Symmetry in Particle Physics, Problem Sheet 5

1. Consider a scalar quantum field $\phi(x)$. Show that

$$\langle 0|\phi(x)|\vec{p}\rangle = e^{-ip\cdot x}\langle 0|\phi(0)|\vec{p}\rangle \,.$$

   Then show that, if we perform a Lorentz transformation Λ,

$$\langle 0|\phi(0)|\vec{p}\rangle = \langle 0|\phi(0)|\Lambda\vec{p}\rangle \,.$$

2. Consider the function

$$\Delta(x-y,\mu^2) = -i\int \frac{d^4q}{(2\pi)^3}\left(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right)\Theta(q^0)\delta(q^2-\mu^2)\,.$$

   Show that

   (a) $(\Box+\mu^2)\Delta(x,\mu^2) = 0$;
   (b) $\Delta(\Lambda x,\mu^2) = \Delta(x,\mu^2)$ if $\Lambda \in L_+^\uparrow$;
   (c) $\Delta(-x,\mu^2) = -\Delta(x,\mu^2)$;
   (d) $\Delta(x,\mu^2)$ is a function of x^2 and $\epsilon(x^0)$ only;
   (e) $\frac{\partial}{\partial x^0}\Delta(x-y,\mu^2)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y})$;
   (f) $\Delta(x,\mu^2) = 0$ if $x^2 < 0$.

3. Consider the annihilation operator $a(\vec{p})$, acting as follows

$$a(\vec{p})|0\rangle = 0\,, \qquad a(\vec{p})|\vec{p}'\rangle = (2\pi)^3 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}')\,.$$

   Show that $a^\dagger(\vec{p})|0\rangle = |\vec{p}\rangle$.

   Hint. The vector $a^\dagger(\vec{p})|0\rangle$ can be written as the following superposition

$$a^\dagger(\vec{p})|0\rangle = c_0|0\rangle + \int\frac{d^3\vec{p}'}{(2\pi)^3 2E_{\vec{p}'}}c_1(\vec{p}')|\vec{p}'\rangle + \int\frac{d^3\vec{p}_1'}{(2\pi)^3 2E_{\vec{p}_1'}}\frac{d^3\vec{p}_2'}{(2\pi)^3 2E_{\vec{p}_2'}}c_2(\vec{p}_1',\vec{p}_2')|\vec{p}'\rangle + \ldots$$

4. Let $\phi(x)$ be a hermitian scalar quantum field, given by

$$\phi(x) = \int\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}\left(e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}a^\dagger(\vec{p})\right)\,.$$

   Show that imposing the commutation rules

$$[a(\vec{p}),a(\vec{p}')] = [a^\dagger(\vec{p}),a^\dagger(\vec{p}')] = 0\,, \qquad [a(\vec{p}),a^\dagger(\vec{p}')] = (2\pi)^3 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}')\,,$$

   gives $[\phi(x),\phi(y)] = 0$ for $(x-y)^2 < 0$.