Symmetry in Particle Physics, Problem Sheet 7

1. Consider a massless Dirac spinor ψ with Lagrangian density $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi$, and the global axial transformation

$$\psi \to \psi' = \Omega_5(\theta)\,\psi\,,$$

where the axial transformation is defined via

$$\Omega_5(\theta) = \exp(-i\gamma^5\theta)$$

with transformation parameter θ.

   (a) Use the Weyl representation for γ^μ and γ^5 to compute Ω_5 explicitly. Recall that the exponential of a matrix is defined via its series expansion.

   (b) Compute the transformation law for $\bar{\psi}$. Then, write the transformation in infinitesimal form and show that it is a symmetry of the Lagrangian.

   (c) Show that the corresponding conserved Noether current is given by

$$J_5^\mu = \bar{\psi}\,\gamma^\mu\,\gamma^5\,\psi\,.$$

2. Consider a four-component Dirac spinor $\psi = \psi_L + \psi_R$ with left- and right-handed components

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi\,, \qquad \psi_R = \frac{1}{2}(1+\gamma^5)\psi\,,$$

with $\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$, and define $\bar{\psi} = \psi^\dagger\gamma^0$.

   (a) Under a global chiral symmetry transformation with real parameter α, left- and right-handed fermion fields transform as

$$\psi_R \to e^{i\alpha}\psi_R\,, \qquad \psi_L \to e^{-i\alpha}\psi_L\,.$$

   Write down the transformation law for $\bar{\psi}_L$ and $\bar{\psi}_R$.

   (b) Show that the Dirac mass term

$$\mathcal{L}_D = m_D\left(\bar{\psi}_R\,\psi_L + \bar{\psi}_L\,\psi_R\right)$$

   is not invariant under chiral symmetry.

   (c) Consider a complex scalar field ϕ which under chiral symmetry transforms as

$$\phi \to e^{-2i\alpha}\phi\,.$$

   Write down the transformation law for ϕ^* and show that the interaction term

$$\mathcal{L}_{\text{int}} = \lambda(\phi\,\bar{\psi}_L\,\psi_R + \phi^*\,\bar{\psi}_R\,\psi_L)$$

   between the scalar and the fermions is invariant under chiral symmetry.

3. Consider the Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi + \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) + \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) - g\bar{\psi}(\sigma + i\pi\gamma^5)\psi - V(\sigma^2 + \pi^2)\,,$$

where $\sigma(x)$ and $\pi(x)$ are two real scalar fields, $\psi(x)$ is a Dirac spinor field, $\bar{\psi} = \psi^\dagger \gamma^0$, g is a constant, and $V(\sigma^2+\pi^2)$ means that the potential depends on this combination of fields.

(a) Compute the classical equations of motion for the fields $\sigma(x), \pi(x), \psi(x), \bar{\psi}(x)$.

(b) Consider the infinitesimal chiral transformation $\psi \to \psi + \delta\psi$, $\sigma \to \sigma + \delta\sigma, \pi \to \pi + \delta\pi$, where

$$\delta\psi = i\beta\gamma^5\psi\,, \quad \delta\sigma = 2\beta\pi\,, \qquad \delta\pi = -2\beta\sigma\,.$$

and β is the parameter of the transformation. Check with an explicit calculation that the Lagrangian is invariant under the above infinitesimal transformation.

(c) Compute the Noether current J^μ associated with the infinitesimal transformation of part (b), and show that it is conserved.

4. Consider two Dirac spinor fields $\psi_i(x)$ $(i = 1, 2)$, and three real scalar fields $\phi_a(x)$ $(a = 1, 2, 3)$, with the Lagrangian

$$\mathcal{L} = i\bar{\psi}_i\gamma^\mu \partial_\mu \psi_i - m\bar{\psi}_i\psi_i + \frac{1}{2}(\partial_\mu \phi_a)(\partial^\mu \phi_a) - V(\phi_a\phi_a) - g\bar{\psi}_i \frac{(\sigma_a)_{ij}}{2}\psi_j\phi_a\,,$$

where γ^μ $(\mu = 0, 1, 2, 3)$ are the four-by-four matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$ with $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and $\bar{\psi}_i = \psi_i^\dagger\gamma^0$. Also, m is a real parameter, the potential $V(\phi_a\phi_a)$ is a function of the combination $\phi_a\phi_a \equiv \phi_1^2+\phi_2^2+\phi_3^2$, and $\sigma_1, \sigma_2, \sigma_3$ are the three Pauli matrices.

(a) Show that the Lagrangian $\mathcal{L}$ is invariant under the infinitesimal transformation

$$\psi_i \to \psi_i - i\,\alpha_a \frac{(\sigma_a)_{ij}}{2}\psi_j\,, \qquad \phi_a \to \phi_a + \epsilon_{abc}\alpha_b\phi_c\,,$$

wiht ϵ_{abc} the totally antisymmetric symbol in three dimensions, with $\epsilon_{123} = +1$.

(b) Compute the Noether currents associated to the global symmetry defined in part (a).