Symmetry in Particle Physics, Problem Sheet 9 [SOLUTIONS]

1. Consider a vector q transforming to the *fundamental* representation of $SU(N_c)$:

$$q_i \to (q_i)' = U_{ij}\, q_j\,, \qquad U = e^{-i\theta^a t^a}\,, \quad a = 1, \dots N_c^2 - 1\,.$$

The vector $q^\dagger$ transforms according to the *conjugate* representation.

(a) Show that the generators of the conjugate representation are $\bar{t}^a = -(t^a)^T$.

By construction

$$q^\dagger \to q^\dagger U^\dagger \implies q_i^* \to q_j^* (U^\dagger)_{ji} = U_{ij}^* q_j^*$$

Therefore, the column vector $(q^\dagger)^T = q^*$ *transforms according to* U^*. *Hence*

$$U^* = (U^\dagger)T = (U^{-1})^T = \left[e^{i\theta^a t^a}\right]_{ji} = \left[e^{-i\theta^a(-t^a)^T}\right]_{ij}\,.$$

By definition of conjugate representation

$$q_i^* \to \left[e^{-i\theta^a \bar{t}^a}\right]_{ij} q_j^* \implies \bar{t}^a = (-t^a)^T\,.$$

(b) Consider the tensor $T_{ij} = q_i q_j^*$, and decompose it as follows

$$T_{ij} = A_{ij} + S_{ij}\,, \qquad A_{ij} \equiv \frac{1}{3}\delta_{ij}(q_k q_k^*) \qquad S_{ij} \equiv \left[q_i q_j^* - \frac{1}{3}\delta_{ij}(q_k q_k^*)\right]\,.$$

Show that A_{ij} is invariant under $SU(N_c)$, while S_{ij} transforms according to the adjoint representation.

The tensor A_{ij} *can be written in the alternative form*

$$A_{ij} \equiv \frac{1}{3}\delta_{ij}(q^\dagger q) \to \frac{1}{3}\delta_{ij}(q^\dagger U^\dagger U q) = \frac{1}{3}\delta_{ij}(q^\dagger q)\,.$$

We now show that S_{ij} *is an element of the Lie algebra of* $SU(N_c)$. *First we show that* $\mathrm{Tr}(S) = 0$. *In fact*

$$S_{ii} \equiv \left[q_i q_i^* - \frac{1}{3}\delta_{ii}(q_k q_k^*)\right] = 0\,.$$

Then we show that $S^\dagger = S$. *In fact*

$$S_{ji}^* = \left[q_j^* q_i - \frac{1}{3}\delta_{ij}(q_k q_k^*)\right] = S_{ij}\,.$$

Therefore, S_{ij} *transforms according to the adjoint representation.*

The relation we have shown can be interpreted in terms of a direct product as follows

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}\,,$$

where representations are labelled according to their dimensionality.

(c) Let $N_c > 2$ and construct the tensor

$$T_i = \epsilon_{ijk} q_j q_k \,.$$

Using $\det(U) = 1$, show that T_i transforms according to the conjugate representation.

The tensor T_i has three components and transforms as follows

$$T_i \to \epsilon_{ijk} U_{jj'} U_{kk'} q_{j'} q_{k'} \,.$$

Since $\det(U) = 1$, we have

$$\det(U)\epsilon_{123} = \epsilon_{ljk} U_{i1} U_{j2} U_{k3} \implies \epsilon_{ljk} U_{li'} U_{jj'} U_{kk'} = \epsilon_{i'j'k'} \det(U) = \epsilon_{i'j'k'} \,.$$

We now multiply the above equation by $U^*_{ii'}$ and sum over i'. Using

$$\mathbb{1} = UU^\dagger \implies U_{li'} U^\dagger_{i'i} = U_{li'} U^*_{ii'} = \delta_{il} \,.$$

This gives

$$\epsilon_{ijk} U_{jj'} U_{kk'} = U^*_{ii'} \epsilon_{i'j'k'} \,.$$

Therefore, the transformation rule for T_i can be written in the form

$$T'_i \equiv \epsilon_{ijk} U_{jj'} U_{kk'} q_{j'} q_{k'} = U^*_{ii'} \epsilon_{i'j'k'} q_{j'} q_{k'} = U^*_{ii'} T_{i'} \,,$$

which implies that T_i transforms according according to the conjugate representation.

2. For each representation R of a compact Lie group G, consider the *quadratic Casimir* operator $T^2(R) = T^a(R) T^a(R)$.

