1 Groups and their representations

1.1 Groups

A group is a set of elements $G = \{a, b, \dots\}$ with a binary operation (e.g. multiplication)

$$\begin{aligned} \cdot : G \times G &\to G \\ (a, b) &\mapsto a \cdot b \equiv ab \in G \text{ (closure)} \end{aligned}$$

with the following properties

(G1) associativity: $\forall a, b, c \in G$, $a(bc) = (ab)c$;

(G2) unit element: $\exists e \in G : ea = ae = a, \forall a \in G$;

(G3) inverse: $\forall a \in G, \exists a^{-1} \in G : aa^{-1} = a^{-1}a = e$.

In general, multiplication is *not* commutative, i.e. $ab \neq ba$.
If $ab = ba$, $\forall a, b \in G$, then G is called an *Abelian* group.
If $ab \neq ba$, $\forall a, b \in G$, then G is called a *non-Abelian* group.
The number of the elements of a group G is called the *order* of G.

Examples of groups.

- The set with one element $\{e\}$ under multiplication.
- The set of integers $\mathbb{Z}$ under ordinary addition.
- Positive rational numbers under ordinary multiplication.
- S_n, the permutations of n objects, under composition.

$$\begin{aligned} \sigma \in S_n \ : \{1, 2, \dots, n\} &\to \{1, 2, \dots, n\} \\ \{1, 2, \dots, n\} &\mapsto \{\sigma(1), \sigma(2), \dots, \sigma(n)\} \\ \forall i \in \{1, 2, \dots, n\}, (\sigma_1\sigma_2)(i) &\equiv \sigma_1(\sigma_2(i)) \end{aligned}$$

  The order of S_n is $n!$.

- Translations in D-dimensional Euclidean space, under composition.
- Rotations in D-dimensional Euclidean space, under composition.
- $\mathbb{Z}_2 = e, a$ (order 2), with $a^2 = e$ (example: mirror symmetry, rotation of π around an axis).

Counter-examples

- $\mathbb{Z}$ is not a group under multiplication ($n^{-1} \notin \mathbb{Z}$ for $n \neq 1$).
- $\mathbb{R}$ is not a group under multiplication: 0 has no inverse (see Problem Sheet 1 to understand why it has to be so)

Note that $\mathbb{R}/\{0\}$ is a group under multiplication.

1.2 Vector spaces and linear operators

In physics, we typically deal with *linear transformations* on *vector spaces*.

Fields. A set $\mathbb{K}$ with two operations $(+, \cdot)$ is a *field* iff

(F1) $(\mathbb{K}, +)$ is an Abelian group ($e_+ \equiv 0$);

(F2) $(\mathbb{K}/\{0\}, \cdot)$ is an Abelian group ($e_\cdot \equiv 1$);

(F3) $\cdot$ is distributive with respect to $+$: $\forall a, b, c \in \mathbb{K},\ (a + b)c = ab + bc$.

Examples: $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers).

Vector spaces. A set V with two operations $(+, \cdot)$ is a *vector space* over a field $\mathbb{K}$ iff

(V1) "addition" $+ : V \times V \to V$

$(V, +)$ is an Abelian group;

(V2) "multiplication by a scalar" $\cdot : V \times \mathbb{K} \to V$ satisfying

(i) $a(u + v) = au + av$, $\forall a \in \mathbb{K}$ and $\forall u, v \in V$;

(ii) $(a + b)u = au + bu$, $\forall a, b \in \mathbb{K}$ and $\forall u \in V$;

(iii) $a(bu) = (ab)u$, $\forall a, b \in \mathbb{K}$ and $\forall u \in V$;

(iv) $1 \cdot u = u$, $\forall u \in V$.

