5 Scalar fields

5.1 Kallen-Lehmann representation of a two-point function

Consider for simplicity the case of one particle with mass m and spin 0. Suppose that, at each point x, we can define a hermitian operator $\phi(x)$, a *quantum field*. If we change coordinates, $x \to x' = \Omega(\Lambda, a)x$, then physical states will change accordingly

$$|\psi\rangle \to |\psi'\rangle = U(\Lambda, a)|\psi\rangle \,.$$

What is then $\phi(x')$? We impose it is a scalar field, i.e. its expectation values do not change with coordinates

$$\langle\chi'|\phi(x')|\psi'\rangle = \langle\chi|U^{-1}(\Lambda, a)\phi(x')U(\Lambda, a)|\psi\rangle = \langle\chi|\phi(x)|\psi\rangle \,.$$

This gives the transformation rule

$$\phi(x') = U(\Lambda, a)\phi(x)U^{-1}(\Lambda, a) \,.$$

In particular, note that

$$\phi(x + a) = e^{iP_\mu a^\mu}\phi(x)e^{-iP_\mu a^\mu} \,.$$

Consider now the vacuum expecation value of the commutator of two scalar hermitian quantum fields at two different space-time points x and y:

$$G(x - y) \equiv \langle 0|[\phi(x), \phi(y)]|0\rangle \,.$$

The fact that G depends only on the difference $x - y$ is a consequence of the invariance of the vacuum under translations, in fact

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}e^{iP\cdot y}\phi(0)e^{-iP\cdot y}|0\rangle = \langle 0|\phi(0)e^{-iP\cdot(x-y)}\phi(0)|0\rangle \,.$$

We study then the function

$$G(x) = \langle 0|[\phi(x), \phi(0)]|0\rangle \,.$$

We insert between the two fields a complete set of states:

$$\mathbb{1} = \sum_n |n\rangle\langle n| = |0\rangle\langle 0| + \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}|\vec{p}\rangle\langle\vec{p}| + \ldots$$

and we get

$$\begin{aligned}\langle 0|\phi(x)\phi(0)|0\rangle &= \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(0)|0\rangle \\ &= \sum_n \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|n\rangle\langle n|\phi(0)|0\rangle = \sum_n e^{-ip_n\cdot x}|\langle 0|\phi(0)|n\rangle|^2 \,.\end{aligned}$$

This gives

$$\langle 0|[\phi(x),\phi(0)]|0\rangle \sum_n \left(e^{-ip_n\cdot x}-e^{ip_n\cdot x}\right)|\langle 0|\phi(0)|n\rangle|^2 = \int \frac{d^4q}{(2\pi)^3}\left(e^{-iq\cdot x}-e^{iq\cdot x}\right)\tilde{\rho}(q)\,,$$

where

$$\tilde{\rho}(q) = (2\pi)^3 \sum_n \delta^4(q-p_n)|\langle 0|\phi(0)|n\rangle|^2\,.$$

We now use the fact that the eigenvalues of P^2 and P^0 are non-negative, and that $\tilde{\rho}(q)$ is invariant under proper orthochronous Lorentz transformations, to recast $\tilde{\rho}(q)$ in terms of a *spectral density* $\rho(q^2)$, as follows:

$$\tilde{\rho}(q) = \rho(q^2)\Theta(q^0)\,.$$

This implies

$$\langle 0|[\phi(x),\phi(0)]|0\rangle = \int \frac{d^4q}{(2\pi)^3}\left(e^{-iq\cdot x}-e^{iq\cdot x}\right)\rho(q^2)\Theta(q^0)\,.$$

We now change variable to $\mu^2 \equiv q^2$, and obtain the *Kallen-Lehmann* representation of the vacuum expectation value of the commutator of two scalar fields at different space-time points

$$\langle 0|[\phi(x),\phi(0)]|0\rangle = \int_0^\infty d\mu^2\,\rho(\mu^2)\,i\Delta(x,\mu^2)\,.$$

The function $\Delta(x,\mu^2)$ is defined as

$$i\Delta(x,\mu^2) = \int \frac{d^4q}{(2\pi)^3}\left(e^{-iq\cdot x}-e^{iq\cdot x}\right)\Theta(q^0)\delta(q^2-\mu^2) = \int \frac{d^4q}{(2\pi)^3}\,e^{-iq\cdot x}\,\epsilon(q^0)\,,$$

with $\epsilon(q^0)\equiv\Theta(q^0)-\Theta(-q^0)$ the sign of q^0. The function $\Delta(x,\mu^2)$ has a set of remarkable propertiess

(i) $(\Box+\mu^2)\Delta(x,\mu^2)=0$;

(ii) $\Delta(\Lambda x,\mu^2)=\Delta(x,\mu^2)$ if $\Lambda\in L_+^\uparrow$;

(iii) $\Delta(x,\mu^2)$ is a function of x^2 and $\epsilon(x^0)$ only;

(iv) $\Delta(x,\mu^2)=-\Delta(-x,\mu^2)$;

(v) $\frac{\partial}{\partial x^0}\Delta(x-y,\mu^2)|_{x^0=y^0}=-\delta^3(\vec{x}-\vec{y})$;

(vi) $\Delta(x,\mu^2)=0$ if $x^2<0$.

