4 Representations of the Poincaré group

4.1 The Poincaré group

The Poincaré group is the set of transformations that preserve the distance between pairs of vectors x, y:

$$(x-y)^2 = (x^0-y^0)^2 - \sum_{i=1}^{d-1}(x^i-y^i)^2 \,.$$

It contains Lorentz transformations and translations by a fixed vector a^μ. The latter transformations act as follows

$$x^\mu \to x'^\mu = x^\mu + a^\mu \,.$$

The generators of translations are the d "momenta" P_μ ($\mu = 0, 1, \ldots, d-1$), and a general group element is given by

$$\Omega(\Lambda, a) = \exp\left[\frac{i}{2}\omega^{\mu\nu}J_{\mu\nu} + ia^\mu P_\mu\right] ,$$

where $J_{\mu\nu}$ are the generators of Lorentz transformations, of which the matrices $(M_{\mu\nu})^\rho_\sigma$ constitute a concrete representation. Of course, $\Omega(\mathbb{1}, a)$ are translations, whereas $\Omega(\Lambda, 0)$ are Lorentz transformations.

$$\text{Parameters:} \qquad \underbrace{\frac{d(d-1)}{2}}_{\text{Lorentz transformations}} + \underbrace{d}_{\text{translations}} = \frac{d(d+1)}{2} \,.$$

Irrespective of the representation of the Poincaré group, we can infer the commutation rules between generators from the multiplication law of the group. Consider then two Poincaré transformations $\Omega(\Lambda_1, a_1)$ and $\Omega(\Lambda_2, a_2)$ and construct the transformation $\Omega(\Lambda_1, a_1)\Omega(\Lambda_2, a_2)$, obtained by acting with $\Omega(\Lambda_2, a_2)$ on a four-vector x^μ, and then with $\Omega(\Lambda_1, a_1)$ on the result of that transformation.

$$x \to \Lambda_2 x + a_2 \to \Lambda_1(\Lambda_2 x + a_2) + a_1 = \Lambda_1\Lambda_2 x + (\Lambda_1 a_2 + a_1) \,.$$

This gives us the rules for group muliplication

$$\Omega(\Lambda_1, a_1)\Omega(\Lambda_2, a_2) = \Omega(\Lambda_1\Lambda_2, \Lambda a_2 + a_1)\,.$$

The inverse of $\Omega(\Lambda, a)$ is $\Omega(\Lambda^{-1}, -\Lambda^{-1}a)$. In fact

$$x \xrightarrow{\Omega(\Lambda,a)} \Lambda x + a \xrightarrow{\Omega(\Lambda^{-1},-\Lambda^{-1}a)} \Lambda^{-1}\Lambda x + \Lambda^{-1}a - \Lambda^{-1}a = x\,.$$

The subgroup of translations in an *Abelian* group, from which we obtain the commutation rules for the generators P_μ:

$$[P_\mu, P_\nu] = 0\,, \qquad \mu = 0, 1, \ldots, d-1\,.$$

The remaining commutation rules can be obtained by considering the product

$$\Omega(\Lambda, 0)\Omega(\mathbb{1}, a)\Omega(\Lambda^{-1}, 0) = \Omega(\Lambda, 0)\Omega(\Lambda^{-1}, a) = \Omega(\mathbb{1}, \Lambda a)$$

If all transformation are infinitesimal, we have $\Omega(\mathbb{1}, a) = \mathbb{1} + a^\mu P_\mu$, and

$$\begin{aligned}\Omega(\Lambda, 0)a^\mu P_\mu \Omega(\Lambda^{-1}, 0) \simeq a^\mu P_\mu + \frac{i}{2}\omega^{\rho\sigma} a^\mu [J_{\rho\sigma}, P_\mu] = \Lambda^\mu_\nu a^\nu P_\mu \\ = a^\mu P_\mu + \frac{i}{2}\omega^{\rho\sigma}(M_{\rho\sigma})^\mu_\nu a^\nu P_\mu\,.\end{aligned}$$

Using the explicit form of $(M_{\rho\sigma})^\mu_\nu$, we find

$$[J_{\mu\nu}, P_\rho] = (M_{\mu\nu})^\sigma_\rho P_\sigma = i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right)\,.$$

