Symmetry in Particle Physics, Problem Sheet 7 [SOLUTIONS]

1. Consider a massless Dirac spinor ψ with Lagrangian density $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi$, and the global axial transformation

$$\psi \to \psi' = \Omega_5(\theta)\,\psi\,,$$

where the axial transformation is defined via

$$\Omega_5(\theta) = \exp(-i\gamma^5\theta)$$

with transformation parameter θ.

(a) Use the Weyl representation for γ^μ and γ^5 to compute Ω_5 explicitly. Recall that the exponential of a matrix is defined via its series expansion.

In the series expansion of $\Omega_5(\theta)$, we separate even and odd powers of γ^5, as follows:

$$\Omega_5(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n\,\theta^{2n}}{(2n)!}(\gamma^5)^{2n} - i\sum_{n=0}^{\infty}\frac{(-1)^n\,\theta^{2n+1}}{(2n+1)!}(\gamma^5)^{2n+1}\,.$$

Since $(\gamma^5)^2 = 1$, we obtain

$$\Omega_5(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n\,\theta^{2n}}{(2n)!} - i\gamma^5\sum_{n=0}^{\infty}\frac{(-1)^n\,\theta^{2n+1}}{(2n+1)!} = \cos\theta - i\sin\theta\,\gamma^5\,.$$

Inserting the explicit Weyl representation of the matrix γ_5 we get

$$\Omega_5(\theta) = \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\,.$$

(b) Compute the transformation law for $\bar{\psi}$. Then, write the transformation in infinitesimal form and show that it is a symmetry of the Lagrangian.

The transformation law for $\bar{\psi}$ is given by

$$\bar{\psi} \to (\Omega_5(\theta)\psi)^\dagger\gamma^0 = \psi^\dagger(\Omega_5(\theta))^\dagger\gamma^0 = \bar{\psi}\gamma^0(\cos\theta + i\sin\theta\,\gamma^5)\gamma^0 = \bar{\psi}(\cos\theta - i\sin\theta\,\gamma^5) = \bar{\psi}\,\Omega_5(\theta)\,.$$

The axial transformation in infinitesimal form is

$$\begin{aligned} \psi &\to \psi + \delta\psi\,, \qquad & \delta\psi &= -i\theta\gamma^5\psi\,, \\ \bar{\psi} &\to \bar{\psi} + \delta\bar{\psi}\,, \qquad & \delta\bar{\psi} &= -i\theta\bar{\psi}\gamma^5\,. \end{aligned}$$

Substituting into the Lagrangian we get

$$\begin{aligned} \mathcal{L} &\to i\,\bar{\psi}\,\gamma^\mu\,\partial_\mu\,\psi + \theta(\bar{\psi}\,\gamma^5\,\gamma^\mu\,\partial_\mu\,\psi + \bar{\psi}\,\gamma^\mu\,\gamma^5\,\partial_\mu\,\psi) \\ &= i\,\bar{\psi}\,\gamma^\mu\,\partial_\mu\,\psi + \theta(\bar{\psi}\,\{\gamma^\mu,\gamma^5\}\,\partial_\mu\,\psi) = i\,\bar{\psi}\,\gamma^\mu\,\partial_\mu\,\psi\,. \end{aligned}$$

(c) Show that the corresponding conserved Noether current is given by

$$J_5^\mu = \bar{\psi}\,\gamma^\mu\,\gamma^5\,\psi\,.$$

The Noether current is given by

$$J_5^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\partial\delta\psi}{\partial\theta} = (i\bar{\psi}\gamma^\mu)(-i\gamma^5\,\psi) = \bar{\psi}\,\gamma^\mu\,\gamma^5\,\psi\,.$$

2. Consider a four-component Dirac spinor $\psi = \psi_L + \psi_R$ with left- and right-handed components

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi\,, \qquad \psi_R = \frac{1}{2}(1+\gamma^5)\psi\,,$$

with $\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$, and define $\bar{\psi} = \psi^\dagger\gamma^0$.

(a) Under a global chiral symmetry transformation with real parameter α, left- and right-handed fermion fields transform as

$$\psi_R \to e^{i\alpha}\psi_R\,, \qquad \psi_L \to e^{-i\alpha}\psi_L\,.$$

Write down the transformation law for $\bar{\psi}_L$ and $\bar{\psi}_R$. *From direct inspection*

$$\bar{\psi}_R = \psi_R^\dagger\gamma^0 \to \bar{\psi}_R\,e^{-i\alpha}\,.$$

Similarly

$$\bar{\psi}_L = \psi_L^\dagger\gamma^0 \to \bar{\psi}_L\,e^{i\alpha}\,.$$

(b) Show that the Dirac mass term

$$\mathcal{L}_D = m_D\left(\bar{\psi}_R\,\psi_L + \bar{\psi}_L\,\psi_R\right)$$

is not invariant under chiral symmetry.

