2 Lie groups and Lie algebras

2.1 Lie groups

In physics, we are often interested in transformations that depend in a smooth way on a number of parameters α_a, $a = 1, 2, \ldots, N$.
Example: rotations in 3D depend smoothy on the three Euler angles α, β, γ, i.e. $R(\alpha, \beta, \gamma) = R(\theta_i), i = 1, 2, 3$.
Let us call $\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $g(\alpha)$ a generic element of one of such groups. Then, we can set

$$g(\alpha)|_{\alpha=0} = e\,.$$

If the group elements depend smoothly on the parameters, we can perform a Taylor expansion around $\alpha = 0$, and obtain

$$g(\alpha) = \underbrace{g(0)}_{=e} + \underbrace{\frac{\partial}{\partial \alpha_a} g(\alpha)|_{\alpha=0}}_{\equiv i X_a} \alpha_a + \ldots$$

X_a are N linear operators called the *generators* of the group
A *Lie group* G is a group which depends smoothly on some (real) parameters, i.e.

(LG1) its elements form a smooth (∞-differentiable) manifold

(LG2) group multiplication (in fact $g(\alpha)g^{-1}(\beta)$) is a smooth function $G \times G \to G$

A representation D of a Lie group G is a *smooth* map $D : G \to L(V)$, with $L(V)$ the set of linear transformations over a vector space V.
If D is a representation of al Lie group G, we denote $D(g(\alpha)) \equiv D(\alpha)$, with $D(0) = \mathbb{1}$.

Theorem 2.1 *For a Lie group, the representation $D(\alpha)$ of an element $g(\alpha)$ continuously connected to the identity can be written as*

$$D(\alpha) = \exp\left[i \alpha_a X_a\right]\,, \qquad a = 1, 2, \ldots, N\,,$$

where α_a are real parameters and X_a are linearly independent matrices. The dimension of V is called the dimension *of the representation.*

The linear operators X_a are called the *generators* of the representation D. They span a vector space, called the *Lie algebra* of the group (in the representation D). Note that, if $D(\alpha)$ is unitary, then X_a are hermitian operators (see Problem Sheet 2).

2.2 Lie algebra of the generators

Consider two group elements g_1 and g_2. Their product is still a group element:

$$\begin{aligned} g_1 &= \exp\left(i\alpha_a^1 X_a\right) \\ g_2 &= \exp\left(i\alpha_a^2 X_a\right) \end{aligned} \implies g_1 g_2 \equiv g_3 = \exp\left[i\gamma_a X_a\right] .$$

Since, in general, given two operators u, v such that their commutator $[u, v]z \equiv uv - vu \neq 0$, we have

$$\exp[u]\exp[v] = \exp\left(u + v + \frac{1}{2}[u, v] + \dots\right) ,$$

we have

$$g_1 g_2 \neq \exp\left(i(\alpha_a^1 + \alpha_a^2)X_a\right) .$$

However, let us consider $u \equiv i\alpha_a^1 X_a$ and $v \equiv i\alpha_a^2 X_a$, and the product

$$\underbrace{\exp[u]}_{=g_1}\underbrace{\exp[v]}_{=g_2}\underbrace{\exp[-u]}_{=g_1^{-1}}\underbrace{\exp[-v]}_{=g_2^{-1}} = \exp\left([u, v] + \dots\right) = \exp(w) , \quad w \equiv i\beta_a X_a .$$

For infinitesimal transformations, $u, v \to 0$, we can expand the exponentials, and get

$$(1 + i\alpha_a^1 X_a)(1 + i\alpha_b^2 X_b)(1 - i\alpha_a^1 X_a)(1 - i\alpha_b^2 X_b) + \dots = 1 + i\beta_c X_c .$$

This implies

$$-\alpha_a^1 \alpha_b^2 [X_a, X_b] = i\beta_c T_c ,$$

which means that the commutator $[X_a, X_b]$ is a linear combination of the generators X_a, conventionally cast in the form

$$[X_a, X_b] = i f_{abc} X_c .$$

The numbers f_{abc} are called the *structure constants* of the Lie algebra. By construction, they are independent of the representation of the Lie group. We have then proven that the generators X_a in any representation form a vector space that is closed under commutation

(LA1) $[X_a, X_b] = i f_{abc} X_c$

From a direct calculation, one can also show that

(LA2) $\underbrace{[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]]}_{\text{cyclic permutations}} = 0$

A vector space with a product satisfying (LA1) and (LA2) is called a *Lie Algebra*

