdot\vec{\sigma}\right]$$

$$\sigma_2 \Omega_L^{-1} \sigma_2 = \exp\left[\frac{i}{2}(\vec{\alpha} - i\vec{\beta})\cdot\vec{\sigma}^*\right] =$$

$$= \exp\left[\frac{i}{2}(\vec{\alpha} - i\vec{\beta})\cdot\vec{\sigma}^T\right] = (\Omega_L)^T$$

Therefore

$$\sigma_2 \Omega_L^T \sigma_2 \Omega_L = \mathbb{1} \qquad \text{same for } \Omega_R$$

$$\Omega_L^\dagger \sigma_2 \Omega_L = \sigma_2$$

Consider now a LT acting on $\sigma_2 \psi_L^*$

$$\sigma_2 \Omega_L^* \psi_L^* = \underbrace{\sigma_2 \Omega_L^* \sigma_2}\ \sigma_2 \psi_L^* =$$

$$= \Omega_R\ \sigma_2 \psi_L^*$$

i.e. $\sigma_2 \psi_L^*$ is a right-handed spinor

- Generators of the representation

Consider now the matrix $\sigma^\mu = (\sigma^0 = \mathbb{1}, \vec{\sigma})$

$$\Omega_L\ \sigma^0 \Omega_L^\dagger = \Omega_L \Omega_L^\dagger \simeq$$

$$\simeq \left(\mathbb{1} + \frac{i}{2}\vec{\sigma}\cdot(\vec{\alpha}+i\vec{\beta})+\dots\right)\left(\mathbb{1} - \frac{i}{2}\vec{\sigma}\cdot(\vec{\alpha}+i\vec{\beta})+\dots\right) =$$
$$= \mathbb{1} - \vec{\beta}\cdot\vec{\sigma} = \sigma^0 + \omega^0{}_i\,\sigma^i$$

$$\Omega_L \sigma^i \Omega_L^\dagger \simeq \sigma^i + \frac{i}{2}\alpha^k \underbrace{[\sigma^k, \sigma^i]}_{2i\varepsilon_{kil}\sigma^l} - \frac{1}{2}\beta^k \underbrace{\{\sigma^k, \sigma^i\}}_{2\delta^{ki}\sigma^0} =$$
$$= \sigma^i + \varepsilon_{ikl}\,\alpha^k \sigma^l - \beta^i \sigma^0$$
$$= \sigma^i + \omega^{il}\sigma_l + \omega^{i0}\sigma_0$$

Altogether

$$\Omega_L \sigma^\mu \Omega_L^\dagger \simeq \sigma^\mu + \omega^{\mu\nu}\sigma_\nu + \dots = \Lambda^\mu{}_\nu \sigma^\nu$$

Analogously, if we define $\bar{\sigma}^\mu = (\sigma^0, -\vec{\sigma})$, we get

$$\Omega_R \bar{\sigma}^\mu \Omega_R^\dagger = \Lambda^\mu{}_\nu \bar{\sigma}^\nu$$

Finally, by comparing

$$\Omega_L = \exp\left(\frac{i}{2}(\vec{\alpha} - i\vec{\beta})\cdot\vec{\sigma}\right) = \exp\left(\frac{i}{2}\omega^{\mu\nu}\Sigma^L_{\mu\nu}\right)$$

$$\Omega_R = \exp\left(\frac{i}{2}(\vec{\alpha} + i\vec{\beta})\cdot\vec{\sigma}\right) = \exp\left(\frac{i}{2}\omega^{\mu\nu}\Sigma^R_{\mu\nu}\right)$$

one finds

$$\Sigma^L_{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu\right)$$

$$\Sigma^R_{\mu\nu} = \frac{i}{4}\left(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu\right)$$

Scalars out of spinor fields

(iv) Spinor fields

Consider the two combinations of left and right spinor fields

$$\chi_L^T \sigma_2 \chi_L \qquad \chi_R^T \sigma_2 \chi_R$$

Under a Lorentz transformation

$$\chi_L^T \sigma_2 \chi_L \rightarrow (\chi_L')^T \sigma_2 \chi_L' = \chi_L \Omega_L^T \sigma_2 \Omega_L \chi_L = \chi_L^T \sigma_2 \chi_L$$

$$\chi_L^T \sigma_2 \chi_L \to (\chi_L')^T \sigma_2 \chi_L' = \chi_L \underbrace{\Omega_L^T \sigma_2 \Omega_L}_{=\sigma_2} \chi_L = \chi_L^T \sigma_2 \chi_L$$

