3 Representations of the Lorentz group

3.1 Reducible and irreducible representations

Direct sum representations. Given two representations D_1 and D_2 of a group G, with dimensions n_1 and n_2 respectively, the *direct sum* $D_1 \oplus D_2$ is the block-diagonal matrix

$$(D_1 \otimes D_2)(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} .$$

The generators of $D_1 \otimes D_2$ are

$$X_a^{(D_1 \oplus D_2)} = \begin{pmatrix} X_a^{(D_1)} & 0 \\ 0 & X_a^{(D_2)} \end{pmatrix} ,$$

and $\dim(D_1 \oplus D_2) = \dim(D_1) + \dim(D_2)$.

Direct product representations. Let $|i\rangle$, $i = 1, 2, \ldots, n_1$ a basis for the vector space V_1 over which the representation D_1 acts, and $|\alpha\rangle$, $\alpha = 1, 2, \ldots n_2$ a basis for V_2 over which D_2 acts. Let us define $V_1 \otimes V_2$ the vector space generated by $|i\rangle|\alpha\rangle$. We can define the direct product representation $D_1 \otimes D_2$ as

$$(D_1 \otimes D_2)(g)|i\rangle|\alpha\rangle = (D_1(g)|i\rangle)\,(D_2(g)|\alpha\rangle) .$$

Consequently

$$[(D_1 \otimes D_2)(g)]_{i\alpha, j\beta} = [D_1(g)]_{ij}[D_2(g)]_{\alpha\beta} .$$

The corresponding generators are given by

$$[X_a^{(D_1 \otimes D_2)}]_{i\alpha, j\beta} = [X_a^{(D_1)}]_{ij}\, \delta_{\alpha\beta} + \delta_{ij}\, [X_a^{(D_2)}]_{\alpha\beta} .$$

With a change of basis, direct product representations can generally be written in a block-diagonal form, i.e. as a direct sum of other representations.
A representation that is the direct sum of other representations is called *reducible*
An *irreducible* representation is one that is not reducible.

Example. With the method of the maximum eigenvalue of J_3 we have constructed irreducible representations of $SU(2)$ of finite dimension.
We can construct reducible representations of $SU(2)$ by taking the direct product of n two-dimensional representations D of $SU(2)$, as follows:

$$\mathcal{D} \equiv \underbrace{D \otimes D \otimes \cdots \otimes D}_{n \text{ times}} \implies D_{i_1 \ldots i_n j_1, \ldots j_n} = D_{i_1 j_1} D_{i_2 j_2} \ldots D_{i_n j_n} .$$

Such direct product representation can be decomposed into irreducible representations by looking for the highest eigenvalue of J_3, e.g.

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 .$$

Sometimes representations are labelled by their respective dimensions, so the above relation might be found in the form $2 \otimes 2 = 1 \oplus 3$.

3.2 The Lorentz group

The *Lorentz group* is the group of linear transformation on coordinates $x^\mu = 0, 1, \ldots, d-1$ preserving the space-time interval

$$\begin{aligned} & s^2 = \eta_{\mu\nu} x^\mu x^\nu \qquad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \ldots, -1)_{\mu\nu} \\ & x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu \\ & s^2 \to s'^2 = \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho x^\sigma = s^2 \end{aligned}$$

The above relation has to be valid for an arbitrary x^μ, which implies

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \, .$$

In matrix notation

$$\begin{aligned} & s^2 = x^T \eta \, x \qquad && x' = \Lambda x \\ & \Lambda^T \eta \, \Lambda = \eta \qquad && \text{Lorentz group condition} \end{aligned}$$

The transformations leaving $\eta = \mathrm{diag}(1, -1, -1, -1)$ constitute the group $O(1, 3) \approx O(3, 1)$, acting on Minkowsy space-time.
The transformations leaving $\mathbb{1}$ invariant constitute the group $O(4)$, acting on 4-dimensional Euclidean space. In fact

$$\Omega^T \mathbb{1} \, \Omega = \Omega^T \Omega = \mathbb{1} \qquad \text{orthogonal group condition}$$