(a) Show that T^2 commutes with every generator T^a, and hence $T^2(R) = C_R \mathbb{1}_R$, where $\mathbb{1}_R$ is the identity matrix in the vector space spanning representation R.

From an explicit calculation of $[T^2(R), T^a(R)]$, we obtain

$$\begin{aligned}[T^a(R)T^a(R), T^b(R)] &= T^a(R)[T^a(R), T^b(R)] + [T^a(R), T^b(R)]T^a(R) \\ &= i f^{abc}(T^a(R)T^c(R) + T^c(R)T^a(R)) = i f^{abc}\{T^a(R), T^c(R)\} \,.\end{aligned}$$

For a compact Lie group, f^{abc} is antisymmetric in the indexes a, c. Since $\{T^a(R), T^c(R)\}$ is symmetric in a, c, the product of the two gives zero.

(b) Given that the generators of each representation are normalised as follows

$$\mathrm{Tr}[T^a(R)\, T^b(R)] = T_R\, \delta^{ab} \,.$$

where T_R depends on the representation, show that T_R and C_R are related by

$$C_R \dim(R) = T_R \dim(G) \,.$$

where $\dim(G)$ is the dimension of the group.

Let us set $a = b$ in $\mathrm{Tr}[T^a(R)\,T^b(R)]$. We obtain

$$\mathrm{Tr}[T^a(R)\,T^a(R)] = C_R\,\mathrm{Tr}(\mathbb{1}_R) = C_R\,\dim(R) = T_R\,\delta^{aa} = T_R\,\dim(G)\,.$$

(c) Given the normalisation $T_F = 1/2$, derive

$$C_F = \frac{N_c^2 - 1}{2N_c} \qquad C_A = T_A\,.$$

For the fundamental representation, we have

$$C_F\,N_c = T_F\,(N_c^2 - 1) \implies C_F = \frac{N_c^2 - 1}{2N_c}\,.$$

For the adjoint representation, we have

$$C_F\,(N_c^2 - 1) = T_A\,(N_c^2 - 1) \implies C_A = T_A\,.$$

Note that, from the lectures, we know that $C_A = N_c$.

3. Consider the Lagrangian

$$\mathcal{L} = i\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 + i\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 - m\left(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2\right)\,,$$

where $\psi_1(x)$ and $\psi_2(x)$ are two Dirac spinor fields, with $\bar{\psi}_i = \psi_i^\dagger\gamma^0$ $(i = 1, 2)$, and m is a real parameter. We have seen in the lectures that the Lagrangian is invariant under a global $U(1) \times SU(2)$ transformation of the form

$$\psi_i(x) \to e^{-i\alpha}U_{ij}\psi_j(x)\,, \qquad U = \exp\left[-i\,\alpha_a\,\frac{\sigma_a}{2}\right]\,,$$

where $\alpha, \alpha_1, \alpha_2, \alpha_3$ are real constant parameters, and σ_a, $a = 1, 2, 3$ are the three Pauli matrices.

(a) Consider now the following **local** $U(1) \times SU(2)$ transformation

$$\psi_i(x) \to e^{-i\frac{g_1}{2}\alpha(x)}U_{ij}(x)\psi_j(x) \qquad U = \exp\left[-i\,g_2\,\alpha_a(x)\,\frac{\sigma_a}{2}\right]\,,$$

where g_1 and g_2 are constants, whereas $\alpha(x)$ and $\alpha_a(x), a = 1, 2, 3$ are arbitrary functions of the space-time point x. Show that the Lagrangian is not invariant any more under such transformation, and compute the corresponding variation $\delta\mathcal{L}$

The Lagrangian changes as follows

$$\mathcal{L} \to i\bar{\psi}e^{i\frac{g_1}{2}\alpha}U^\dagger\gamma^\mu e^{-i\frac{g_1}{2}\alpha}\left(U\left(1 - i\frac{g_1}{2}(\partial_\mu\alpha)\right) + (\partial_\mu U)\right)\psi = \mathcal{L} + \delta\mathcal{L}\,,$$

where

$$\delta\mathcal{L} = \bar{\psi}\gamma^\mu\left(\frac{g_1}{2}(\partial_\mu\alpha) + iU^\dagger(\partial_\mu U)\right)\psi\,.$$

(b) We can modify the Lagrangian so that it is invariant under a local $U(1)\times SU(2)$ transformation by promoting the ordinary derivative ∂_μ to a covariant derivative D_μ as follows

$$D_\mu = \partial_\mu + i\frac{g_1}{2}B_\mu + i\frac{g_2}{2}(W_a)_\mu\sigma_a\,,$$

where B^μ, and W_i^μ are vector gauge fields. How should B^μ, and W_a^μ transform so that $\mathcal{L}$ is still invariant under local $U(1)\times SU(2)$ transformations?

