**Symmetry in Particle Physics, Problem Sheet 5 [SOLUTIONS]**

1. Consider a scalar quantum field $\phi(x)$. Show that

$$\langle 0|\phi(x)|\vec{p}\rangle = e^{-ip\cdot x}\langle 0|\phi(0)|\vec{p}\rangle\,.$$

Then show that, if we perform a Lorentz transformation $\Lambda$,

$$\langle 0|\phi(0)|\vec{p}\rangle = \langle 0|\phi(0)|\Lambda\vec{p}\rangle\,.$$

*From the transformation properties of $\phi(x)$, we have*

$$\phi(x) = e^{iP_\mu x^\mu}\phi(0)e^{-iP_\mu x^\mu}\,.$$

*Using the above, and the fact that $P_\mu|0\rangle = 0$, we obtain*

$$\langle 0|\phi(x)|\vec{p}\rangle = \langle 0|e^{iP_\mu x^\mu}\phi(0)e^{-iP_\mu x^\mu}|\vec{p}\rangle = \langle 0|\phi(0)e^{-ip\cdot x}|\vec{p}\rangle = e^{-ip\cdot x}\langle 0|\phi(0)|\vec{p}\rangle\,.$$

*Also,*

$$\langle 0|\phi(0)|\Lambda\vec{p}\rangle = \langle 0|\phi(0)U(\Lambda,0)|\vec{p}\rangle = \langle 0|U^{-1}(\Lambda,0)\phi(0)U(\Lambda,0)|\vec{p}\rangle = \langle 0|\phi(\Lambda 0)|\vec{p}\rangle = \langle 0|\phi(0)|\vec{p}\rangle\,.$$

2. Consider the function

$$\Delta(x-y,\mu^2) = -i\int\frac{d^4q}{(2\pi)^3}\left(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right)\Theta(q^0)\delta(q^2-\mu^2)\,.$$

Show that

(a) $(\Box+\mu^2)\Delta(x,\mu^2) = 0$;
*From a direct calculation*

$$(\Box+\mu^2)e^{-iq\cdot(x-y)}\delta(q^2-\mu^2) = (-q^2+\mu^2)e^{-iq\cdot(x-y)}\delta(q^2-\mu^2) = 0\,.$$

(b) $\Delta(\Lambda x,\mu^2) = \Delta(x,\mu^2)$ if $\Lambda\in L_+^\uparrow$;
*We perform the change of variable $q' = \Lambda^{-1}q$ with $\Lambda\in L_+^\uparrow$. Then $d^4q' = d^4q$. This gives*

$$\begin{aligned}\Delta(\Lambda x,\mu^2) &= -i\int\frac{d^4q'}{(2\pi)^3}\left(e^{-i(\Lambda q')\cdot(\Lambda x)} - e^{i(\Lambda q')\cdot(\Lambda x)}\right)\Theta(q'^0)\delta(q'^2-\mu^2)\\ &= -i\int\frac{d^4q'}{(2\pi)^3}\left(e^{-iq'\cdot x} - e^{iq'\cdot x}\right)\Theta(q'^0)\delta(q'^2-\mu^2) = \Delta(\Lambda x,\mu^2)\,.\end{aligned}$$

(c) $\Delta(-x,\mu^2) = -\Delta(x,\mu^2)$;
*From a direct calculation*

$$\begin{aligned}\Delta(-x,\mu^2) &= -i\int \frac{d^4q}{(2\pi)^3}\left(e^{-iq\cdot(-x)} - e^{iq\cdot(-x)}\right)\Theta(q^0)\delta(q^2-\mu^2)\\ &= i\int \frac{d^4q}{(2\pi)^3}\left(e^{-iq\cdot x} - e^{iq\cdot x}\right)\Theta(q^0)\delta(q^2-\mu^2) = -\Delta(x,\mu^2)\end{aligned}$$

(d) $\Delta(x,\mu^2)$ is a function of $x^2$ and $\epsilon(x^0)$ only;
*Since $\Delta(\Lambda x,\mu^2) = \Delta(\Lambda x,\mu^2)$ for $\Lambda \in L_+^\uparrow$, it has to be a function of two functions of $x$ that are left invariant by such transformations, and these are $x^2$ and $\epsilon(x^0)$.*

(e) $\frac{\partial}{\partial x^0}\Delta(x-y,\mu^2)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y})$;

$$\begin{aligned}\frac{\partial}{\partial x^0}\Delta(x-y,\mu^2)|_{x^0=y^0} &= -i\int \frac{d^4q}{(2\pi)^3}(-i)q_0\left(e^{i\vec{q}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{q}\cdot(\vec{x}-\vec{y})}\right)\Theta(q^0)\delta(q^2-\mu^2)\\ &= -\int \frac{d^3\vec{q}}{2q_0(2\pi)^3}q_0\left(e^{i\vec{q}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{q}\cdot(\vec{x}-\vec{y})}\right) = -\delta^3(\vec{x}-\vec{y})\,.\end{aligned}$$

(f) $\Delta(x,\mu^2) = 0$ if $x^2 < 0$.
*If $x$ is space-like, we can perform an orthochronous Lorentz transformation, and set it into the form $x = (0,\vec{x})$. This gives*

$$\Delta(x,\mu^2) = -i\int \frac{d^3\vec{q}}{2q_0(2\pi)^3}\left(\underbrace{e^{-i\vec{q}\cdot\vec{x}} - e^{i\vec{q}\cdot\vec{x}}}_{\text{odd function of }\vec{q}}\right) = 0\,.$$

3. Consider the annihilation operator $a(\vec{p})$, acting as follows

$$a(\vec{p})|0\rangle = 0\,,\qquad a(\vec{p})|\vec{p}'\rangle = (2\pi)^3 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}')\,.$$

Show that $a^\dagger(\vec{p})|0\rangle = |\vec{p}\rangle$.