Taking the determinant of the Lorentz group condition gives

$$\det(\Lambda^T) \det(\eta) \det(\Lambda) = (\det(\Lambda))^2 \det(\eta) = \det(\eta) \implies (\det(\Lambda))^2 = 1 \, ,$$

which implies

$$\det \Lambda = \begin{cases} +1 & \text{proper LT, e.g. identity, rotations, boost} \\ -1 & \text{improper LT, e.g. parity } \Lambda = \mathrm{diag}(1, -1, -1, -1) \end{cases}$$

The 00-component of the Lorentz group condition also gives

$$(\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^0_i)^2 = 1 \implies \Lambda^0_0 = 1 + \sum_{i=1}^{3} (\Lambda^0_i)^2 \geq 1 \, .$$

This gives

$$\begin{cases} \Lambda^0_0 \geq 1 & \text{orthochronous LT} \\ \Lambda^0_0 \leq -1 & \text{non-orthochronous LT, e.g. time-reversal } \Lambda = \mathrm{diag}(-1, 1, 1, 1) \end{cases}$$

To summarise

Λ^0_0 \ $\det(\Lambda)$	≥ 1	≤ -1
1	$L_+^{\uparrow}$: proper orthochronous (e.g. identity, rotations, boosts)	$L_+^{\downarrow}$: proper non-orthochronous (e.g. PT)
-1	$L_-^{\uparrow}$: improper orthochronous (e.g. parity P)	$L_-^{\downarrow}$: improper non-orthochronous (e.g. time-reversal T)

3.3 Lie algebra of the Lorentz group

The Lorentz group in d-dimensions is the subgroup $O(1, d-1) \approx O(d-1, 1)$ of $GL(d, \mathbb{R})$. As such, it has $d(d-1)/2$ generators (6 in four dimensions), as follows

$$M_{\mu\nu} = -M_{\nu\mu}\,, \qquad \mu = 0, 1, \ldots d-1\,, \qquad \Lambda = \exp\left[\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\right].$$

To find $M_{\mu\nu}$ we consider an infinitesimal Lorentz transformation

$$\Lambda^\mu_\nu \simeq \eta^\mu_\nu + \omega^\mu_\nu \quad (\eta^\mu_\nu = \delta^\mu_n u)\,.$$

From the definition of the Lorentz group we get $\omega_{\mu\nu} = -\omega_{\nu\mu}$. In fact

$$\eta_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma = \eta_{\rho\sigma} \implies \eta_{\rho\sigma} = \eta_{\mu\nu}(\eta^\mu_\rho + \omega^\mu_\rho)(\eta^\nu_\sigma + \omega^\nu_\sigma) \simeq \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} \implies \omega_{\rho\sigma} = -\omega_{\sigma\rho}\,.$$

This gives the explicit form of $M_{\mu\nu}$, as follows (see Problem Sheet 3)

$$\Lambda^\mu_\nu \simeq \eta^\mu_\nu + \omega^\mu_\nu = \eta^\mu_\nu + \left[\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right]^\mu_\nu \implies (M_{\rho\sigma})^\mu_\nu = -i\left(\eta^\mu_\rho\eta_{\nu\sigma} - \eta^\mu_\sigma\eta_{\nu\rho}\right).$$

Using the explicit form of $M_{\mu\nu}$, we obtain their commutation rules (see Problem Sheet 3)

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\right).$$

Lorentz algebra in 4 dimensions. The explicit form of $M_{\mu\nu}$ in four dimensions is

$$M_{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ & 0 & -J_3 & J_2 \\ & & 0 & -J_1 \\ & & & 0 \end{pmatrix} \qquad \begin{aligned} M_{0i} &= K_i \\ M_{ij} &= \epsilon_{ijk}J_k \end{aligned}$$