Both combinations behave as <u>scalars</u> under Lorentz transformations

Let us consider the explicit form of each combination

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\chi_L^T \sigma_2 \chi_L = -i(\chi_{L,1}\chi_{L,2} + \chi_{L,2}\chi_{L,1})$$

Such combination vanishes if χ_{Li} are real numbers

However, if they are Grassmann variables which <u>anti</u>-commute between themselves, such scalars are non zero

Consider now

$$\psi_L^+ \bar{\sigma}^\mu B_\mu \psi_L$$

where B_μ transforms as a vector

Under a Lorentz transformation

$$\psi_L^+ \bar{\sigma}^\mu B_\mu \psi \to (\psi_L')^+ \bar{\sigma}^\mu B_\mu' \psi_L' =$$

$$= \psi^+ \Omega_L^+ \bar{\sigma}^\mu \Lambda_\mu{}^\nu B_\nu \Omega_L \psi =$$

$$= \psi^+ \underbrace{\Omega_L^+ \bar{\sigma}^\mu \Omega_L}_{=\Lambda^\mu{}_\rho \sigma^\rho} \Lambda_\mu{}^\nu B_\nu \psi = \psi^+ \sigma^\nu B_\nu \psi$$

Similarly $\chi_R^+ \sigma^\mu B_\mu \chi_R$ also transforms as a scalar

Two-component spinors are also known as <u>Weyl fermions</u>

(v) Dirac spinors $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad J^i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad K^i = -\frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

γ-matrices in Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \end{pmatrix} \quad \to \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \text{ or } \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \sigma^{\mu\nu}$$

with $\sigma_{ij} = \varepsilon_{ijk} J^k \qquad \sigma_{0i} = k_i$

Lorentz transformation: $\Omega = \exp[i(\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{k})]$

$$\Omega^\dagger \gamma^0 \gamma^\mu \Omega = \Lambda^\mu{}_\nu \gamma^0 \gamma^\mu$$

Next, let us define the parity operator P ~~that~~ exchanges ψ_L and ψ_R

$$P\psi = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \gamma^0 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

Let us define also

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3$$

The two projectors

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma^5) \qquad P_R = \frac{1}{2}(\mathbb{1} + \gamma^5)$$

project onto the left- and right-handed components of a spinor respectively

To define Lorentz-invariant quantities we need to define the spinor

$$\bar{\Psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \psi_R^* \\ \psi_L^* \end{pmatrix}^T = (\psi_R^\dagger \;\; \psi_L^\dagger)$$

This gives the two invariants

$$\bar{\Psi}\gamma^\mu \partial_\mu \Psi = \psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + \psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R$$

$$\bar{\Psi}\Psi = \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$$

hermitian conjugation

Charge conjugation

A generic Dirac spinor can be written in the form

$$\psi = \begin{pmatrix} \psi_L \\ -i\sigma_2 \chi_L^* \end{pmatrix} \qquad \psi_L, \chi_L \in \left(\tfrac{1}{2}, 0\right)$$

We wants to construct the "charge conjugate" spinor that exchanges ψ_L and χ_L, as follows

$$\psi^c = \begin{pmatrix} \chi_L \\ -i\sigma_2 \psi_L^* \end{pmatrix} = i\gamma^2 \psi^* \equiv C\bar{\psi}^T$$

$$-i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ is real}$$

Here we have defined the "charge conjugation" matrix

$$C \equiv i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$$

This gives rise to another invariant

$$\psi^T C \psi = i\psi_L^T \sigma_2 \psi_L - i\psi_R^T \sigma_2 \psi_R$$

Quadratic Lagrangians for fermionic fields

iv) Spinor fields [see e.g. arXiv:0812.1594]

Consider a left-handed spinor field $\psi_L(x)$ and a right-handed spinor field $\psi_R(x)$

- kinetic terms

Neglecting terms that can be eliminated with an integration by part in the action, we get

$$\mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$$

The normalization is again chosen in such a way that the kinetic energy is positive (check: exercise)

The Fock space obtained from quantisation of this theory corresponds to two massless fermions.
The fermionic nature of the fields is due to the fact that, to achieve causality, we need to impose canonical anti-commutation rules, so that

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = 0 \quad \text{for } (x-y)^2 < 0$$

- Mass terms

If we construct Lorentz-invariant terms out of ψ_L and ψ_R separately, we can add two Majorana mass terms

$$\mathcal{L} \supset \mu_L \chi_L^T \sigma_2 \chi_L + \mu_R \chi_R^T \sigma_2 \chi_R + \text{h.c.}$$