The commutation rules for $J_{\mu\nu}$ can be obtained in a similar way, or from the explicit representation $(M_{\mu\nu})^\rho_\sigma$. In summary, we have

$$\begin{aligned}&[P_\mu, P_\nu] = 0\,,\\ &[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}\right)\,,\\ &[J_{\mu\nu}, P_\rho] = i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right)\,.\end{aligned}$$

4.2 Poincaré transformations on classical fields.

A classical field is a map from each point x in Minkowsky space-time into a vector space V which hosts a finite-dimensional representation of the Lorentz group:

$$\begin{aligned}f :& \mathbb{R}^4 \to V\\ & x \;\mapsto f_a(x)\end{aligned}$$

This is useful to establish the transformation rules of the quantity that determines the dynamics of the fields, the action

$$S = \int d^4x\,\mathcal{L}\left[\underbrace{\phi(x)}_{\text{scalar}}, \underbrace{\psi(x)}_{\text{spinor}}, \underbrace{A_\mu(x)}_{\text{vector}}, \underbrace{h_{\mu\nu}(x)}_{\text{spin-2 tensor}}, \dots\right].$$

where $\mathcal{L}$ is the Lagrangian (density), a function of all the fields involved, and more rarely also of space-time coordinates. The action S is a number. If the theory described by S is to be invariant under Poicaré transformations, then S has to be invariant. This means that, under a Poincaré transformation $x \to x' = \Lambda x + a$, S must stay invariant, i.e.

$$S' = \int d^4x'\,\mathcal{L}'(x') = S = \int d^4x\,\mathcal{L}(x)\,.$$

since $d^4x' = d^4x$, this can happen if and only if $\mathcal{L}'(x') = \mathcal{L}(x)$. With the notation $\mathcal{L}'(x')$ we mean that, when x changes to x', all fields change in form, so that the functional form of the Lagrangian changes. Once we have identified such Lagrangian, the dynamics of each field $f_a(x)$ is obtained by making the action S stationary, which requires solving the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu f_a(x))} - \frac{\partial\mathcal{L}}{\partial f_a(x)} = 0\,.$$

We now investigate the transformation properties of $\mathcal{L}(x)$ under Poincaré transformations.

(a) Translations. A translation corresponds to a shift in the origin of the coordinates. No field should change its value after such change of coordinates, because it would mean there is a preferred reference frame. This means that, under a translation $x \to x' = x + a$, for every component of a field $f_a(x)$, we have

$$f'_a(x') = f'_a(x + a) = f_a(x) .$$

Example: field $p(z)$ of water pressure. It is known that, setting the origin of a z-axis at the sea level, and letting z increase as we go deeper and deeper in the ocean, we have

$$p(z) = p_0 + \rho g z ,$$

where p_0 is atmospheric pressure, ρ the density of water, and g the acceleration of gravity. A scuba diver, located at a depth z_0, will measure depth according to a coordinate $z' = z - z_0$. Nevertheless, at the same point in the ocean, the pressure measured by the scuba diver will be the same as that measured by a boat on the surface of the sea, i.e.

$$p(z) = p_0 + \rho g z = \underbrace{p_0 + \rho g z_0}_{\equiv p'_0} + \rho g \underbrace{(z - z_0)}_{\equiv z'} = p'_0 + \rho g z' = p'_0(z') .$$

Therefore, under translations

$$\mathcal{L}'(x') = \mathcal{L}'(x + a) = \mathcal{L}(x) .$$

(b) Lorentz transformations. Let us consider a generic Lorentz transformation $x \to x' = \Lambda x$. Fields have different transformation properties with respect to Lorentz transformations.

(i) Scalar fields. A *scalar* field $\phi(x)$ is invariant under Lorentz transformations, i.e.

$$\phi(x) \to \phi'(x') = \phi'(\Lambda x) = \phi(x) .$$

The Lagrangian density has to be a scalar field if we want that the action is Lorentz invariant. Note that, for fixed x, $\phi(x)$ belongs to the vector space that hosts the $(0, 0)$ representation of the Lorentz group.