- *anti-hermitian* generators of boosts

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

- *hermitian* generators of rotations

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the generators of boost are not hermitian, the fundamental representation of the Lorentz group is not unitary. This poses problems for Quantum Mechanics, because we need unitary representations acting on wave functions to leave transition amplitudes invariant. We might then ask whether there exist other unitary representations of the Lorentz group, different from the fundamental one. Let us consider the analogy with ordinary quantum mechanics, which is invariant under rotations. There, the fundamental representation of the group of rotations $SO(3)$ was the adjoint representation of its double cover $SU(2)$, which has an isomorphic Lie algebra. Suppose we are able to find a group whose Lie algebra isomorphic to that of $SO(1,3)$, of which $SO(1,3)$ constitutes the adjoint representation. Note that, a group generated by the Lie algebra of a group G is called the *universal cover* (or covering) of group G. Our task is that of finding the universal cover of $SO(1,3)$. First, we observe that

$$\mathrm{Tr}(J_i J_j) = 2\delta_{ij}\,, \quad \mathrm{Tr}(J_i K_j) = 0\,, \quad \mathrm{Tr}(K_i K_j) = -2\delta_{ij}\,.$$

The Lie algebra is then non-compact. Unfortunately, it can be shown that the only finite-dimensional representations of any (semi-simple) non-compact Lie group is the trivial one, where all elements are mapped into the identity. So, if we are to construct a relativistic quantum mechanics, we need to consider infinite-dimensional unitary representations of the Lorentz group.
Finite dimensional representations are nevertheless very useful. In fact, suppose we want to find relativistically invariant equations for classical fields. Then, there is no need for these fields to be wave functions and have a probabilistic interpretation.

3.4 Finite-dimensional representations of the Lorentz group

The explicit commutation rules for the generators of the Lorentz group are

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k\,, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k\,.$$

We now consider the two linear combinations

$$S_i = \frac{1}{2}(J_i + iK_i)\,, \qquad A_i = \frac{1}{2}(J_i - iK_i)\,.$$

The operators S_i and A_i are hermitian and give two independent $SU(2)$ algebras:

$$[S_i, S_j] = i\epsilon_{ijk}S_k\,, \qquad [A_i, A_j] = -i\epsilon_{ijk}A_k\,, \qquad [S_i, A_j] = 0\,.$$

So, the Lie algebra of $SO(1,3)$ is isomorphic the direct sum of two $SU(2)$ Lie algebras, or $\mathfrak{so}(1,3) \approx \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Therefore, each finite-dimensional representations of the Lie algebra of $SO(1,3)$ is labelled by the pair

$$(s_1, s_2)\,, \qquad s_i = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$$

The Casimir operators are

$$\vec{S}^{\,2} = s_1(s_1+1)\,, \qquad \vec{A}^2 = s_2(s_2+1)\,.$$

The number s_1+s_2 is called the *total spin* of the representation and $\dim(s_1, s_2) = (2s_1+1)(2s_2+1)$. Given a representation (s_1, s_2), from S_i and A_i we can compute J_i and K_i. Then, given a set of three parameters for rotations $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and three parameters for boosts $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$, the generic element of the representation can be written in the form

$$\Omega^{(s_1,s_2)}(\vec{\alpha}, \vec{\beta}) = \exp\left[i(\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K})\right] = \exp\left[\frac{i}{2}\omega^{\mu\nu} M^{(s_1,s_2)}_{\mu\nu}\right]\,,$$

with $M^{(s_1,s_2)}_{\mu\nu}$ the generators of the representation in covariant form.

$(\mathbf{0},\mathbf{0})$: scalars. This is the trivial representation, i.e.

$$S_i = 0\,, \quad A_i = 0\,, \implies J_i = 0\,, \quad K_i = 0\,.$$

so that $\Omega^{(0,0)}(\vec{\alpha}, \vec{\beta}) = \mathbb{1}$. Objects that transform under $(0,0)$ are invariant under Lorentz transformations, and are called *scalars*. Example: the Higgs field ϕ.

$(1/2, 0)$ and $(0, 1/2)$: Weyl spinors. The two representations $(1/2, 0)$ and $(0, 1/2)$ of the Lie algebra of the Lorentz group are called *spinor* representations of the Lorentz group. These representations are two-dimensional, and defined by

$$\begin{aligned} (1/2, 0): &\quad S_i = \frac{\sigma_i}{2}\,, \quad A_i = 0\,, \quad (\text{“left”})\,, \\ (0, 1/2): &\quad S_i = 0\,, \quad A_i = \frac{\sigma_i}{2}\,, \quad (\text{“right”})\,. \end{aligned}$$