From direct inspection

$$\bar{\psi}_R\,\psi_L \to e^{-2i\alpha}\,\bar{\psi}_R\,\psi_L\,,$$

and

$$\bar{\psi}_L\,\psi_R \to e^{2i\alpha}\,\bar{\psi}_L\,\psi_R\,,$$

therefore a Dirac mass term is not invariant under chiral transformations.

(c) Consider a complex scalar field ϕ which under chiral symmetry transforms as

$$\phi \to e^{-2i\alpha}\phi\,.$$

Write down the transformation law for ϕ^* and show that the interaction term

$$\mathcal{L}_{\text{int}} = \lambda(\phi\,\bar{\psi}_L\,\psi_R + \phi^*\,\bar{\psi}_R\,\psi_L)$$

between the scalar and the fermions is invariant under chiral symmetry.
The transformation law for ϕ^* is

$$\phi^* \to e^{2i\alpha}\phi^*\,.$$

Using this result we find

$$\phi\,\bar{\psi}_L\,\psi_R \to e^{-2i\alpha}e^{2i\alpha}\phi\,\bar{\psi}_L\,\psi_R = \phi\,\bar{\psi}_L\,\psi_R\,,$$

and

$$\phi^*\,\bar{\psi}_R\,\psi_L \to e^{2i\alpha}e^{-2i\alpha}\phi^*\,\bar{\psi}_R\,\psi_L = \phi^*\,\bar{\psi}_R\,\psi_L\,,\,.$$

Hence the interaction term is invariant.

3. Consider the Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) + \frac{1}{2}(\partial_\mu\pi)(\partial^\mu\pi) - g\bar{\psi}(\sigma + i\pi\gamma^5)\psi - V(\sigma^2+\pi^2)\,,$$

where $\sigma(x)$ and $\pi(x)$ are two real scalar fields, $\psi(x)$ is a Dirac spinor field, $\bar{\psi} = \psi^\dagger\gamma^0$, g is a constant, and $V(\sigma^2+\pi^2)$ means that the potential depends on this combination of fields.

(a) Compute the classical equations of motion for the fields $\sigma(x), \pi(x), \psi(x), \bar{\psi}(x)$.
The equations of motions are obtained from the Euler-Lagrange equations

$$\begin{aligned}
\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\sigma)} - \frac{\partial\mathcal{L}}{\partial\sigma} &= \Box\sigma + g\bar{\psi}\,\psi + 2\sigma\,V'(\sigma^2+\pi^2) = 0\,,\\
\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi)} - \frac{\partial\mathcal{L}}{\partial\pi} &= \Box\pi + ig\bar{\psi}\,\gamma^5\psi + 2\pi\,V'(\sigma^2+\pi^2) = 0\,,\\
\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} &= -i\gamma^\mu\partial_\mu\psi + g(\sigma + i\pi\gamma^5)\psi = 0\,,\\
\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} &= i\partial_\mu\bar{\psi}\gamma^\mu + g\bar{\psi}(\sigma + i\pi\gamma^5) = 0\,,
\end{aligned}$$

(b) Consider the infinitesimal chiral transformation $\psi \to \psi + \delta\psi$, $\sigma \to \sigma + \delta\sigma, \pi \to \pi + \delta\pi$, where

$$\delta\psi = i\beta\gamma^5\psi\,,\quad \delta\sigma = 2\beta\pi\,,\qquad \delta\pi = -2\beta\sigma\,.$$

and β is the parameter of the transformation. Check with an explicit calculation that the Lagrangian is invariant under the above infinitesimal transformation. *Taking the hermitian conjugate of $\delta\psi$ we find $\bar{\psi} \to \bar{\psi} + \delta\bar{\psi}$, where*

$$\delta\bar{\psi} = (\delta\psi)^\dagger \gamma^0 = (-i\beta\psi^\dagger\gamma^5)\gamma^0 = i\beta\bar{\psi}\gamma^5 \,.$$