(ii) Vector fields. Given a scalar field, let us consider the transformation rules of the field $\partial^\mu \phi(x)$, with

$$\partial^\mu = \eta^{\mu\nu} \partial_\nu , \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} .$$

Given a Lorentz transformation, we have

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \implies x^\mu = \eta^{\mu\rho} \eta_{\sigma\nu} \Lambda^\sigma{}_\rho x'^\nu = \Lambda_\nu{}^\mu x'^\nu .$$

This in turn implies

$$\frac{\partial}{\partial x'^\nu} = \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial}{\partial x^\rho} = \Lambda_\nu{}^\rho \partial_\rho .$$

Therefore

$$\partial^\mu \phi(x) \to \eta^{\mu\nu} \frac{\partial \phi'(x')}{\partial x'^\nu} = \eta^{\mu\nu} \Lambda_\nu{}^\rho \partial_\rho \phi(x) = \Lambda^\mu{}_\rho \partial^\rho \phi(x) \,.$$

The transformation rules of $\partial^\mu \phi(x)$ are the transformation rules for *vector* fields, i.e. fields $A^\mu(x)$ that transform as follows

$$A^\mu(x) \to A'^\mu(x') = A'^\mu(\Lambda x) = \Lambda^\mu{}_\nu A^\nu(x) \,.$$

Note that, for every point x, $A^\mu(x)$ lives in a vector space that hosts the $(1/2, 1/2)$ representation of the Lorentz group.

(iii) Tensor fields. Taking one more derivative of $\partial^\mu \phi(x)$, one can construct a field $h_{\mu\nu}(x)$ with two indexes that transform as a tensor:

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h'_{\mu\nu}(\Lambda x) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x) \,.$$

As explained at the end of previous week, $\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma$ can be further decomposed into irreducible representations. Correspondingly, $h_{\rho\sigma}(x)$ will be made up of different fields transforming in a different way with respect to Lorentz transformations.

(iv) Spinor fields. Similarly, one can consider a left-handed Weyl spinor field $\psi(x)$ transforming as follows

$$\psi(x) \to \psi'(x') = \psi'(\Lambda(\vec{\alpha}, \vec{\beta})x) = \Omega_L(\vec{\alpha}, \vec{\beta})\, \psi(x) \,.$$

Similarly, a right-handed Weyl spinor field $\chi(x)$ transforms as follows:

$$\chi(x) \to \chi'(x') = \chi'(\Lambda(\vec{\alpha}, \vec{\beta})x) = \Omega_R(\vec{\alpha}, \vec{\beta})\, \chi(x) \,.$$

As one can see, for each point x, the fields $\psi(x)$ and $\chi(x)$ belong to $\mathbb{C}^2$, which host the $(1/2, 0)$ and $(0, 1/2)$ representations of the Lorentz group. Dirac spinor fields can be defined in a similar way.

In general, a *classical* field $f_a(x)$ is said to transform according to the (s_1, s_2) representation of the Lorentz group if and only if

$$f_a(x) \to f'_a(x') = f'_a(\Lambda x + a) = \Omega^{(s_1,s_2)}_{ab}(\Lambda)\, f_b(x) \,,$$

where $\Omega^{(s_1,s_2)}_{ab}(\Lambda)$ is a matrix belonging to the (s_1, s_2) representation of the Lorentz group, and corresponding to the Lorentz matrix Λ.

A *local* Lagrangian is made up of products of such fields at the same point x, arranged in such a way as to obtain a Lorentz scalar. Once we have the Lagrangian, we can compute the corresponding Euler-Lagrange equations for the fields. Their solutions give the dynamics of the classical fields.

Infinitesimal transformations. It is useful to determine how classical fields transform under infinitesimal transformations. Suppose now that

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu \,,$$

where δx^μ is infinitesimal. Then the total change $\delta f_a(x)$ of a field $f_a(x)$ is given by

$$\begin{aligned}
\delta f_a(x) &\equiv f'_a(x') - f_a(x) = f'_a(x+\delta x) - f_a(x) \\
&= f'_a(x) - f_a(x) + f'_a(x+\delta x) - f'_a(x) \\
&= f'_a(x) - f_a(x) + \partial_\mu f'_a(x)\delta x^\mu + O(\delta x^2) \\
&= \underbrace{\delta_0 f_a(x)}_{\text{functional change}} + \underbrace{\partial_\mu f_a(x)\delta x^\mu}_{\text{transport form}} + O(\delta x^2)
\end{aligned}$$

(a) Translations: $\delta x^\mu = a^\mu$.

$$\delta f_a = 0 \implies \delta_0 f_a = -a^\mu \partial_\mu f_a = i a^\mu P_\mu f_a \,,$$

where $P_\mu = i\partial_\mu$ gives a representation of the generators of translations on smooth functions.