This gives

$$J_i = \frac{1}{2}\sigma_i\,, \qquad K_i = \begin{cases} -\dfrac{i}{2}\sigma_i & (\text{left}) \\ +\dfrac{i}{2}\sigma_i & (\text{right}) \end{cases}\,.$$

Let us consider now an operator $\Omega_L \equiv \Omega^{(1/2,0)}$ generated by the left algebra:

$$\Omega_L(\vec{\alpha}, \vec{\beta}) = \exp\left[i(\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K})\right] = \exp\left[i(\vec{\alpha} - i\vec{\beta})\cdot\frac{\vec{\sigma}}{2}\right]\,.$$

Due to the identity

$$\ln\det\Omega = \mathrm{Tr}\left(\ln\Omega\right) .$$

and the fact that σ_i are traceless, we have

$$\det\left(\Omega_L(\vec{\alpha},\vec{\beta})\right) = \exp\left(\mathrm{Tr}\left[\frac{i}{2}(\alpha_i - i\beta_i)\mathrm{Tr}(\sigma_i)\right]\right) = 1 .$$

Therefore, Ω_L is a 2×2 complex matrix with unit determinant, hence $\Omega_L \in SL(2,\mathbb{C})$. Note that $SL(2,\mathbb{C})$ can be obtained by exponentiating the Lie algebra $\mathfrak{su}(2)$ with *complex* parameters. Similarly, an operator $\Omega_R \equiv \Omega^{(0,1/2)}$ generated by the right algebra is given by

$$\Omega_R(\vec{\alpha},\vec{\beta}) = \exp\left[i(\vec{\alpha} + i\vec{\beta})\cdot\frac{\vec{\sigma}}{2}\right] .$$

There are some very important relations between these matrices:

- $SL(2,\mathbb{C})$ is not unitary, but
$$\Omega_L^\dagger = \Omega_R^{-1} .$$
The above relation implies
$$\Omega_R^\dagger = \Omega_L^{-1} .$$
Since Ω_R^{-1} belongs to $(0,1/2)$, this means that $\Omega_L^* = (\Omega_L^\dagger)^T$ also belongs to $(0,1/2)$. Therefore, if we regard $(1/2,0)$ as the fundamental representation of $SL(2,\mathbb{C})$, then $(0,1/2)$ is the so-called *conjugate* representation.

- Due to the relation between Pauli matrices
$$\sigma_2\sigma_i\sigma_2 = -\sigma_i^* ,$$
we have
$$\sigma_2\Omega_L\sigma_2 = \Omega_R^* .$$
This implies:
$$\sigma_2\Omega_L^T\sigma_2\Omega_L = \sigma_2\Omega_R^T\sigma_2\Omega_R = \mathbb{1} ,$$
and that, if ψ transforms according to $(1/2,0)$, then $\sigma_2\psi^*$ transforms according to $(0,1/2)$. Similarly, if χ transforms according to $(0,1/2)$, then $\sigma_2\chi^*$ transforms according to $(1/2,0)$.

A two-component complex vector that transforms according to $(1/2,0)$ (equivalently according to the fundamental representation of $SL(2,\mathbb{C})$ is called a *left-handed (Weyl) spinor* ψ_α, $\alpha = 1,2$. Example: the field ν_L describing a left-handed neutrino.
A two-component complex vector that transforms according to $(0,1/2)$ (equivalently according to the conjugate representation of $SL(2,\mathbb{C})$ is called a *right-handed (Weyl) spinor* $\psi_{\dot{\alpha}}$, $\dot{\alpha} = 1,2$. Example: the field ν_R describing a right-handed neutrino.
The generators of the left- and right-handed spinor representations are given in terms of the matrices $\sigma_\mu = (\mathbb{1},\sigma_i)$ and $\bar{\sigma}_\mu = (\mathbb{1},-\sigma_i)$ as follows

$$\Sigma^L_{\mu\nu} \equiv M^{(1/2,0)}_{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu\right) ,$$
$$\Sigma^R_{\mu\nu} \equiv M^{(0,1/2)}_{\mu\nu} = \frac{i}{4}\left(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu\right) .$$