From the algebraic structure of the generators we infer the following properties for f_{abc}:

(LA1) $\implies$ $f_{bac} = -f_{abc}$ (antisymmetric in a and b)

(LA2) $\implies$ $f_{ade} f_{bcd} + f_{cde} f_{abd} + f_{bde} f_{cad} = 0$

Note that the structure constans and the exponential map are enough to ontain the full structure of a Lie group, even beyond infinitesimal transformations. Also, if a Lie group admits a unitary representation, f_{abc} are real (see Problem Sheet 2)

2.3 Adjoint representation

The generators of the adjoint representation are defined by

$$(T_a)_{bc} = -if_{abc} = if_{bac} \,.$$

Due to the Jacobi identity for f_{abc}, we have $[T_a, T_b] = if_{abc}T_c$, which means that indeed T_a generate a representation of a Lie group.
The dimension of the adjoint representation is the number of generators, hence the number of *real* parameters needed to uniquely specify a group element.
Consider the real, symmetric matrix

$$g_{ab} \equiv \mathrm{Tr}\,(T_a T_b) \ .$$

This matrix can be diagonalised and recast in the form

$$g_{ab} = k\,\mathrm{diag}(\underbrace{+1,\ldots,+1}_{m},\underbrace{-1,\ldots,-1}_{N-m}) \,.$$

If $m = n$, the Lie algebra is said to be *compact*. A Lie group generated by a compact Lie algebra is said to be a *compact* Lie group.
In a compact Lie algebra, $\mathrm{Tr}\,(T_a T_b) = \lambda\delta_{ab}$. In this case, we have

$$[T_a, T_b] = if_{abc}T_c \implies \mathrm{Tr}\,([T_a, T_b]T_c) = if_{abd}\mathrm{Tr}\,(T_d T_c) = i\lambda f_{abc} \implies f_{abc} = -i\lambda^{-1}\mathrm{Tr}\,([T_a, T_b]T_c) \ .$$

This in turn implies that, for compact Lie algebras, f_{abc} is totally antisymmetric. In fact

$$\begin{aligned}\mathrm{Tr}\,([T_a, T_b]T_c) &= \mathrm{Tr}(T_a T_b T_c) - \mathrm{Tr}(T_b T_a T_c)\\ &= \mathrm{Tr}(T_b T_c T_a) - \mathrm{Tr}(T_C T_b T_a) = \mathrm{Tr}\,([T_b, T_c]T_a)\end{aligned}$$

This implies $f_{abc} = f_{bca} \implies f_{abc}$ is antisymmetric also in a, b.
Note that, if f_{abc} are real, then the adjoint representation is a real representation. Therefore, if it is unitary, is also orthogonal.

2.4 Examples of Lie groups and their algebras

General linear group GL($\mathbf{n}, \mathbb{K}$). This is the group of invertible linear transformations on $\mathbb{K}^n$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. These are all $n \times n$ matrices with entries in $\mathbb{K}$ with determinant different from zero.

real parameters:	n^2	$GL(n, \mathbb{R})$
	$2n^2$	$GL(n, \mathbb{C})$

Dilatations. $x \mapsto \lambda x$, with $x \in \mathbb{K}^n, \lambda \in \mathbb{K}$

parameters:	1	$GL(1, \mathbb{R})$
	2	$GL(1, \mathbb{C})$

Dilatations form a subgroup of $GL(n, \mathbb{K})$.

Special linear group SL(n, $\mathbb{K}$). These are all $n\times n$ matrices with entries in $\mathbb{K}$ with determinant equal to one. By construction $SL(n,\mathbb{K}) \subset GL(n,\mathbb{K})$.

$$\text{parameters:}\quad \underbrace{n^2}_{\text{real entries}} - \underbrace{1}_{\det M=1} \qquad SL(n,\mathbb{R})$$
$$\underbrace{2n^2}_{\text{real entries}} - \underbrace{2 = 2(n^2-1)}_{\det M=1+i\cdot 0} \qquad SL(n,\mathbb{C})$$

Orthogonal groups. Let M be an orthogonal transformation for a scalar product $g:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$. Let us fix an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ such that the matrix $g_{ij} = g(e_i,e_j)$ is $g = \text{diag}(\underbrace{+1,\dots,+1}_{s},\underbrace{-1,\dots,-1}_{n-s})$. The orthogonal group on such space is a Lie group denoted by $O(s,n-s)$. We can associate to M the matrix $M_{ij} = g(e_i, Me_j)$, that has the property $M^TgM = g$. This fixes all the elements on the diagonal and above the diagonal of M, thus giving $n(n+1)/2$ constraints. Therefore

$$\text{parameters:}\quad \underbrace{n^2}_{\text{real entries}} - \underbrace{\frac{n(n+1)}{2}}_{M^TgM=g} = \frac{n(n-1)}{2}\,.$$