Property (vi) ensures that the commutator of two fields at points separated by a space-like distance, i.e. that cannot be connected by a light-ray, vanishes. This ensures the *causality* of the theory.

Let us isolate the contribution of one-particle states

$$|\langle 0|\phi(0)|1\rangle|^2 = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} |\langle 0|\phi(0)|\vec{p}\rangle|^2 \,.$$

Since $\phi(x)$ is a scalar quantum field

$$\langle 0|\phi(0)|\vec{p}\rangle = \sqrt{Z_\phi} \quad \text{(independent of } \vec{p}) \,.$$

This gives

$$(2\pi)^3 \delta^4(q - p_1) |\langle 0|\phi(0)|1\rangle|^2 = \frac{Z_\phi}{2E_{\vec{q}}} \delta(q^0 - E_{\vec{q}}) \Theta(q^0) = Z_\phi \delta(q^2 - m^2) \Theta(q^0) \,.$$

This, and the fact that, for states with more than one particle, $\mu^2 \geq (2m)^2$, gives the final form of the K'allen-Lehmann representation for the vacuum expectation value of the commutator of two scalar fields:

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = iZ_\phi \, \Delta(x - y, m^2) + \int_{4\mu^2}^{\infty} d\mu^2 \rho(\mu^2) \, i\Delta(x - y, \mu^2) \,.$$

Suppose that $\phi(x)$ is normalised so as to satisfy *canonical commutation rules*

$$[\dot{\phi}(x), \phi(y)]|_{x^0 = y^0} = -i\delta^3(\vec{x} - \vec{y}) \,.$$

We then obtain

$$\begin{aligned} \frac{\partial}{\partial x^0} \langle 0|[\phi(x), \phi(y)]|0\rangle|_{x^0 = y^0} &= -i\delta^3(\vec{x} - \vec{y}) \\ &= \left(Z_\phi + \int_{4\mu^2}^{\infty} d\mu^2 \rho(\mu^2) \right) - i\delta^3(\vec{x} - \vec{y}) \,. \end{aligned}$$

This gives the normalisation condition

$$1 = Z_\phi + \int_{4\mu^2}^{\infty} d\mu^2 \rho(\mu^2) \,.$$

Since $\rho(\mu^2)$ is non-negative, and Z_ϕ is by construction non-negative, we obtain the constraint $0 \leq Z_\phi \leq 1$. If $Z_\phi = 1$, then the field $\phi(x)$ creates states that do not overlap with two- or more-particle states. It is therefore a *free field.*

Infinitesimal transformations.

5.2 Free scalar fields

Consider now a free quantum scalar field $\phi(x)$, satisfying

$$U(\Lambda, a)\phi(x)U^{-1}(\Lambda, a) = \phi(\Lambda x + a)\,,$$

with the additional condition

$$\langle 0|\phi(0)|\vec{p}\rangle = 1 \implies \langle 0|\phi(x)|\vec{p}\rangle = e^{-ip\cdot x}\,.$$

The effect of this field is to transform $|\vec{p}\rangle$ into the vacuum. We then construct an *annihilation operator* $a(\vec{p})$ that "destroys" a particle of momentum $\vec{p}$, as follows:

$$\begin{aligned} &a(\vec{p})|0\rangle = 0\,,\\ &a(\vec{p})|\vec{p}\,'\rangle = (2\pi)^3 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}\,')|0\rangle\,. \end{aligned}$$

Given this operator, a free hermitian scalar quantum field $\phi(x)$ is given by

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^2 2E_{\vec{p}}}\left(e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}a^\dagger(\vec{p})\right)\,.$$

From the definition of an adjoint operator, we get that $a^\dagger(\vec{p})$ "creates" a particle of momentum $\vec{p}$, as follows

$$a^\dagger(\vec{p})|0\rangle = |\vec{p}\rangle\,,$$

and is therefore called *creation operator*. We can construct two-particle states by successive application of creationoperators

$$a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle = a^\dagger(\vec{p}_1)|\vec{p}_2\rangle = |\vec{p}_1, \vec{p}_2\rangle\,.$$