$(1/2,0)\oplus(0,1/2)$: Dirac spinors. So far we have not been able to make a link between the S_i and the A_i. This link is provided by parity. Consider the Lorentz transformation in Minkowsky space corresponding to parity. This is represented by the 4×4 matrix $P=\mathrm{diag}(1,-1,-1,-1)$. The action of parity on the generators of the Lorentz group is given by

$$PJ_iP^{-1}=J_i\,,\quad PK_iP^{-1}=-K_i \implies S_i \overset{P}{\leftrightarrow} A_i\,.$$

This relation must hold in any representation of the Lorentz group. Therefore, if we need a theory that is invariant under parity, we need to represent this operators. Since the parity operator swaps S_i and A_i, it maps the $(1/2,0)$ representation into the $(0,1/2)$ representation. Therefore, in such a theory we need both representations. This is accomplished by constructing the direct sum of the two representations $(1/2,0)\oplus(0,1/2)$. An object transforming according to this representation is a four-dimensional vector ψ_D, a pair of Weyl spinors, one left-handed and the other right-handed, as follows

$$\psi_D \equiv \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix}, \qquad \psi_L\in(1/2,0)\,,\quad \chi_R\in(0,1/2)\,.$$

Example: since electrodynamics is parity invariant, the field ψ_e describing an electron has to be a Dirac spinor.

Note that, since $\sigma_2\psi_L^*$ transforms according to $(0,1/2)$, one can write a Weyl spinor as a Dirac spinor as follows

$$\psi_D \equiv \begin{pmatrix} \psi_L \\ \sigma_2\psi_L^* \end{pmatrix}, \qquad \psi_L\in(1/2,0)\,.$$

A four-dimensional spinor that depends as above only on a single Weyl spinor is called a Majorana spinor.

(1/2,1/2): vectors. Once we have the fundamental and conjugate representations of $SL(2,\mathbb{C})$, we can obtain all other representations by taking tensor products of these, and decomposing the results in irreducible representations by using the same technique used for $SU(2)$. We start with

$$(1/2,0)\otimes(0,1/2)=(1/2,1/2)\,.$$

This is the fundamental representation of $SO(1,3)$. Objects that transform according to this representation are called *vectors*. In fact, using the matrix σ_μ, we can associate to each four-vector $x^\mu=(x^0,x^1,x^2,x^3)$ the complex 2×2 matrix

$$\hat{x}\equiv x^\mu\sigma_\mu = \begin{pmatrix} x^3 & x^1-ix^2 \\ x^1+ix^2 & -x^3 \end{pmatrix}.$$

This matrix transforms according to $(1/2,1/2)$ as follows:

$$\hat{x}\to\hat{x}'=\Omega_L\,\hat{x}\,\Omega_L^\dagger\,,$$

with $\Omega_L\in SL(2,\mathbb{C})$ an element of the $(1/2,0)$ representation, and $\Omega_L^\dagger$ an element of the $(0,1/2)$ representation that corresponds to the *same* group parameters. Since σ_μ is a basis for the vector space of 2×2 matrices, then $\hat{x}'=x'^\mu\sigma_\mu$. It is possible to show that

$$x'^\mu=\Lambda^\mu_\nu x^\nu\,,$$

where Λ^μ_ν is a Lorentz transformation. Since there are two matrices of $SL(2,\mathbb{C})$ that give rise to the same Lorentz transformation, $SL(2,\mathbb{C})$ is the double cover of the Lorentz group.
Note that one can also associate to a four-vector x^μ a 2×2 matrix using $\bar{\sigma}_\mu$. In this case

$$\hat{x} \to \hat{x}' = \Omega_R \, \hat{x} \, \Omega_R^\dagger = x'^\mu \bar{\sigma}_\mu \,,$$

with Ω_R an element of the $(0,1/2)$ representation. Again, x'^μ and x^μ are related by a Lorentz transformation.
Objects transforming with respect to the $(1/2,1/2)$ representations are called *vectors*. Example: the electromagnetic field A^μ.

