Symmetry in Particle Physics, Problem Sheet 4

1. Consider operators or matrices A and B which obey $[A, B] = 1$.

   (a) Show that $e^{aA}\, B\, e^{-aA} = B + a$ for a some real or complex number. Hint. Recall that, if we define

   $$f(A) \equiv \sum_{n=0}^{\infty} f_n A^n\,, \qquad f'(A) \equiv \sum_{n=0}^{\infty} n f_n A^{n-1}\,,$$

   we have

   $$[f(A), B] = f'(A)[A, B]\,,$$

   (b) Use the result above to show that $e^{aA}\, f(B)\, e^{-aA} = f(B + a)$ for functions $f(x)$ which are Taylor-expandable.

   (c) Show that a representation of the above algebra in terms of operators acting on smooth functions $g(x)$ is given by

   $$(Ag)(x) = \frac{dg}{dx}\,, \qquad (Bg)(x) = x\, g(x)\,,$$

   i.e. $A = d/dx$ and $B = x$. Show then that d/dx is the infinitesimal generator for translations for any multiplication operator $f(x)$, i.e.

   $$\exp\left(a\frac{d}{dx}\right)\, f(x)\, \exp\left(-a\frac{d}{dx}\right) = f(x + a)\,.$$

2. Consider the a generic representation of the Poincaré group, with generators $J_{\mu\nu}$ and P_μ. Define the two operators

   $$P^2 \equiv P_\mu P^\mu\,, \qquad W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\, P^\nu\, J^{\rho\sigma}\,.$$

   Using the commutation rules of the Lie algebra of the Poincaré group, show that P^2 and $W^2 \equiv W_\mu W^\mu$ commute with all the generators of the Poincaré group.

3. Let $\Omega(\Lambda, a)$ be a Poincaré transformation, in an arbitrary representation with generators $J_{\mu\nu}$ and P_μ. Show that

   $$\Omega(\Lambda, a) J_{\mu\nu}\, \Omega^{-1}(\Lambda, a) = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \left(J_{\rho\sigma} - a_\rho P_\sigma + a_\sigma P_\rho\right)\,,$$
   $$\Omega(\Lambda, a) P_\mu\, \Omega^{-1}(\Lambda, a) = \Lambda^\rho{}_\mu P_\rho\,.$$

   Hint. Consider the product $\Omega(\Lambda, a)\, \Omega(\bar{\Lambda}, \epsilon)\, \Omega^{-1}(\Lambda, a)$, where $\Omega(\bar{\Lambda}, \epsilon)$ is an infinitesimal Poincaré transformation.

4. Consider the operators

$$M_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\,, \quad P_\mu = -i\partial_\mu\,,$$

acting on smooth functions $f(x)$. Note that $[x_\mu, P_\nu] = i\eta_{\mu\nu}$.

(a) Show that $[P_\mu, P_\nu] = 0$. Hence show that the Pauli-Ljubanski vector $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\, P^\nu\, M^{\rho\sigma} = 0$ vanishes in this representation, i.e. $W_\mu = 0$.

(b) Compute the remaining commutators $[M_{\mu\nu}, M_{\tau\sigma}]$, $[M_{\mu\nu}, P_\rho]$ and $[P_\mu, P_\nu]$ and compare the result with the definition of the Poincaré algebra given in the lectures. Conclude that $M_{\mu\nu}$ and P_ν give a representation thereof.