Symmetry in Particle Physics, Problem Sheet 8 [SOLUTIONS]

1. Consider two *quantum* free scalar fields $\phi_i(x)$, $i = 1, 2$, given by

$$\phi_i(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left(a_i(\vec{p}) e^{-ip\cdot x} + a_i^\dagger(\vec{p}) e^{ip\cdot x} \right) , \quad E_{\vec{p}} = \sqrt{p^2 + m^2} ,$$

as well as the conserved current

$$J^\mu = (\partial^\mu \phi_1)\phi_2 - (\partial^\mu \phi_2)\phi_1 .$$

(a) Compute the conserved charge Q corresponding to the current J^μ in terms of creation and annihilation operators.
<u>Hint</u>. Creation and annihilation operators for different fields commute.
The conserved charge Q is given by

$$Q = \int d^3x J^0(x) = \int d^3x \left(\pi_1 \phi_2 - \pi_2 \phi_1 \right) ,$$

where $\pi_i(x) = \dot{\phi}_i(x)$ is given by

$$\pi_i(x) = -i \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} \left(a_i(\vec{p}) e^{-ip\cdot x} - a_i^\dagger(\vec{p}) e^{ip\cdot x} \right)$$

Since Q is time-independent, we compute it at $t = 0$. We then have

$$\int d^3x\, \pi_1(0,\vec{x})\phi_2(0,\vec{x}) = -i \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3 2E_{\vec{p}'}} \times$$
$$\times \int d^3x \frac{1}{2} \left(a_1(\vec{p}) e^{i\vec{p}\cdot\vec{x}} - a_1^\dagger(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right) \left(a_2(\vec{p}') e^{i\vec{p}'\cdot\vec{x}} + a_2^\dagger(\vec{p}') e^{-i\vec{p}'\cdot\vec{x}} \right)$$

Using the relation

$$\int d^3x e^{i(\vec{p}-\vec{q})\cdot\vec{x}} = (2\pi^3)\delta^{(3)}(\vec{p}-\vec{q}) ,$$

we obtain

$$\int d^3x\, \pi_1(0,\vec{x})\phi_2(0,\vec{x}) = \frac{i}{2} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \times$$
$$\times \left(a_1^\dagger(\vec{p}) a_2(\vec{p}) - a_1(\vec{p}) a_2^\dagger(\vec{p}) + a_1^\dagger(\vec{p}) a_2^\dagger(-\vec{p}') - a_1(\vec{p}) a_2(-\vec{p}) \right)$$

Similarly

$$\int d^3x\, \pi_2(0,\vec{x})\phi_1(0,\vec{x}) = \frac{i}{2} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \times$$
$$\times \left(a_2^\dagger(\vec{p}) a_1(\vec{p}) - a_2(\vec{p}) a_1^\dagger(\vec{p}) + a_2^\dagger(\vec{p}) a_1^\dagger(-\vec{p}') - a_2(\vec{p}) a_1(-\vec{p}) \right)$$

Adding the two together we obtain

$$Q = i\int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left(a_1^\dagger(\vec{p}) a_2(\vec{p}) - a_2^\dagger(\vec{p}) a_1(\vec{p}) \right) .$$

(b) Consider the field

$$\phi(x) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + i\phi_2(x) \right) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left(a(\vec{p}) e^{-ip\cdot x} + b^\dagger(\vec{p}) e^{ip\cdot x} \right) .$$

Show that the one-particle states created by $a^\dagger(\vec{p})$ and $b^\dagger(\vec{p})$ are eigenstates of the charge operator Q. What are the corresponding eigenvalues?
Using the explicit expressions for ϕ_1 and ϕ_2 we obtain

$$a(\vec{p}) = \frac{1}{\sqrt{2}} \left(a_1(\vec{p}) + i a_2(\vec{p}) \right) ,$$
$$b(\vec{p}) = \frac{1}{\sqrt{2}} \left(a_1(\vec{p}) - i a_2(\vec{p}) \right) .$$