(b) Lorentz transformations. $\delta x^\mu = \omega^\mu{}_\nu x^\nu$.

$$f'_a(x') = \Omega^{(s_1,s_2)}_{ab} f_b(x) = \left(\delta_{ab} + \frac{i}{2}\omega^{\rho\sigma}\left(\Sigma^{(s_1,s_2)}_{\rho\sigma}\right)_{ab}\right) f_b(x)\,.$$

This gives

$$\delta f_a = \frac{i}{2}\omega^{\rho\sigma}\left(\Sigma^{(s_1,s_2)}_{\rho\sigma}\right)_{ab} f_b \implies \delta_0 f_a = \frac{i}{2}\omega^{\rho\sigma}\left(\Sigma^{(s_1,s_2)}_{\rho\sigma}\right)_{ab} f_b - \delta x^\mu \partial_\mu f_a\,.$$

The transport form then becomes

$$-\delta x^\mu \partial_\mu f_a(x) = -\omega^\mu{}_\nu x^\nu \partial_\mu f_a(x) = -\omega^{\mu\nu} x_\nu \partial_\mu f_a(x) = \frac{1}{2}\omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial\mu)\, f_a(x) = \frac{i}{2}\omega^{\mu\nu} L_{\mu\nu}\, f_a(x)\,,$$

where

$$L_{\mu\nu} \equiv -i(x_\mu\partial_\nu - x_\nu\partial\mu) = x_\nu P_\mu - x_\mu P_\nu$$

give a representation of the Lorentz algebra on the space of smooth functions. Altogether, for an arbitrary (s_1, s_2) field, we have

$$\delta_0 f_a(x) = \frac{i}{2}\omega^{\rho\sigma}\left(L_{\rho\sigma}\delta_{ab} + \left(\Sigma^{(s_1,s_2)}_{\rho\sigma}\right)_{ab}\right) f_b(x)\,.$$

The quantities $L_{\mu\nu}$ are referred to as *orbital angular momentum*, and is independent of the representation of the field $f_a(x)$. The matrices $\Sigma^{(s_1,s_2)}_{\rho\sigma}$ are the generator of the (s_1, s_2) representation of the Lorentz group, and are refereed to as *spin angular momentum*.

4.3 Unitary representations of the Poincaré group

A unitary representation of the Poincaré group is a map

$$\Omega(\Lambda, a) \mapsto U(\Lambda, a),$$

where $U(\Lambda, a)$ is a unitary operator on a Hilbert space $\mathcal{H}$ that satisfies the same multiplication rules of the Poincaré group, i.e.

$$U(\mathbb{1}, 0) = \mathbb{1}, \text{ and } \quad U(\Lambda_1, a_1)U(\Lambda_2, a_2) = U(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1).$$

Let $\mathcal{H}$ be the Hilbert space containing physical states. Then, if $|\psi\rangle \in \mathcal{H}$ represents a physical state for a given observer O', then the same physical state for an observer related to O' by a Poincaré transformation $\Omega(\Lambda, a)$ is represented by the vector $|\psi'\rangle = U(\Lambda, a)|\psi\rangle$. The fact that U is unitary ensures that transition amplitudes between physical states are left invariant by Poincaré transformations:

$$\langle\phi'|\psi'\rangle = \langle U(\Lambda, a)\phi|U(\Lambda, a)\psi\rangle = \langle\phi|\psi\rangle.$$

Actually, physical states are such that $U(\Lambda, a)$ can be a unitary representation of the double cover of the Poincaré group. As we will see, this implies the existence of states with half-integer spin. The aim of this section is to classify all irreducible representations of the double cover of the Poincaré group in four dimensions, and from them to construct the Hilbert space that describes physical states.

The two Casimir operators P^2 and W^2. There are two operators that commute with all the generators of the Poincaré group.