Examples: ordinary vectors in D-dimensional Euclidean space, wave functions in Quantum Mechanics (QM).
Note: in physics we *always* consider vector spaces on $\mathbb{R}$ or $\mathbb{C}$, because they are complete sets (see Analysis 1 for definition of completeness).
A *basis* $\{e_i\}$ of a vector space V is a set of linearly independent vectors that span the whole vector space, i.e.

$$\forall u \in V, \qquad u = \sum_{i=1}^{n} u_i e_i \qquad \text{(span } V\text{)}$$

$$\sum_{i=1}^{n} u_i e_i = 0 \Leftrightarrow c_i = 0\,, i = 1, 2, \ldots, n \qquad \text{(linearly independent)}$$

The number of elements of a basis is called the *dimension* of the vector space. The numbers $\{u_1, u_2, \ldots, u_n\}$ are called the "components" of the vector u in the basis $\{e_i\}$.
Note that vector spaces can have both finite and infinite dimension. Ordinary Euclidean vectors are a finite-dimensional vector space, wave functions in QM are not.

Linear transformations. A transformation (or map) $T : V \to V$ on a vector space V on a field $\mathbb{K}$ is *linear* iff

$$\forall u, v \in V,\ \forall a, b \in \mathbb{K}, \quad T(au + bv) = aT(u) + bT(v)\,.$$

Consider a finite-dimensional vector space V of dimension n. Given a basis $\{e_i\}_{i=1,2,\ldots,n}$, any linear transformation T is represented by a matrix D_{ij}, as follows

$$T(e_j) = \sum_{i=1}^{n} D_{ij}\, e_i \equiv D_{ij}\, e_i \qquad \text{(Einstein's notation)}$$

In this representation, the j-th column of D_{ij} contains the components of $T(e_j)$. For any vector $u \in V$, we have

$$u = u_j e_j \implies T(u) = u_j T(e_j) = u_j (D_{ij} e_i) = v_i e_i \implies v_i = D_{ij} u_j\,.$$

Note: a linear transformation is the generalisation of the multiplication by a scalar, so it is customary to use the short-hand notation $Tu \equiv T(u)$.

1.3 Representations of a group

A representation of a group G is a map $D : G \to L(V)$, where $L(V)$ is the set of linear operators on a vector space V. The map D satisfies the properties:

(R1) $D(e) = \mathbb{1}$, where e is the identity of G, and $\mathbb{1}$ the identity of $L(V)$;

(R2) $\forall g_1, g_2 \in G$, $D(g_1 g_2) = D(g_1) D(g_2)$.

Example.

$$\begin{aligned} D :&(\mathbb{Z}, +) \to L(\mathbb{C}) = (\mathbb{C}, \cdot) \\ &n \mapsto D(n) = e^{in\theta}, \quad \theta \in \mathbb{R} \end{aligned}$$

(R1) $D(0) = e^{i0\theta} = 1$;

(R2) $\forall n, m \in \mathbb{Z}\ D(n + m) = e^{i(n+m)\theta} = e^{in\theta} e^{im\theta} = D(n) D(m)$.

If V is finite-dimensional, we can express $D(g)$ in matrix form. In a basis $\{e_i\}_{i=1,2,\ldots,n}$ we have

$$D(g) e_j = [D(g)]_{ij} e_i$$

Two representations D_1 and D_2 are called *equivalent* if they can be related by a similarity transformation, i.e.

$$\exists S \in L(V) : D_1(g) = S D_2(g) S^{-1}, \quad \forall g \in G.$$

Note that the matrix S has to be invertible.
A complex or real representation of order n is a representation D of the group G onto the sets $GL(n, \mathbb{C})$ or $GL(n, \mathbb{R})$ of non-singular $n \times n$ complex or real matrices. The matrix $D(g)$ has to be invertible because, if D is a representation of a group G, then $D(g^{-1}) = D(g)^{-1}$, $\forall g \in G$. In fact, from (R1)

$$D(gg^{-1}) = D(e) = \mathbb{1}.$$

But, from (R2), we have

$$D(gg^{-1}) = D(g)D(g^{-1}) = \mathbb{1} \implies D(g^{-1}) = D(g)^{-1}.$$

Example. The group $\mathbb{Z}_3$ is a finite group with three elements, $\mathbb{Z}_3 = \{e, a, b\}$. For finite groups we can construct the so-called multiplication table, e.g.