Examples:
$O(n)$: group of isometries in n dimensions ($M^TM = \mathbb{1}$)
$O(3,1)$: Lorentz group

Special orthogonal groups. Note that $M^TgM = g \implies (\det M)^2 = 1$. This implies $\det M = \pm1$, hence orthogonal groups are divided into two connected components according to the value of $\det M$. The component that contains the identity has $\det M = 1$, and is called *special orthogonal group $SO(s,n-s)$*.

$$\text{parameters:}\quad \frac{n(n-1)}{2}\,.$$

Example: $SO(n)$ is the group of rotations in n dimensions. Note that in three dimensions , $SO(3)$ has $3\times 2/2 = 3$ parameters, which are in fact the three Euler's angles.

Unitary groups. Consider a unitary operator M with respect to a product $g:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$. The groups of unitary operators in n dimensions is denoted by $U(n)$. Recall that, if we fix an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ such that $g(e_i,e_j) = \delta_{ij}$, and we associate to M a matrix $M_{ij} = g(e_i, Me_j)$, we have $M^\dagger M = \mathbb{1}$. This condition gives n^2 real conditions (see Problem Sheet 2).

$$\text{parameters:}\quad \underbrace{2n^2}_{\text{real entries}} - \underbrace{n^2}_{M^\dagger M=\mathbb{1}} = n^2\,.$$

Also,

$$M^\dagger M = \mathbb{1} \implies |\det M| = 1 \implies \det M = e^{i\phi}\,.$$

The operators with $\det M = 1$ form the *special unitary group $SU(n)$*.

$$\text{parameters:}\quad \underbrace{n^2}_{\text{parameters of } U(n)} - \underbrace{1}_{\det M=1}$$

2.5 The group SU(2)

The group $SU(2)$ is made up of 2×2 complex matrices with unit determinant.

Fundamental representation. Consider the Pauli matrices σ_a, $a = 1, 2, 3$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From them construct $\tau_a \equiv \sigma_a/2$. The matrix

$$U(\vec{\theta}) = \exp[i\theta_a \tau_a] = \cos(\theta/2) + i \sin(\theta/2)\hat{\theta}_a \sigma_a , \quad \theta \equiv |\vec{\theta}| , \quad \hat{\theta} = \vec{\theta}/\theta .$$

is an element of $SU(2)$. In fact, since τ_a are hermitian, U is unitary. Also, since τ_a are traceless

$$\det U = \det\left(\exp[i\theta_a \tau_a]\right) = \exp[\mathrm{Tr}(i\theta_a \tau_a)] = 1 .$$

So the τ_a are the generators of the representation that defines the group, which is called the *fundamental* representation. The Lie algebra of the group can be obtained from an explicit calculation:

$$[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c , \qquad a, b, c \in \{1, 2, 3\} ,$$

where ϵ_{abc} is the Levi-Civita symbol, defined by

$$\epsilon_{abc} = \begin{cases} +1 & \text{cyclic permuations} \qquad \epsilon_{123} = \epsilon_{312} = \epsilon_{231} = +1 \\ -1 & \text{anti-cyclic permutations} \quad \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1 \end{cases}$$

Adjoint representation. The generators are $(T_a)_{bc} = -i\epsilon_{abc}$:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the adjoint representation is real, $\exp[i\theta_a T_a]$ is not only a unitary matrix, but it is also an orthogonal matrix with unit determinant, so it is a member of $SO(3)$. Therefore, the adjoint representation of $SU(2)$ is the fundamental representation of $SO(3)$. This is the group of rotations in three dimensions, parameterised in terms of the Euler's angles as

$$\Omega(\alpha, \beta, \gamma) = \exp[i\alpha T_3] \exp[i\beta T_2] \exp[i\gamma T_3] .$$

Once we have the adjoint representation, we compute

$$\mathrm{Tr}(T_a T_b) = 2\delta_{ab} .$$

Therefore, the Lie algebra of $SU(2)$ is compact.