1. $P^2 = P_\mu P^\mu$ (mass). Since P_μ commute, $[P^2, P_\mu] = 0$. Due to the commutation rules of P_μ with $J_{\mu\nu}$ we have $[P^2, J_{\mu\nu}] = 0$ (see Problem Sheet 4).

2. Pauli-Ljubanski vector (polarisation). The Pauli-Ljubanski vector is defined as

   $$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma},$$

   where $\epsilon_{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric symbol in four dimensions, with $\epsilon_{0123} = +1$. From the definition, one obtains $W_\mu P^\mu = 0$. Recasting $J^{\rho\sigma}$ in terms of the generators of rotations J^i and the generators of boosts K^i, we have

   $$\begin{aligned} W_0 &= -P^i J_i = -\vec{P}\cdot\vec{J} \\ W_i &= P_0 J_i - \epsilon_{ijk}P^j J^k \implies \vec{W} = P_0\vec{J} - \vec{P}\times\vec{J}. \end{aligned}$$

   From W_μ we can construct $W^2 \equiv W_\mu W^\mu$. Due to the antisymmetry properties of W_μ one obtains

   $$[W_\mu, P_\nu] = 0 \implies [W^2, P_\mu] = 0.$$

A direct calculation (see Problem Sheet 4) gives

$$[W_\rho, J_{\mu\nu}] = -i\left(\eta_{\rho\mu} W_\nu - \eta_{\rho\nu} W_\mu\right) .$$

Since W_μ commutes with $J_{\mu\nu}$ like P_μ, then W^2 commutes with $J_{\mu\nu}$ like P^2. Therefore, $[W^2, J_{\mu\nu}] = 0$.

In conclusion, P^2 and W^2 are Casimir operator, hence we can decompose the Hilbert space into orthogonal Hilbert spaces according to the eigenvalues of these operators. Each of these spaces hosts an irreducible representation of the Poincaré group.

Spin-0 particles. We have already seen an infinite-dimensional unitary representation of the Poincaré group. Consider the space of all smooth functions $f(x)$. There, a unitary representation of the Lie algebra of the Poincaré group is given by the operators

$$P^\mu \equiv i\partial^\mu , \quad J^{\mu\nu} = L^{\mu\nu} \equiv x^\mu P^\nu - x^\nu P^\mu .$$

From the definition of the Pauli-Ljubanski vector one obtains $W_\mu = 0$, which means that all these functions have $W^2 = 0$. The physical interpretation is that this space contains states corresponding to a single particle of spin 0.

We now split the space of smooth functions in subspaces corresponding to different eigenvalues m^2 of P^2, and we restrict ourselves to those states for which $m^2 \geq 0$. We interpret the eigenvalue m^2 as the mass squared of the particle.

Suppose we are in the space with the a given value of m^2. A complete set of states is given by the eigenvalues of P^μ:

$$P^\mu f_{\vec{p}}(x) = p^\mu f_{\vec{p}}(x) \implies f_{\vec{p}}(x) = N(\vec{p})\, e^{-ip\cdot x} ,$$

where $N(\vec{p})$ is an appropriate normalisation. Note that

$$p^2 = m^2 \implies p^0 = \pm\sqrt{\vec{p}^2 + m^2} .$$

We now interpret $\vec{p}$ as the particle momentum and p^0 as the particle energy. Therefore, we only consider eigenvectors that have $p^0 \geq 0$. Using Dirac notation, we identify $f_{\vec{p}}(x) = |\vec{p}\rangle$. A generic single-particle state $|\psi\rangle$ is given by a superposition of eigenvectors of P^μ, as follows:

$$|\psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \psi(\vec{p})\,|\vec{p}\rangle , \qquad E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2} .$$

The eigenvectors of P^μ are conventionally normalised as

$$\langle\vec{p}|\vec{p}'\rangle = (2\pi)^2 (2E_{\vec{p}}) \delta^3(\vec{p} - \vec{p}') .$$

Note also that the integration measure $d^3\vec{p}/E_{\vec{p}}$ is invariant under Poicaré transformations.