Together with the kinetic term, this Lagrangian describes a left-handed and a right-handed fermionic particle with mass squared m_L^2 and m_R^2 respectively, given by

$$m_L^2 = \mu_L^* \mu_L > 0 \qquad m_R^2 = \mu_R^* \mu_R > 0$$

Another possibility is to add a Dirac mass term that couples the two fermions

$$\mathcal{L} \supset \mu \psi_L^\dagger \psi_R + \mu^* \psi_R^\dagger \psi_L$$

Together with the kinetic term, this Lagrangian describes a left-handed and a right-handed fermion particle with the same squared mass m^2, given by

$$m^2 = \mu^* \mu > 0$$

v) Dirac fields

Let us consider a Dirac spinor field

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \qquad \bar{\psi}(x) = \psi^\dagger(x)\gamma^0$$

- kinetic terms

Up to terms that can be eliminated via an integration by parts, we get

$$\mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R = i\bar{\psi}\gamma^\mu \partial_\mu \psi$$

From what we have seen from two-component spinor fields, this Lagrangian describes massless right-handed and left-handed fermions

For causality reasons, we must impose canonical anti-commutation relations, so that

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = 0 \quad \text{for } (x-y)^2 < 0$$

- Mass terms

A <u>Dirac mass term</u> can be written in the form

$$\mathcal{L} \supset \mu \psi_L^\dagger \psi_R + \mu^* \psi_R^\dagger \psi_L = \bar{\psi} M \psi \qquad M \equiv \begin{pmatrix} \mu^* & 0 \\ 0 & \mu \end{pmatrix}$$

A <u>Majorana mass term</u> can be constructed by introducing the charge-conjugation matrix $C = i\gamma^2\gamma^0$

$$\mathcal{L} \supset \mu \psi^T C \psi - \mu^* \bar{\psi} C \bar{\psi}^T =$$

$$= \mu \left(i\psi_L^T \sigma_2 \psi_L - i \psi_R^T \sigma_2 \psi_R \right) + h.c.$$

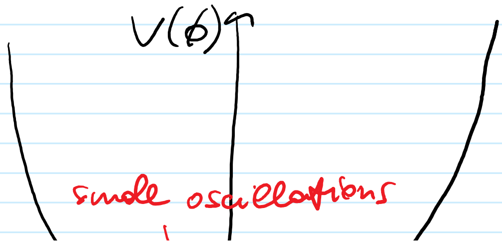
If one has a mixture of Dirac and Majorana mass terms, one needs to perform suitable linear combinations of the fields (see the literature for details)

<u>General local Lagrangians</u>

In general, we can construct a Lagrangian with an arbitrary number of <u>local</u> products of fields and of their derivatives

Consider an example with one real scalar field

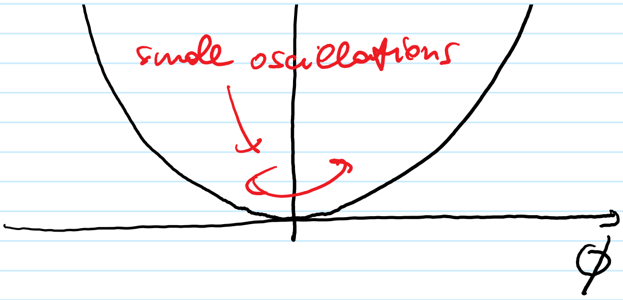
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$



Some guiding principles

- The kinetic energy has to be positive.

Some guiding principles



- The kinetic energy has to be positive
- Terms with higher derivatives give rise to Fock spaces with unphysical degrees of freedom: leave them to the experts
- The potential $V(\phi)$ is a power series in ϕ (locality)
- $V(\phi)$ has to be bounded from below (positive energy), hence $\lim_{\phi \to \pm\infty} V(\phi) = +\infty$
- Assume $V(\phi)$ has only one minimum for $\phi = 0$

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\frac{\partial^2 V}{\partial \phi^2}\Big|_{\phi=0} \phi^2}_{\text{quadratic Lagrangian}} + \mathcal{L}_{int}$$

If we assume that $\mathcal{L}_{int}$ does not make the field ϕ grow too much, the Lagrangian can be approximated with its quadratic term

$$\frac{\partial^2 V}{\partial \phi^2}\Big|_{\phi=0} = m^2 > 0 \quad \text{because } \phi=0 \text{ is a minimum}$$

The quadratic Lagrangian describes a spin-0 particle of mass m

Note: the case in which the minimum of $V(\phi)$ is for $\phi \neq 0$ will be considered later in this module

- If $\mathcal{L}_{int}$ is a small perturbation of the quadratic Lagrangian, (hopefully) the Fock space of the interacting theory is the same as that of the free theory (i.e. that obtained by quantising the quadratic part of the Lagrangian)