Symmetry in Particle Physics, Problem Sheet 6 [SOLUTIONS]

1. Consider a Dirac field ψ with a Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi\,,$$

where m is a real, not necessarily positive.

(a) Compute the equation of motion for the field ψ.

The Euler-Lagrange equation for ψ are given by

$$0 = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = -i\gamma^\mu\partial_\mu\psi + m\psi \implies (i\gamma^\mu\partial_\mu - m)\psi = 0\,.$$

(b) Show that the field ψ satisfies Klein-Gordon equation, and from that determine the mass of the spin-1/2 fermions described by the quantised field ψ.

Acting on the equation of motion with the operator $(i\gamma_\mu\partial_\mu + m)$ we obtain

$$(i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi = (-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = 0$$

The term with two gamma matrices can be written in the form

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \Box\,.$$

This gives

$$(\Box + m^2)\psi = 0$$

This means that the mass of the fermions is in fact $\sqrt{m^2} = |m|$.

2. Consider the four-by-four matrices γ^μ ($\mu = 0, 1, 2, 3$) in the Weyl representation,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\,,$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. The three-dimensional vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ contains the three Pauli matrices satisfying $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. Note that $\{\sigma^\mu, \bar{\sigma}^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$, with $\eta^{\mu\nu}$ the metric of Minkowsy space.

(a) The generators of Lorentz transformations for a left-handed Weyl spinor are

$$S_L^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)\,,$$

while for a right-handed spinor the generators are

$$S_R^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)\,.$$

Deduce that for a Dirac spinor the generators of the Lorentz group are $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

A Dirac spinor is the direct sum of a left-handed and a right-handed spinor. Therefore

$$S^{\mu\nu} = \begin{pmatrix} S_L^{\mu\nu} & 0 \\ 0 & S_R^{\mu\nu} \end{pmatrix} .$$

With an explicit calculation

$$\gamma^\mu \gamma^\nu = \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu \end{pmatrix} .$$

Therefore

$$\frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4} \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix} = \begin{pmatrix} S_L^{\mu\nu} & 0 \\ 0 & S_R^{\mu\nu} \end{pmatrix} .$$

(b) Show that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and that the same holds for the matrices $U^\dagger \gamma^\mu U$ provided U is unitary.

With an explicit calculation

$$\gamma^\mu \gamma^\nu = \begin{pmatrix} \{\sigma^\mu, \bar{\sigma}^\nu\} & 0 \\ 0 & \{\bar{\sigma}^\mu, \sigma^\nu\} \end{pmatrix} = \begin{pmatrix} 2\eta^{\mu\nu}\mathbb{1} & 0 \\ 0 & 2\eta^{\mu\nu}\mathbb{1} \end{pmatrix} = 2\eta^{\mu\nu}\mathbb{1} .$$

With another explicit calculation

$$\{U^\dagger \gamma^\mu U, U^\dagger \gamma^\nu U\} = U^\dagger \gamma^\mu \underbrace{U U^\dagger}_{=\mathbb{1}} \gamma^\nu U + U^\dagger \gamma^\nu \underbrace{U U^\dagger}_{=\mathbb{1}} \gamma^\mu U = U^\dagger (2\eta^{\mu\nu}\mathbb{1}) U = 2\eta^{\mu\nu} .$$

(c) Show that the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$$

transforms the Weyl representation into the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

3. Consider the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

(a) Show that γ^5 is diagonal in the Weyl representation,

$$\gamma^5 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} ,$$

and that, in any representation, $(\gamma^5)^\dagger = \gamma^5$.

A possible way to proceed:

$$\gamma^0\gamma^1 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \gamma^2\gamma^3 = \begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix}.$$

Use now

$$\sigma_2\sigma_3 = \frac{1}{2}\underbrace{\{\sigma_2,\sigma_3\}}_{=0} + \frac{1}{2}\underbrace{[\sigma_2,\sigma_3]}_{=2i\sigma_1} = i\sigma_1 \implies \gamma^2\gamma^3 = -i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}.$$

This gives

$$i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{pmatrix} -\sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix} = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}.$$

By direct inspection, the matrix γ^5 in the Weyl representation satisfies $(\gamma^5)^\dagger = \gamma^5$. In any other representation, the matrix γ^5 is related to the one in the Weyl representation by $U^\dagger\gamma^5 U$, with U a unitary matrix. This gives

$$(U^\dagger\gamma^5 U)^\dagger = U^\dagger(\gamma^5)^\dagger U = U^\dagger\gamma^5 U\,.$$

(b) Show by direct inspection, or otherwise, that the matrices γ^μ anti-commute with γ^5, i.e. $\{\gamma^\mu, \gamma^5\} = 0$.

This solution uses the anti-commutation rules of γ^μ. If we compute $\gamma^\mu\gamma^5$, we observe that, in order to move γ^μ to the right, we need to perform three swaps with γ^ν, with $\nu \neq \mu$. This gives:

$$\gamma^\mu\gamma^5 = (-1)^3\gamma^5\gamma^\mu = -\gamma^5\gamma^\mu\,.$$