We then have

$$\begin{aligned}
\delta\mathcal{L} &= i\delta\bar{\psi}\gamma^\mu\partial_\mu\psi + i\bar{\psi}\gamma^\mu\partial_\mu\delta\psi + (\partial_\mu\delta\sigma)(\partial^\mu\sigma) + (\partial_\mu\delta\pi)(\partial^\mu\pi) \\
&\quad - g\left[\delta\bar{\psi}(\sigma + i\pi\gamma^5)\psi + \bar{\psi}(\sigma + i\pi\gamma^5)\delta\psi + \bar{\psi}(\delta\sigma + i\delta\pi\gamma^5)\psi\right] - 2(\sigma\delta\sigma + \pi\delta\pi)V'(\sigma^2+\pi^2) \\
&= i\beta\bar{\psi}\underbrace{\{\gamma^5,\gamma^\mu\}}_{=0}\psi + 2\beta\left[\underbrace{(\partial_\mu\pi)(\partial^\mu\sigma) - (\partial_\mu\sigma)(\partial^\mu\pi)}_{=0}\right] \\
&\quad - 2g\beta\left[\underbrace{\bar{\psi}(i\sigma\gamma^5 - \pi)\psi + \bar{\psi}(\pi - i\sigma\gamma^5)\psi}_{=0}\right] - 4\beta(\underbrace{\sigma\pi - \pi\sigma}_{=0})V'(\sigma^2+\pi^2) = 0\,.
\end{aligned}$$

(c) Compute the Noether current J^μ associated with the infinitesimal transformation of part (b), and show that it is conserved.

$$\begin{aligned}
J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\partial\delta\psi}{\partial\beta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\frac{\partial\delta\bar{\psi}}{\partial\beta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\sigma)}\frac{\partial\delta\sigma}{\partial\beta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi)}\frac{\partial\delta\pi}{\partial\beta} \\
&= -\bar{\psi}\gamma^\mu\gamma^5\psi + 2\left[(\partial^\mu\sigma)\pi - (\partial^\mu\pi)\sigma\right]\,.
\end{aligned}$$

Let us check that the current is indeed conserved:

$$\partial_\mu J^\mu = -(\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^5\gamma^\mu(\partial_\mu\psi) + 2\left[(\Box\sigma)\pi - (\Box\pi)\sigma\right]\,.$$

Now we impose the equations of motions, and obtain

$$\begin{aligned}
\partial_\mu J^\mu &= -ig\bar{\psi}(\sigma + i\pi\gamma^5)\gamma^5\psi - ig\bar{\psi}\gamma^5(\sigma + i\pi\gamma^5)\psi + 2\left[-(g\bar{\psi}\psi + 2\sigma V')\pi + (ig\bar{\psi}\gamma^5\psi + 2\pi V')\sigma\right] \\
&= 2g\bar{\psi}(\pi - i\sigma\gamma^5)\psi - 2g\bar{\psi}(\pi - i\sigma\gamma^5)\psi = 0\,.
\end{aligned}$$

4. Consider two Dirac spinor fields $\psi_i(x)$ $(i = 1, 2)$, and three real scalar fields $\phi_a(x)$ $(a = 1, 2, 3)$, with the Lagrangian

$$\mathcal{L} = i\bar{\psi}_i\gamma^\mu\partial_\mu\psi_i - m\bar{\psi}_i\psi_i + \frac{1}{2}(\partial_\mu\phi_a)(\partial^\mu\phi_a) - V(\phi_a\phi_a) - g\bar{\psi}_i\frac{(\sigma_a)_{ij}}{2}\psi_j\phi_a\,,$$

where γ^μ $(\mu = 0, 1, 2, 3)$ are the four-by-four matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$ with $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and $\bar{\psi}_i = \psi_i^\dagger\gamma^0$. Also, m is a real parameter, the potential $V(\phi_a\phi_a)$ is a function of the combination $\phi_a\phi_a \equiv \phi_1^2 + \phi_2^2 + \phi_3^2$, and $\sigma_1, \sigma_2, \sigma_3$ are the three Pauli matrices.

(a) Show that the Lagrangian $\mathcal{L}$ is invariant under the infinitesimal transformation

$$\psi_i \to \psi_i - i\,\alpha_a \frac{(\sigma_a)_{ij}}{2}\psi_j\,, \qquad \phi_a \to \phi_a + \epsilon_{abc}\alpha_b\phi_c\,,$$

wiht ϵ_{abc} the totally antisymmetric symbol in three dimensions, with $\epsilon_{123} = +1$.