Massive particles. Consider vectors such that $P^2 = m^2 > 0$. There, we consider the eigenvectors of P^μ such that

$$P^\mu|\vec{k}\rangle = k^\mu|\vec{k}\rangle\,, \qquad k^\mu = (m,0,0,0)\,.$$

We now consider the Pauli-Ljubanski vector for these "reference" vectors. Before we do this, we set

$$J^{\mu\nu} = L^{\mu\nu} + \Sigma^{\mu\nu} \implies W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu\Sigma^{\rho\sigma}\,.$$

The operator $L^{\mu\nu}$ plays the role of an orbital angular momentum, and $\Sigma^{\mu\nu}$ of the spin angular momentum. Note that the value of W^2 depends only on $\Sigma^{\mu\nu}$. For our reference vectors

$$W_0 = 0\,, \qquad W_i = -\frac{m}{2}\epsilon_{ijk}\Sigma^{jk} = mS_i\,,$$

where

$$S_i = -\frac{1}{2}\epsilon_{ijk}\Sigma^{jk} \implies [S_i, S_j] = i\epsilon_{ijk}S_k\,.$$

The operators S_i obey the algebra of $SU(2)$! Therefore, the Hilbert space generated by $|\vec{k}\rangle$ can be further decomposed into subspaces, each hosting an irreducible representation of $SU(2)$. These are labelled by the integer s corresponding to the largest eigenvalue of S_3 in that subspace. This integer can take the value

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

Also, for these reference vectors

$$W^2 = -\vec{W}^2 = -m^2\vec{S}^2\,.$$

Therefore, for each value of m (mass) and s (spin), the two Casimirs P^2 and W^2 have definite eigenvalues

$$P^2 = m^2\,, \qquad W^2 = -m^2 s(s+1)\,,$$

and hence the Hilbert space $\mathcal{H}$ corresponding to a given value of m and s hosts an irreducible representation of the Poincaré group.

Consider now the subspace generated by $|\vec{k}\rangle$, for a fixed value of m and s. This is a vector space of dimension $2s+1$, and a basis for this vector space is given by the eigenvalues of S_3, which we denote by $|\vec{k}, s_3\rangle$, with $s_3 = -s, -s+1, \ldots, +s$. A complete set of states that span the whole Hilbert space $\mathcal{H}$ is then given by $|\vec{p}, s_3\rangle$. Since any vector p^μ can be obtained from k^μ by means of a Lorentz transformation Λ, then $|\vec{p}, s_3\rangle$ can be obtained from $|\vec{k}, s_3\rangle$ by applying the corresponding unitary operator $U(\Lambda)$ (the details of this are not relevant for the moment). A generic one-particle state $|\psi\rangle$ of mass m and spin s can then be written as the superposition

$$|\psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \sum_{s_3} \psi(\vec{p}, s_3)\,|\vec{p}, m_s\rangle\,, \qquad E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}\,,$$

with the vectors $|\vec{p}, m_s\rangle$ normalised as follows

$$\langle \vec{p}, s_3|\vec{p}', s_3'\rangle = (2\pi)^2(2E_{\vec{p}})\delta_{s_3 s_3'}\delta^3(\vec{p}-\vec{p}')\,.$$

The subgroup of Poincaré transformations that keeps a vector k^μ fixed is called the *little group*. In this case, the little group corresponding to $k^\mu = (m, 0, 0, 0)$ is $SO(3)$. With our construction we have obtained the unitary representations of the double cover of the little group $SO(3)$, which is in fact $SU(2)$.

Examples:

$$\begin{array}{llll}
\text{spin } 0: & \text{Higgs boson} & s = 0 & \\[1em]
\text{spin } 1/2: & \text{electron} & s = 1/2 & s_3 = \pm 1/2 \\[1em]
\text{spin } 1: & \text{W/Z boson} & s = 1\,, & \overbrace{s_3 = \underbrace{\pm 1}_{\text{transverse}}\,,\ \underbrace{0}_{\text{longitudinal}}}^{\text{polarisations}} \\[1em]
\text{spin } 2: & \text{KK graviton} & s = 2\,, & s_3 = \pm 2, \pm 1, 0
\end{array}$$

Massless particles. If $P^2 = 0$, can find no state with $\vec{k} = 0$. We can consider the eigenvectors of P^μ such that

$$P^\mu|\vec{k}\rangle = k^\mu|\vec{k}\rangle\,, \qquad k^\mu = (E, 0, 0, E)\,.$$