Relation between $SU(2)$ and $SO(3)$. Since the Lie algebra of the adjoint representation of $SU(2)$ is the Lie algebra of $SO(3)$, we can establish a one-to-one correspondence (isomorphism)

$$\text{Lie algebra of } SU(2) \approx \text{Lie algebra of } SO(3)\,.$$

Is there also an isomorphism between $SU(2)$ and $SO(3)$? The answer is no. Heuristically, if we interpret θ as a rotation angle, for a rotation of 2π around an axis, say the z-axis, we have

$$\exp[i(2\pi)T_3] = \mathbb{1}\,, \qquad \exp[i(2\pi)\tau_3] = -\mathbb{1}\,,$$

so not all representations of $SU(2)$ are also representations of $SO(3)$. More precisely, consider $\vec{v} \in \mathbb{R}^3 = (v_1, v_2, v_3)$, and construct the 2×2 matrix $v_a\sigma_a$. Then, $\forall U \in SU(2)$, we have (see Problem Sheet 2)

$$U(v_a\sigma_a)U^{-1} = [\Omega\vec{v}]_a\sigma_a\,,$$

with $\Omega \in SO(3)$. Note that this is not a one-to-one correspondence, because if $U \in SU(2)$ corresponds to $\Omega \in SO(3)$, so does $-U$. In mathematical terms, we say that $SU(2)$ is the *double cover* of $SO(3)$.

2.6 Finite-dimensional representations of SU(2)

We now want to characterise all the finite-dimensional unitary representations of $SU(2)$. We consider then the generators J_a of a finite-dimensional representation. They satisfy of course the abstract $SU(2)$ algebra $[J_a, J_b] = i\epsilon_{abc}J_c$. Each operator J_a is hermitian, so we can diagonalise the operator J_3.
We then define "raising" and "lowering" operators as follows

$$J_\pm = \frac{1}{\sqrt{2}}(J_1 + iJ_2) \implies [J_3, J_\pm] = \pm J_\pm\,,\ [J_+, J_-] = J_3\,,\ J_- = (J_+)^\dagger$$

Now, if $J_3|m\rangle = m|m\rangle$, we have

$$J_3J_\pm|m\rangle = [J_3, J_\pm]|m\rangle + J_\pm J_3|m\rangle = (m \pm 1)J_\pm|m\rangle\,.$$

Therfore, $J_\pm|m\rangle \propto |m \pm 1\rangle$.
We consider the state $|j\rangle$ corresponding to the highest eigenvalue j of J_3. By construction $J_+|j\rangle = 0$. Applying J_+ to this state we obtain

$$J_-|j\rangle = N_j|j-1\rangle\,.$$

Squaring the above we can compute the normalisation factor N_j, as follows:

$$\langle j|J_+J_-|j\rangle = \langle j|[J_+, J_-]|j\rangle = \langle j|J_3|j\rangle = j = N_j^2 \implies N_j = \sqrt{j}$$

This allows us to show that

$$J_+|j-1\rangle = N_j|j\rangle$$

as follows:

$$J_+|j-1\rangle = \frac{1}{N_j} J_+ J_- |j\rangle = \frac{1}{N_j}[J_+, J_-]|j\rangle = \frac{1}{N_j} J_3|j\rangle = \frac{j}{N_j}|j\rangle = N_j|j\rangle$$

Applying J_- to $|j-1\rangle$ gives

$$J_-|j-1\rangle = N_{j-1}|j-2\rangle\,,$$

with a coefficient N_{j-1} that can be determined by squaring the above equation, as follows:

$$N_{j-1}^2 = \langle j-1|J_+J_-|j-1\rangle = \langle j-1|J_-J_+ + [J_+, J_-]|j-1\rangle = N_j^2 + \langle j-1|J_3|j-1\rangle = N_j^2 + j - 1\,.$$

This gives the relation

$$N_{j-1}^2 - N_j^2 = j - 1\,.$$

Also, if we apply J_+ to $|j-2\rangle$ we obtain

$$\begin{aligned} J_+|j-2\rangle &= \frac{1}{N_{j-1}} J_+J_-|j-1\rangle = \frac{1}{N_{j-1}}(J_-J_+ + [J_+, J_-])|j-1\rangle = \frac{1}{N_{j-1}}\Big(J_-N_j|j\rangle + J_3|j-1\rangle\Big) \\ &= \frac{1}{N_{j-1}}\underbrace{(N_j^2 + j - 1)}_{=N_{j-1}^2}|j-1\rangle = N_{j-1}|j-1\rangle\,. \end{aligned}$$

