Symmetry in Particle Physics, Problem Sheet 6

1. Consider a Dirac field ψ with a Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi\,,$$

where m is a real, not necessarily positive.

(a) Compute the equation of motion for the field ψ.

(b) Show that the field ψ satisfies Klein-Gordon equation, and from that determine the mass of the spin-1/2 fermions described by the quantised field ψ.

2. Consider the four-by-four matrices γ^μ ($\mu = 0, 1, 2, 3$) in the Weyl representation,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. The three-dimensional vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ contains the three Pauli matrices satisfying $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. Note that $\{\sigma^\mu, \bar{\sigma}^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$, with $\eta^{\mu\nu}$ the metric of Minkowsy space.

(a) The generators of Lorentz transformations for a left-handed Weyl spinor are

$$S_L^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)\,,$$

while for a right-handed spinor the generators are

$$S_R^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)\,.$$

Deduce that for a Dirac spinor the generators of the Lorentz group are $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

(b) Show that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and that the same holds for the matrices $U^\dagger\gamma^\mu U$ provided U is unitary.

(c) Show that the matrix

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$$

transforms the Weyl representation into the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

3. Consider the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

(a) Show that γ^5 is diagonal in the Weyl representation,

$$\gamma^5 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix},$$

and that, in any representation, $(\gamma^5)^\dagger = \gamma^5$.

(b) Show by direct inspection, or otherwise, that the matrices γ^μ anti-commute with γ^5, i.e. $\{\gamma^\mu, \gamma^5\} = 0$.