How is $|\vec{p}_1, \vec{p}_2\rangle$ related to $|\vec{p}_2, \vec{p}_1\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle$? This depends on the relation between $a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)$ and $a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)$. If we impose commutation relations

$$[a(\vec{p}), a(\vec{p}\,')] = 0\,, \quad \text{and} \quad [a(\vec{p}), a^\dagger(\vec{p}\,')] = (2\pi)^3 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}\,')\,,$$

we obtain $\langle 0|[\phi(x), \phi(y)]|0\rangle = 0$ if $(x-y)^2 < 0$, which preserves causality. These commutation relations imply

$$|\vec{p}_2, \vec{p}_1\rangle = |\vec{p}_1, \vec{p}_2\rangle\,,$$

which implies these particles are *bosons*.

We now construct an operator that gives us the total energy of a state $|\vec{p}_1, \vec{p}_2, \ldots, p_N\rangle$. This is the Hamiltonian

$$\begin{aligned} &H = \int \frac{d^3\vec{p}}{(2\pi)^2 2E_{\vec{p}}} E_{\vec{p}}\, a^\dagger(\vec{p})a(\vec{p})\,,\\ &H|\vec{p}_1, \vec{p}_2, \ldots, p_N\rangle = \sum_{i=1}^{N} E_{\vec{p}_i}\, |\vec{p}_1, \vec{p}_2, \ldots, p_N\rangle \implies \langle\psi|H|\psi\rangle \geq 0\,. \end{aligned}$$

Also, a direct calculation yields

$$[H, \phi(x)] = -i\partial_0\phi(x) \implies H = P_0 \text{ (generator of time translations)}\,.$$

Similarly, we can construct the *momentum operator*:

$$\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^2 2E_{\vec{p}}} \vec{p}\, a^\dagger(\vec{p})a(\vec{p})\,.$$

If we build the operator $P^\mu \equiv (H, \vec{P})$, we obtain $[P^\mu, \phi(x)] = -i\partial^\mu\phi(x)$, which means that P^μ is the generator of translations. Similarly, $J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu$ is the generator of Lorentz transformations, and since these particles have spin zero, there is no other contribution to $J_{\mu\nu}$. Consistently, $W_\mu = 0$. Note that this representation of fields is a approrpriate for both massive and massless particles.
A direct computation show that the Hamiltonian H can be expressed in terms of a Hamiltonian density $\mathcal{H}(x)$ as follows:

$$H = \int d^3\mathcal{H}(x)\,, \qquad \mathcal{H}(x) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \ldots\,.$$

where the dots give a constant contribution to the energy, which we can neglect. Since $\mathcal{H}(x)$ can be expressed in terms of local fields

$$[\phi(x), \phi(y)] = 0\,, \quad (x-y)^2 < 0 \implies [\mathcal{H}(x), \mathcal{H}(y)] = 0\,, \quad (x-y)^2 < 0\,.$$

Therefore, Hamilitonian densities made of local operators are causal.
Last, we observe that $\phi(x)$ satisfies the equation $(\Box + m^2)\phi(x) = 0$. But this is the classical equation of motion obtained from the most general Lorentz-invariant quadratic Lagrangian for a classical scalar field:

$$\mathcal{L} = c_1(\partial_\mu\phi)(\partial^\mu\phi) + c_2\phi^2\,,$$

by computing the corresponding equations of motions, and setting

$$c_1 = \frac{1}{2}\,, \quad c_2 = -\frac{1}{2}m^2\,.$$

In fact, one could start from $\mathcal{L}$, solve the corresponding equation of motion, and promote the solution $\phi(x)$ to an operator by introducing the operators of creation and annihilation, and interpreting their action as before. Such a procedure is called *canonical quantisation.* Note that, if we wish to interpret the solutions of a classical Lagrangian as a quantum field, we obtain important constraints on the coefficients c_1 and c_2. In fact, from the Lagrangian, we can compute the Hamiltonian density as follows

$$\pi(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(x))}\,, \quad \text{and} \quad \mathcal{H} = \pi\dot{\phi} - \mathcal{L}\,.$$

In the present case

$$\pi(x) = \dot{\phi}\,, \quad \text{and} \quad \mathcal{H} = c_1\pi^2 + c_1(\vec{\nabla}\phi)^2 - c_2\phi^2\,.$$

If the term containing π^2 is to be interpreted as the kinetic energy, we must have $c_1 > 0$. Also, if we want that the energy is bounded from below, we need to have $c_2 < 0$. The actual value $c_2 = -m^2 c_1$ is set by imposing that this quantised Hamiltonian is the energy of particles of mass m.