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

One representation of $\mathbb{Z}_3$ in $(\mathbb{C}, \cdot)$ is given by the roots of $z^{1/3} = 1$, i.e. $\{1, e^{\frac{2}{3}\pi i}, e^{\frac{4}{3}\pi i}\}$. A representation of $\mathbb{Z}_3$ in $GL(3, \mathbb{C})$ can be found in Problem Sheet 1.

1.4 Scalar products on real vector spaces

Consider a vector space V on $\mathbb{R}$. A scalar product is a map $g : V \times V \to V$ that is

(S1) bilinear: $\forall u, v, w \in V$, and $\forall a, b \in \mathbb{R}$:

$$g(u, av + bw) = ag(u, v) + bg(u, w),$$
$$g(au + bv, w) = ag(u, w) + bg(v, w);$$

(S2) symmetric: $\forall u, v \in V$, $g(v, u) = g(u, v)$.

Examples.

- Scalar products of vectors in three-dimensional Euclidean space $\mathbb{R}^3 \equiv \mathbb{R} \times \mathbb{R} \times \mathbb{R}$:

  $$\forall \vec{u}, \vec{v} \in \mathbb{R}^3, \qquad g(\vec{u}, \vec{v}) \equiv \vec{u} \cdot \vec{v} \equiv |\vec{u}||\vec{v}| \cos\theta.$$

  The above product is strictly positive, i.e.

  (S3) $\forall u \in V, g(u, u) \geq 0$, and $g(u, u) = 0 \implies u = 0$.

- Scalar product of vectors in Minkowsky space:

  $$\forall u \equiv (u^0, \vec{u}), v \equiv (v^0, \vec{v}), \qquad g(u, v) \equiv u \cdot v \equiv u^0 v^0 - \vec{u} \cdot \vec{v}.$$

  The above scalar product is not strictly positive, in fact $\forall u = (|\vec{u}|, u)$ we have $u \cdot u = 0$.

Orthogonal vectors. Given a scalar product g, two vectors u, v are said to be *orthogonal* iff $g(u, v) = 0$.
If $\{e_i\}_{i=1,2,\ldots,n}$ is a basis of V, and $u = u_i e_j$ and $v = v_i e_i$, we have

$$g(u, v) = g_{ij} u_i v_j \qquad g_{ij} \equiv g(e_i, e_j)\,.$$

A basis $\{e_i\}_{i=1,2,\ldots,n}$ is said to be orthonormal iff

$$g_{ij} = \pm\delta_{ij}\,, \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Orthogonal transformations. A linear transformation M onto a real vector space V with a scalar product g is said to be *orthogonal* iff $\forall u, v \in V, g(Mu, Mv) = g(u, v)$.
Let us define the matrix $M_{ij} \equiv g(e_i, Me_j)$. If $\{e_i\}_{i=1,2,\ldots,n}$ is orthonormal basis, and M is orthogonal, we have (see Problem Sheet 1)

$$M^T g M = g\,.$$

Using the fact that $g^2 = \mathbb{1}$, we have

$$M^T g M = g \implies g M^T g M = g^2 = \mathbb{1} \implies M^{-1} = g M^T g\,.$$

For a positive scalar product, $g = \mathbb{1}$. This implies that $M^{-1} = M^T$.

1.5 Scalar products on complex vector spaces

Let us consider a vector space $\mathcal{H}$ on $\mathbb{C}$. A hermitian scalar product is a map $g : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that is:

(H1) sesquilinear: $\forall u, v, w \in \mathcal{H}$, and $\forall a, b \in C$, we have

$$g(u, av + bw) = a g(u, v) + b g(u, w)\,,$$
$$g(au + bv, w) = a^* g(u, w) + b^* g(v, w)\,;$$

(H2) hermitian: $\forall u, v \in \mathcal{H}, g(v, u) = g(u, v)^*$.

In physics applications, we always require strict positivity, i.e.

(H3) $\forall u \in \mathcal{H}, g(u, u) \geq 0$, and $g(u, u) = 0 \implies u = 0$.