Hint. The vector $a^\dagger(\vec{p})|0\rangle$ can be written as the following superposition

$$a^\dagger(\vec{p})|0\rangle = c_0|0\rangle + \int \frac{d^3\vec{p}'}{(2\pi)^3 2E_{\vec{p}'}}c_1(\vec{p}')|\vec{p}'\rangle + \int \frac{d^3\vec{p}_1'}{(2\pi)^3 2E_{\vec{p}_1'}}\frac{d^3\vec{p}_2'}{(2\pi)^3 2E_{\vec{p}_2'}}c_2(\vec{p}_1',\vec{p}_2')|\vec{p}'\rangle + \ldots$$

*The coefficients $c_i$ can be obtained by taking the scalar product with the appropriate basis vectors. Using the normalisation condition*

$$\langle\vec{p}|\vec{p}'\rangle = (2\pi)^2 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}')\,,$$

*this gives*

$$c_0 = \langle 0|a^\dagger(\vec{p})|0\rangle = \langle 0|a(\vec{p})|0\rangle^* = 0\,,$$
$$c_1(\vec{p}\,') = \langle \vec{p}\,'|a^\dagger(\vec{p})|0\rangle = \langle 0|a(\vec{p})|\vec{p}\,'\rangle^* = (2\pi)^3 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}\,')\,,$$

*and all the other scalar products are zero because they involve the vacuum on one side and one- or more-particle states on the other side. Therefore*

$$a^\dagger(\vec{p})|0\rangle = \int \frac{d^3\vec{p}\,'}{(2\pi)^3 2E_{\vec{p}\,'}}(2\pi)^3 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}\,')|\vec{p}\,'\rangle = |\vec{p}\rangle\,.$$

4. Let $\phi(x)$ be a hermitian scalar quantum field, given by

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}\left(e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}a^\dagger(\vec{p})\right)\,.$$

Show that imposing the commutation rules

$$[a(\vec{p}), a(\vec{p}\,')] = [a^\dagger(\vec{p}), a^\dagger(\vec{p}\,')] = 0\,, \qquad [a(\vec{p}), a^\dagger(\vec{p}\,')] = (2\pi)^3 2E_{\vec{p}}\delta^3(\vec{p}-\vec{p}\,')\,,$$

gives $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$.

*We first consider $[\phi(x), \phi(0)]$ with $x^\mu = (0, \vec{x})$. From a direct calculation, we obtain:*

$$\begin{aligned}[\phi(x), \phi(0)] = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}\frac{d^3\vec{p}\,'}{(2\pi)^3 2E_{\vec{p}\,'}}\Bigg[&\Bigg(\underbrace{[a(\vec{p}), a(\vec{p}\,')]}_{=0} + \underbrace{[a(\vec{p}), a^\dagger(\vec{p}\,')]}_{=(2\pi)^3(2E_{\vec{p}})\delta^3(\vec{p}-\vec{p}\,')}\Bigg)e^{i\vec{p}\cdot\vec{x}} \\ &+ \Bigg(\underbrace{[a^\dagger(\vec{p}), a^\dagger(\vec{p}\,')]}_{=0} + \underbrace{[a^\dagger(\vec{p}), a(\vec{p}\,')]}_{=-(2\pi)^3(2E_{\vec{p}})\delta^3(\vec{p}-\vec{p}\,')}\Bigg)e^{-i\vec{p}\cdot\vec{x}}\Bigg] \\ = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}&\left(e^{i\vec{p}\cdot\vec{x}} - e^{-i\vec{p}\cdot\vec{x}}\right) = 0\,.\end{aligned}$$

*We now observe that*

$$\phi(x)\phi(y) = \phi(x)e^{iP\cdot y}\phi(0)e^{-iP\cdot y} = e^{iP\cdot y}e^{-iP\cdot y}\phi(x)e^{iP\cdot y}\phi(0)e^{-iP\cdot y} = e^{iP\cdot y}\phi(x-y)\phi(0)e^{-iP\cdot y}\,.$$

*Similarly,*

$$\phi(x)\phi(y) = e^{iP\cdot y}\phi(0)\phi(x-y)e^{-iP\cdot y}\,.$$

*It is always possible to find a proper orthochronous Lorentz transformation such that*

$$x - y = \Lambda\bar{x}\,, \qquad \bar{x} = (0, \vec{x}-\vec{y})\,.$$

*Therefore*

$$\phi(x-y)\phi(0) = \phi(\Lambda\bar{x})\phi(0) = U(\Lambda,0)\phi(\bar{x})U^{-1}(\Lambda,0)\phi(0) = U(\Lambda,0)\phi(\bar{x})\phi(\Lambda 0)U^{-1}(\Lambda,0) = U(\Lambda,0)\phi(\bar{x})\phi$$

*The same holds for* $\phi(0)\phi(x-y)$, *therefore*

$$[\phi(x),\phi(y)] = e^{iP\cdot y}U(\Lambda,0)\underbrace{[\phi(\bar{x}),\phi(0)]}_{=0}U^{-1}(\Lambda,0)e^{-iP\cdot y} = 0\,.$$