A suitable transformation has to be such that

$$i\bar{\psi}\gamma^\mu U^\dagger\left(i\frac{g_1}{2}B'_\mu + i\frac{g_2}{2}(W'_a)_\mu\sigma_a\right)U\psi - i\bar{\psi}\gamma^\mu\left(i\frac{g_1}{2}B_\mu + i\frac{g_2}{2}(W_a)_\mu\sigma_a\right)\psi = -\delta\mathcal{L}\,.$$

We then impose

$$-\frac{g_1}{2}\bar{\psi}\gamma^\mu(B'_\mu - B_\mu)\psi = -\frac{g_1}{2}\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi \implies B'_\mu = B_\mu + \partial_\mu\alpha\,,$$

and

$$-\frac{g_2}{2}\bar{\psi}\gamma^\mu\left((W'_a)_\mu U^\dagger\sigma_a U - (W_a)_\mu\sigma_a\right)\psi = -i\bar{\psi}\gamma^\mu U^\dagger(\partial_\mu U)\psi\,,$$

which implies

$$(W'_a)_\mu U^\dagger\frac{\sigma_a}{2} = (W_a)_\mu U\frac{\sigma_a}{2}U^\dagger - \frac{i}{g_2}U^\dagger(\partial_\mu U)\,.$$

Using the relation $\mathrm{Tr}[\sigma_a\sigma_b] = 2\delta_{ab}$*, we obtain*

$$(W'_a)_\mu = \frac{1}{2}\mathrm{Tr}\left[\sigma_a U\sigma_b U^\dagger\right](W_b)_\mu - \frac{i}{g_2}\mathrm{Tr}\left[\sigma_a U^\dagger(\partial_\mu U)\right]\,.$$

(c) You want this very same Lagrangian to describe electromagnetism, and you know that the particles described by ψ_2 are electrically neutral. How can you accommodate this in the theory?

Hint. Consider a suitable linear combination of gauge fields.

First, we write the interaction Lagrangian explicitly

$$\mathcal{L}\supset -\bar{\psi}_1\gamma_\mu\left(\frac{g_1}{2}B^\mu + \frac{g_2}{2}W_3^\mu\right)\psi_1 - \bar{\psi}_2\gamma_\mu\left(\frac{g_1}{2}B^\mu - \frac{g_2}{2}W_3^\mu\right)\psi_2 - \frac{g_2}{2}\bar{\psi}_1\gamma_\mu\left(W_1^\mu - iW_2^\mu\right)\psi_2 + \text{h.c}\,.$$

If we consider the interaction term involving ψ_2 *only, we can write it in the form*

$$\mathcal{L}\supset\frac{\sqrt{g_1^2+g_2^2}}{2}\bar{\psi}_2\gamma_\mu Z^\mu\psi\,,$$

with Z^μ *the following linear combination of* B^μ *and* W_3^μ

$$Z^\mu = -\sin\theta\,B^\mu + \cos\theta\,W_3^\mu \qquad \sin\theta = \frac{g_1}{\sqrt{g_1^2+g_2^2}}\,, \qquad \cos\theta = \frac{g_2}{\sqrt{g_1^2+g_2^2}}\,,$$

and interpret the orthogonal combination

$$A^\mu = \cos\theta\, B^\mu + \sin\theta\, W_3^\mu$$

as the electromagnetic field, so that the particle described by ψ_2 does not interact with the electromagnetic field.

(d) What is the electric charge of the particles described by ψ_1?

We consider the interaction term involving ψ_1 only

$$\mathcal{L} \supset -\frac{\sqrt{g_1^2+g_2^2}}{2}\bar{\psi}_2\gamma_\mu\left(\sin\theta B^\mu + \cos\theta W_3^\mu\right)$$

Expressing B^μ and W_3^μ in terms of A^μ and W_3^μ

$$\begin{aligned} B^\mu &= \cos\theta\, A^\mu - \sin\theta\, Z^\mu\,, \\ W_3^\mu &= \sin\theta\, A^\mu + \cos\theta\, Z^\mu\,, \end{aligned}$$

we obtain

$$\mathcal{L} \supset -\sqrt{g_1^2+g_2^2}\sin\theta\cos\theta\,\bar{\psi}_1\gamma_\mu A^\mu\psi_1\,,$$

which implies that the electric charge of the particle described by ψ_1 is

$$\sqrt{g_1^2+g_2^2}\sin\theta\cos\theta = \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}}\,,$$