Example. Wave functions in quantum mechanics.

$$|\psi\rangle \equiv \psi : \mathbb{R}^3 \to \mathbb{C}$$

$$g(|\phi\rangle, |\psi\rangle) \equiv \langle\phi|\psi\rangle \equiv \int_{\mathbb{R}^3} d^3\vec{x}\, \phi^*(\vec{x})\, \psi(\vec{x})$$

This is indeed a scalar product due to the properties of integrals.
A set of vectors $\{|e_i\rangle\}$ is called *orthonormal* if $\langle e_i|e_j\rangle = \delta_{ij}$.
A complex vector space $\mathcal{H}$ with a hermitian scalar product (and the additional property of completeness) is called a *Hilbert space*.

Unitary transformations. Let us consider a a Hilbert space $\mathcal{H}$. A linear transformation (a.k.a. operator) U on $\mathcal{H}$ is unitary iff, $\forall |u\rangle, |v\rangle \in \mathcal{H}$, we have $\langle Uu|Uv\rangle = \langle u|v\rangle$.
The *adjoint* of a linear operator A on $\mathcal{H}$ is the linear operator $A^\dagger$ defined by $\langle u|A^\dagger v\rangle = \langle Au|v\rangle$, $\forall |u\rangle, |v\rangle \in \mathcal{H}$ (with some extra conditions . From the definition $(A^\dagger)^\dagger = A$. A linear operator A is *hermitian* iff $A^\dagger = A$.
If U is a unitary operator, then $U^\dagger = U^{-1}$. In fact, $\forall |u\rangle, |v\rangle \in \mathcal{H}$, we have

$$\langle u|v\rangle = \langle Uu|Uv\rangle = \langle u|U^\dagger U v\rangle \implies U^\dagger U = \mathbb{1} \implies U^\dagger = U^{-1} .$$

If $\mathcal{H}$ has finite dimension n, we can construct an orthonormal basis $\{|e_i\rangle\}_{i=1,2,\ldots,n}$. Then we can associate to each operator A a matrix $A_{ij} = \langle e_i|Ae_j\rangle$, such that $\forall |u\rangle = u_i|e_i\rangle \in \mathcal{H}$, we have

$$|v\rangle \equiv A|u\rangle = v_i|e_i\rangle \implies v_i = A_{ij}u_j .$$

Note that, by construction, $(A^\dagger)_{ij} = A^*_{ij}$ (see Problem Sheet 1).

Unitary representations. A representation D of a group G is unitary if, $\forall g \in G$, $D(g)$ is a unitary operator on a Hilbert space.
All representation of $\mathbb{Z}_3$ considered so far are unitary. The one in Problem Sheet 1 maps each element of $\mathbb{Z}_3$ onto an orthogonal matrix on $\mathbb{R}^3$. We have therefore an *orthogonal* representation of $\mathbb{Z}_3$.

Unitary vs anti-unitary operators. Unitary representations are commonly used to model the action of a symmetry on quantum states:

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle .$$

Unitary transformation leave transition amplitudes unchanged, in fact

$$\langle \phi'|\psi'\rangle = \langle U\phi|U\psi\rangle = \langle \phi|\psi\rangle .$$

In quantum mechanics, probabilities are obtained by squaring transition amplitudes, so we can also have

$$\langle U\phi|U\psi\rangle = \langle \phi|\psi\rangle^* .$$

If an operator U satisfies the above relation $\forall |\phi\rangle, |\psi\rangle \in \mathcal{H}$, the operator U is said to be *anti-unitary*. Anti-unitary operators are anti-linear, i.e.

$$U(\alpha|\phi\rangle + \beta|\psi\rangle) = \alpha^* U|\phi\rangle + \beta^* U|\psi\rangle , \qquad \forall |\phi\rangle, |\psi\rangle \in \mathcal{H}, \text{and } \forall \alpha, \beta \in \mathbb{C} .$$

The adjoint of an anti-linear operator is defined through the relation

$$\langle \phi|A^\dagger \psi\rangle = \langle A\phi|\psi\rangle^* = \langle \psi|A\phi\rangle .$$

Note, if U is anti-unitary, $U^\dagger = U^{-1}$ (see Problem Sheet 1).