As before, we restric ourselves to states with $E > 0$. Since $W_\mu P^\mu = 0$ we obtain that, for this states

$$W^\mu = (\lambda E, W_1, W_2, \lambda E)\,, \qquad W^2 = -W_1^2 - W_2^2 \leq 0\,.$$

For $W^2 = 0$, we obtain $W^\mu = \lambda P^\mu$, where λ is a Lorentz scalar. Then

$$[\lambda, P^\mu] = [\lambda, J^{\mu\nu}] = 0\,.$$

This means that λ is an additional Casimir, so its eigenvalues can be used to classify the irreducible representations of the Poincaré groups.

Since $W_0 = -P^i J_i = \lambda E$, we conclude that

$$\lambda = -\frac{P^i J_i}{E} = -\frac{\vec{P}\cdot\vec{J}}{E}\,.$$

The value of λ specifies the *helicity* of a massless particle. The operator λ is in fact the projection of the angular momentum in the direction of motion of the particle, i.e. J_3. The eigenvalues of J_3 are of the form

$$n_0, n_0 \pm 1, n_0 \pm 2, \ldots$$

The quantity n_0 is not constrained by algebraic means. However, interpreting J_3 as the generator of rotations around an axis constrains n_0 to be 0, so as to make sure that a rotation of 2π is the identity. We also allow the double cover of rotations around an axis, which implies that n_0 can be $1/2$ as well. The magnitude of λ is referred to as the *spin* of the massless particle. However, it is worth to point out that each value of λ corresponds in principle to a different particle

Examples:

spin 0 :	Goldstone boson	$\lambda = \pm 1/2$
spin 1/2 :	neutrino	$\lambda = \pm 1/2$
spin 1 :	photon	$\lambda = \pm 1$
spin 2 :	graviton	$\lambda = \pm 2$

The remaining case $P^2 = 0$ and $W^2 < 0$ would correspond to particles with zero rest mass and continuous polarisations. These states are not realised in nature.

What we have found can be again recast in terms of the little group corresponding to $k^\mu = (E, 0, 0, E)$. The group that leave this vector invariant is $SO(2)$. We have just constructed the finite-dimensional unitary representations of the double cover of $SO(2)$.

The vacuum. There is one unitary representation that corresponds to states with $P^\mu = (0, 0, 0, 0)$. This is the trivial representation $U(\Lambda, a) = \mathbb{1}$. This is also a finite-dimensional representation, and contains a state $|0\rangle$ that is invariant under Poincaré transformations. This state is called the *vacuum*. It is the state of lowest energy of a theory. We require it to be *unique*, but we will discuss the case of degenerate vacua later.

In fact, the little group for $k^\mu = (0, 0, 0, 0)$ is $SO(1, 3)$, and we know that the only finite-dimensional unitary representation of the double cover of $SO(1, 3)$ is the trivial one.

Fock space. Once we have obtained the irreducible representations of the Poincaré group, we can construct the Hilbert space $\mathcal{H}$ for physical states $|\psi\rangle$. This is called the Fock space and is defined as follows

$$\mathcal{H} = \underbrace{\mathcal{H}^{(0)}}_{\ni|0\rangle} \oplus \underbrace{\mathcal{H}^{(1)}}_{\ni|\vec{p},\alpha\rangle} \oplus \underbrace{\mathcal{H}^{(2)}}_{\ni|\vec{p}_1,\vec{p}_2,\alpha_1,\alpha_2\rangle} \oplus \ldots$$

The Hilbert space $\mathcal{H}^{(0)}$ contains the vacuum, and $\mathcal{H}^{(1)}$ contains one-particle states, with internal quantum number α. Two particle-states live in $\mathcal{H}^{(2)}$. If no bound states are formed, then $\mathcal{H}^{(2)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$. Similarly, we will have a Hilbert space $\mathcal{H}^{(3)}$ for three-particle states and so on. Physical states $|\psi\rangle$ are restricted to have non-negative invariant mass squared and positive, energy, i.e.

$$\langle\psi|P^2|\psi\rangle \geq 0\,, \quad \text{and} \quad \langle\psi|P^0|\psi\rangle \geq 0\,.$$