Higher-spin representations: tensors. A *tensor* $T^{\mu_1,\dots,\mu_m}_{\nu_1,\dots,\nu_n}$ of rank (m,n) is an object transforming according to

$$T^{\mu_1,\dots,\mu_m}_{\nu_1,\dots,\nu_n} \to (T')^{\mu_1,\dots,\mu_m}_{\nu_1,\dots,\nu_n} = \Lambda^{\mu_1}_{\alpha_1} \cdots \Lambda^{\mu_m}_{\alpha_m} \Lambda^{\beta_1}_{\nu_1} \cdots \Lambda^{\beta_n}_{\nu_n} T^{\alpha_1,\dots,\alpha_m}_{\beta_1,\dots,\beta_n} \,.$$

For instance, Λ^μ_ν is a $(1,1)$ tensor.
Higher-spin representations are constructed by taking suitable tensor products of the representations we have seen so far, and then decomposing the resulting representation into irreducible representations using the same method used for $SU(2)$ representations.
To discuss the first example, we need the concept of *dual* tensor. Given the rank-$(2,0)$ tensor, $T^{\mu\nu}$, its dual $\tilde{T}^{\mu\nu}$ is obtained from

$$\tilde{T}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} T_{\alpha\beta} \,,$$

where $\epsilon^{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric symbol in four dimensions with $\epsilon_{0123} = +1$.
The first higher-spin representation we can construct is

$$(1/2,0) \otimes (1/2,0) = (0,0) \oplus (1,0) \,.$$

Objects transforming according to $(0,0)$ are scalars. Objects transforming according to $(1,0)$ are spin-1 antisymmetric self-dual tensors $T^{\mu\nu}$, i.e.

$$T^{\nu\mu} = -T^{\mu\nu} \,, \quad \text{and} \quad \tilde{T}^{\mu\nu} = T^{\mu\nu} \,.$$

$$\text{Dimension of } T^{\mu\nu} = \underbrace{6}_{\text{antisymmetri}} - \underbrace{3}_{\text{self-dual}} = 3 = \dim(1,0) \,.$$

Siimilarly

$$(0,1/2) \otimes (0,1/2) = (0,0) \oplus (0,1) \,.$$

Objects transforming according to $(0,1)$ are spin-1 antisymmetric anti-self-dual tensors $T^{\mu\nu}$, i.e.

$$T^{\nu\mu} = -T^{\mu\nu} \,, \quad \text{and} \quad \tilde{T}^{\mu\nu} = -T^{\mu\nu} \,.$$

Every rank-2 tensor $T^{\mu\nu}$ can be constructed by taking the direct product of two vector representations, as follows

$$\begin{aligned}
(1/2,1/2) \otimes (1/2,1/2) &= [(1/2,0) \otimes (0,1/2)] \otimes [(1/2,0) \otimes (0,1/2)] \\
&= [(1/2,0) \otimes (1/2,0)] \otimes [(0,1/2) \otimes (0,1/2)] \\
&= [(0,0) \oplus (1,0)] \otimes [(0,0) \oplus (0,1)] \\
&= (0,0) \oplus (1,0) \oplus (0,1) \oplus (1,1) \,.
\end{aligned}$$

Every rank-2 tensor $T^{\mu\nu}$ contains a spin-0 part $(0,0)$, a spin-1 part $(1,0)\oplus(0,1)$ and a spin-2 part $(1,1)$, corresponding to the decomposition

$$T^{\mu\nu} = a\,\eta^{\mu\nu} + A^{\mu\nu} + S^{\mu\nu}\,,$$

$$a = \frac{1}{4}T^{\mu}_{\ \mu}\,, \qquad \underbrace{A^{\mu\nu} = -A^{\nu\mu}}_{\text{e.g. electromagnetic tensor } F^{\mu\nu}}\,, \quad \underbrace{S^{\mu\nu} = S^{\nu\mu}\,,\ \text{and } S^{\mu}_{\ \mu} = 0}_{\text{e.g. stress-energy tensor } \Theta^{\mu\nu}}\,.$$

$$\text{dimensions of } T^{\mu\nu} = 4\times 4 = 1 + \underbrace{\frac{4(4-1)}{2}}_{\text{anti-symmetric}} + \underbrace{\frac{4(4+1)}{2} - 1}_{\text{symmetric traceless}} = \underbrace{1}_{\dim(0,0)} + \underbrace{6}_{\dim[(1,0)\oplus(0,1)]} + \underbrace{9}_{\dim(1,1)}$$