We first consider the states created by $a^\dagger(\vec{p})$. Then

$$\begin{aligned} Q a^\dagger(\vec{p})|0\rangle &= i\int \frac{d^3\vec{p}'}{(2\pi)^3 2E_{\vec{p}'}} \left(a_1^\dagger(\vec{p}') a_2(\vec{p}') - a_2^\dagger(\vec{p}') a_1(\vec{p}') \right) \frac{1}{\sqrt{2}} \left(a_1^\dagger(\vec{p}) - i a_2^\dagger(\vec{p}) \right) |0\rangle \\ &= \frac{i}{\sqrt{2}} \int \frac{d^3\vec{p}'}{(2\pi)^3 2E_{\vec{p}'}} \left(-i a_1^\dagger(\vec{p}') [a_2(\vec{p}'), a_2^\dagger(\vec{p})] - a_2^\dagger(\vec{p}') [a_1(\vec{p}'), a_1^\dagger(\vec{p})] \right) |0\rangle \\ &= \frac{1}{\sqrt{2}} \left(a_1^\dagger(\vec{p}) - i a_2^\dagger(\vec{p}) \right) |0\rangle = a^\dagger(\vec{p})|0\rangle . \end{aligned}$$

Therefore, defining

$$a^\dagger(\vec{p})|0\rangle \equiv |\vec{p}; +\rangle ,$$

we have $Q|\vec{p};+\rangle = |\vec{p};+\rangle$. Similarly, defining

$$b^\dagger(\vec{p})|0\rangle \equiv |\vec{p}; -\rangle ,$$

we have $Q|\vec{p};+\rangle = -|\vec{p};-\rangle$. We call the single-particle states with positive Q "particles" and those with negative Q "antiparticles".

2. Consider the following Lagrangian for a classical real *classical* vector field A^μ:

$$\mathcal{L} = \frac{c_1}{2} F^{\mu\nu} F_{\mu\nu} + \frac{c_2}{2} (\partial_\mu A^\mu)^2 ,$$

where c_1, c_2 are real parameters and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Compute the equations of motions for the field A^μ.

The Euler-Lagrange equations are

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 2c_1 \partial_\mu F^{\mu\nu} + c_2 \partial^\nu (\partial_\mu A^\mu)\,.$$

In terms of the field A^μ explicitly

$$\Box A^\nu - \left(1 - \frac{c_2}{2c_1}\right) \partial^\nu (\partial_\mu A^\mu) = 0\,.$$

(b) Show that $\Box(\partial_\mu A^\mu) = 0$.

Taking the derivative of the equations of motion with respect to x^ν gives

$$\Box(\partial_\nu A^\nu) - \left(1 - \frac{c_2}{2c_1}\right) \Box(\partial_\mu A^\mu) = \frac{c_2}{2c_1} \Box(\partial_\mu A^\mu) = 0 \implies \Box(\partial_\mu A^\mu) = 0\,.$$

(c) Consider the gauge transformation

$$A^\mu \to A^\mu + \partial^\mu \alpha\,.$$

What is the condition on α such that $\mathcal{L}$ is gauge invariant?

As seen in the lectures, $F^{\mu\nu}$ is invariant under such gauge transformations. Then

$$\partial_\mu A^\mu \to \partial_\mu (A^\mu + \partial^\mu \alpha) = \partial_\mu A^\mu + \Box \alpha = \partial_\mu A^\mu \Leftrightarrow \Box \alpha = 0\,.$$

(d) Let us fix now $c_2 = 0$. Compute the Hamiltonian density $\mathcal{H}$ and show that the kinetic energy is positive if and only if $c_1 < 0$.

First we need to compute π^μ, the momentum conjugate to A^μ. This gives

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\nu)} = c_1 F^{0\mu}\,.$$

Therefore

$$\mathcal{H} = \pi^\mu (\partial_0 A_\mu) - \mathcal{L} = c_1 \left(F^{0\mu} F_{0\mu} - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} \right) = \frac{c_1}{2} \Bigg(\underbrace{F^{0i} F_{0i}}_{<0} - F^{ij} F_{ij} \Bigg)\,.$$

The kinetic energy is the term containing time derivatives, which is positive if and only if $c_1 < 0$.