We now prove by induction that, for every non-negative integer k, once we define

$$J_-|j-k\rangle = N_{j-k}|j-k-1\rangle\,,$$

we have

$$N_{j-k}^2 - N_{j-k+1}^2 = j - k\,,$$

and

$$J_+|j-k-1\rangle = N_{j-k}|j-k\rangle\,.$$

We have already shown both equations to hold for $k = 1$. Assuming they hold for $k-1$ we show they hold for k. In fact, squaring $J_-|j-k\rangle$, we obtain

$$N_{j-k}^2 = \langle j-k|J_+J_-|j-k\rangle = \langle j-k|J_-J_+ + J_3|j-k\rangle\,.$$

Using the induction hypothesis $J_+|j-k\rangle = N_{j-k+1}|j-k+1\rangle$, we have

$$\langle j-k|J_-J_+|j-k\rangle = N_{j-k+1}^2\,.$$

This gives

$$N_{j-k}^2 = N_{j-k+1}^2 + j - k\,,$$

which is the first of the two relations we needed to show. Now, applying J_- to $|j-k-1\rangle$ gives

$$J_+|j-k-1\rangle = \frac{1}{N_{j-k}} J_+J_-|j-k\rangle = \frac{1}{N_{j-k}}(J_-J_+ + [J_+, J_-])|j-k\rangle = \frac{1}{N_{j-k}}(J_-J_+ + J_3)|j-k\rangle\,.$$

Using the induction hypothesis, we obtain

$$J_- J_+ |j-k\rangle = N_{j-k+1} J_- |j-k+1\rangle = N^2_{j-k+1} |j-k\rangle \,.$$

This gives

$$J_+|j-k-1\rangle = \frac{1}{N_{j-k}} (\underbrace{N^2_{j-k+1} + j - k}_{=N^2_{j-k}}) |j-k\rangle = N_{j-k} |j-k\rangle \,.$$

In the end, for each k, we have the two relations

$$\begin{aligned} J_-|j-k\rangle &= N_{j-k}|j-k-1\rangle \\ J_+|j-k-1\rangle &= N_{j-k}|j-k\rangle \,, \end{aligned}$$

as well as the recursion relation

$$N^2_{j-k} - N^2_{j-k+1} = j - k \,.$$

We can now solve the above recursion relation by adding all the rows of the following table

$$\begin{array}{cccccc} N^2_j & & & = & & j \\ N^2_{j-1} & - & N^2_j & = & & j-1 \\ N^2_{j-2} & - & N^2_{j-1} & = & & j-2 \\ \vdots & & \vdots & & & \vdots \\ N^2_{j-k} & - & N^2_{j-k+1} & = & & j-k \\ \hline N^2_{j-k} & & & = & (k+1)j - \frac{k(k-1)}{2} = \frac{1}{2}(k+1)(2j-k) \,. & \end{array}$$

If the representation is finite dimensional, $\exists l : J_-|j-l\rangle = 0$. But this implies

$$\langle j-l|J_+ J_-|j-l\rangle = N^2_{j-k} = \frac{1}{2}(l+1)(2j-l) = 0 \implies l = 2j \,.$$

But this implies $j = l/2$ for some *integer l*.
In conclusion, finite dimensional representations of $SU(2)$ are labelled by the highest egeinvalue j of J_3, which is either integer of half integer. The eigenvalues of J_3 satisfy the relations

$$\begin{aligned} J_3|m\rangle &= m|m\rangle \,, \quad \text{with } -j \le m \le j \quad \text{(dimension } 2j+1) \\ J_-|m\rangle &= N_m|m\rangle \,, \quad \text{with } N_m = \sqrt{\frac{1}{2}(j+m)(j-m+1)} \end{aligned}$$

- According to the value of j, a Hilbert space can be split into orthogonal subspaces having a definite value of j, with
$$\langle m_1, j_1|m_2, j_2\rangle \sim \delta_{m_1 m_2} \delta_{j_1 j_2} \,.$$

- The operator $\vec{J}^2 \equiv J_a J_a = J_+ J_- + J_- J_+ + J_3^2 = 2J_- J_+ + J_3 + J_3^2$ commutes with all J_a, so its eigenvalue J^2 is the same in each representation. This eigenvalue can be computed by applying $\vec{J}^2$ to the state with the highest eigenvalue of J_3, as follows:

$$\vec{J}^2 |j\rangle = j(j+1)|j\rangle \implies J^2 = j(j+1)\,.$$

  An operators that commutes with all the generators of the representation is called the "Casimir" operator. Its eigenvalues can then be used to label the representation.