We first compute the transformation of $\bar{\psi}$:

$$\psi_i^\dagger \to \left(\psi_i - i\,\alpha_a \frac{(\sigma_a)_{ij}}{2}\psi_j\right)^\dagger = \psi_i^\dagger + i\,\alpha_a\,\psi_j^\dagger \frac{(\sigma_a)_{ji}^\dagger}{2}\,.$$

Since $(\sigma_a)^\dagger = \sigma_a$, we obtain

$$\bar{\psi}_i \to \bar{\psi}_i + i\,\alpha_a\,\bar{\psi}_j \frac{(\sigma_a)_{ji}}{2}\,.$$

The invariance f the kinetic and mass terms for the fermionic fields relies on the invariance of $\bar{\psi}_i\psi_i$. Neglecting quadratic terms in α_a, we have

$$\begin{aligned}\bar{\psi}_i\psi_i \to &\left(\bar{\psi}_i\delta_{ij} + i\,\bar{\psi}_i\alpha_a\frac{(\sigma_a)_{ij}}{2}\right)\left(\psi_j - i\,\alpha_a\frac{(\sigma_a)_{jk}}{2}\psi_k\right)\\ &\simeq \bar{\psi}_i\psi_i - i\,\bar{\psi}_i\alpha_a\frac{(\sigma_a)_{ij}}{2}\psi_j + i\,\bar{\psi}_i\alpha_a\frac{(\sigma_a)_{ij}}{2}\psi_j = \bar{\psi}_i\psi_i\,.\end{aligned}$$

Similarly, the invariance of the kinetic and potential terms for the scalar fields relies on the invariance of $\phi_a\phi_a$:

$$\begin{aligned}\phi_a\phi_a &\to (\phi_a + \epsilon_{abc}\alpha_b\phi_c)(\phi_a + \epsilon_{ade}\alpha_d\phi_e)\\ &\simeq \phi_a\phi_a + 2\epsilon_{abc}\alpha_b\phi_a\phi_c = \phi_a\phi_a\,.\end{aligned}$$

Last, the interaction term:

$$\begin{aligned}\bar{\psi}_i\frac{(\sigma_a)_{ij}}{2}\psi_j\phi_a \to &\left(\bar{\psi}_i\delta_{ij} + i\,\bar{\psi}_i\alpha_b\frac{(\sigma_b)_{ij}}{2}\right)\frac{(\sigma_a)_{jk}}{2}\left(\psi_k - i\,\alpha_b\frac{(\sigma_b)_{kl}}{2}\psi_l\right)(\phi_a + \epsilon_{acd}\alpha_c\phi_d)\\ &\simeq \bar{\psi}_i\frac{(\sigma_a)_{ij}}{2}\psi_j\phi_a + i\bar{\psi}_i\alpha_b\underbrace{\left[\frac{\sigma_b}{2},\frac{\sigma_a}{2}\right]_{ij}}_{=i\epsilon_{bac}(\sigma_c)_{ij}/2}\psi_j\phi_a + \bar{\psi}_i\frac{(\sigma_a)_{ij}}{2}\psi_j\epsilon_{acd}\alpha_c\phi_d\\ &= \bar{\psi}_i\frac{(\sigma_a)_{ij}}{2}\psi_j\phi_a + \bar{\psi}_i\left(-\frac{(\sigma_a)_{ij}}{2}\epsilon_{cda}\alpha_c\phi_d + \frac{(\sigma_a)_{ij}}{2}\epsilon_{acd}\alpha_c\phi_d\right)\psi_j = \bar{\psi}_i\frac{(\sigma_a)_{ij}}{2}\psi_j\phi_a\,.\end{aligned}$$

(b) Compute the Noether currents associated to the global symmetry defined in part (a).

We have three Noether currents J_a^μ associated to $SU(2)$, defined as

$$J_a^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\frac{\partial\delta\psi_i}{\partial\alpha_a} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_i)}\frac{\partial\delta\bar{\psi}_i}{\partial\alpha_a} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_b)}\frac{\partial\delta\phi_b}{\partial\alpha_a}\,.$$

Since $\partial\mathcal{L}/\partial(\partial_\mu\bar{\psi}_i) = 0$ *we can consider only the infinitesimal variation of* ψ_i *and* ϕ_a*, which gives*

$$\frac{\partial\delta\psi_i}{\partial\alpha_a} = -i\frac{(\sigma_a)_{ij}}{2}\psi_j\,, \qquad \frac{\partial\delta\phi_b}{\partial\alpha_a} = \epsilon_{bac}\phi_c\,.$$

Inserting this information in the definition of the conserved current, we obtain

$$J^\mu_a = \bar{\psi}_i\gamma^\mu\frac{(\sigma_a)_{ij}}{2}\psi_j - \epsilon_{abc}(\partial^\mu\phi_b)\phi_c\,.$$