Symmetry in Particle Physics, Problem Sheet 4 [SOLUTIONS]

1. Consider operators or matrices A and B which obey $[A, B] = 1$.

   (a) Show that $e^{aA}\, B\, e^{-aA} = B + a$ for a some real or complex number. Hint. Recall that, if we define

   $$f(A) \equiv \sum_{n=0}^{\infty} f_n A^n \,, \qquad f'(A) \equiv \sum_{n=0}^{\infty} n f_n A^{n-1} \,,$$

   we have

   $$[f(A), B] = f'(A)[A, B] \,,$$

   From a direct computation, and using the hint, we find

   $$e^{aA}\, B\, e^{-aA} = B\, \underbrace{e^{aA}\, e^{-aA}}_{=1} + \underbrace{[e^{aA}, B]}_{=a\, e^{aA}\, [A,B]}\; e^{-aA} = B + a \,.$$

   (b) Use the result above to show that $e^{aA}\, f(B)\, e^{-aA} = f(B + a)$ for functions $f(x)$ which are Taylor-expandable.
   Given the Taylor expansion of $f(B)$

   $$f(B) = \sum_{n=0}^{\infty} f_n B^n \,,$$

   what we need to show is that, for every n, we have

   $$e^{aA}\, B^n\, e^{-aA} = (B + a)^n \,.$$

   We can just insert in each factor of the above product a unit matrix as $e^{-aA}\, e^{aA}$ to get

   $$e^{aA}\, B^n\, e^{-aA} = \left(e^{aA}\, B\, e^{-aA}\right)^n = (B + a)^n \,.$$

   (c) Show that a representation of the above algebra in terms of operators acting on smooth functions $g(x)$ is given by

   $$(Ag)(x) = \frac{dg}{dx} \,, \qquad (Bg)(x) = x\, g(x) \,,$$

   i.e. $A = d/dx$ and $B = x$. Show then that d/dx is the infinitesimal generator for translations for any multiplication operator $f(x)$, i.e.

   $$\exp\left(a \frac{d}{dx}\right)\, f(x)\, \exp\left(-a \frac{d}{dx}\right) = f(x + a) \,.$$

First of all, we show that $[A,B]=1$. In fact, for each smooth function $g(x)$ we have

$$\left[\frac{d}{dx},x\right]g(x)=\frac{d}{dx}xg(x)-x\frac{dg}{dx}=g(x)=1\cdot g(x)\,.$$

The desired identity follows from the application of the result of part (b).

2. Consider the a generic representation of the Poincaré group, with generators $J_{\mu\nu}$ and P_μ. Define the two operators

$$P^2\equiv P_\mu P^\mu\,,\qquad W_\mu=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,P^\nu\,J^{\rho\sigma}\,.$$

Using the commutation rules of the Lie algebra of the Poincaré group, show that P^2 and $W^2\equiv W_\mu W^\mu$ commute with all the generators of the Poincaré group.

Let us consider P^2 first. Since $[P_\mu,P_\nu]=0$, then also $[P^2,P_\mu]=0$. For the commutator with $J_{\mu\nu}$ we have

$$\begin{aligned}[P^2,J_{\mu\nu}]&=P_\rho[P^\rho,J_{\mu\nu}]+[P_\rho,J_{\mu\nu}]P^\rho\\&=-iP_\rho(\eta^\rho_\mu P_\nu-\eta^\rho_\nu P_\mu)-i(\eta_{\mu\rho}P_\nu-\eta_{\nu\rho}P_\mu)P^\rho=-2i[P_\mu,P_\nu]=0\,.\end{aligned}$$

We now consider W^2. First, we observe that

$$[W_\mu,P_\tau]=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,P^\nu\,[J^{\rho\sigma},P_\tau]\,.$$

But this is zero, because $[J^{\rho\sigma},P_\tau]$ is proportional either to P_ρ or P_σ, and the multiplication of either with P^ν gives zero, once it is contracted with $\epsilon_{\mu\nu\rho\sigma}$. Therefore, also $[W^2,P_\tau]=0$.

3. Let $\Omega(\Lambda,a)$ be a Poincaré transformation, in an arbitrary representation with generators $J_{\mu\nu}$ and P_μ. Show that

$$\begin{aligned}&\Omega(\Lambda,a)J_{\mu\nu}\,\Omega^{-1}(\Lambda,a)=\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu\left(J_{\rho\sigma}-a_\rho P_\sigma+a_\sigma P_\rho\right)\,,\\&\Omega(\Lambda,a)P_\mu\,\Omega^{-1}(\Lambda,a)=\Lambda^\rho{}_\mu P_\rho\,.\end{aligned}$$

Hint. Consider the product $\Omega(\Lambda,a)\,\Omega(\bar\Lambda,\epsilon)\,\Omega^{-1}(\Lambda,a)$, where $\Omega(\bar\Lambda,\epsilon)$ is an infinitesimal Poincaré transformation.

From the multiplication rules of the Poincaré group:

$$\begin{aligned}\Omega(\Lambda,a)\,\Omega(\bar\Lambda,\epsilon)\,\Omega^{-1}(\Lambda,a)&=\Omega(\Lambda,a)\,\Omega(\bar\Lambda,\epsilon)\,\Omega(\Lambda^{-1},-\Lambda^{-1}a)\\&=\Omega(\Lambda,a)\,\Omega(\bar\Lambda\Lambda^{-1},\epsilon-\bar\Lambda\Lambda^{-1}a)\\&=\Omega(\Lambda\bar\Lambda\Lambda^{-1},\Lambda\epsilon+\Lambda(1-\bar\Lambda)\Lambda^{-1}a)\,.\end{aligned}$$

Since $\Omega(\bar\Lambda,\epsilon)$ is infinitesimal we have

$$\bar\Lambda^\mu{}_{nu}=\eta^\mu{}_\nu+\omega^\mu{}_\nu\implies\Omega(\bar\Lambda,\epsilon)=\mathbb{1}+\frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}+i\epsilon^\mu P_\mu\,.$$

This gives

$$\Omega(\Lambda,a)\left(\mathbb{1}+\frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}+i\epsilon^{\mu}P_{\mu}\right)\Omega^{-1}(\Lambda,a)\,,$$

and

$$\begin{aligned}\Omega(\Lambda\bar{\Lambda}\Lambda^{-1},\Lambda\epsilon+\Lambda(1-\bar{\Lambda})\Lambda^{-1}a)&=\Omega(\Lambda(\mathbb{1}+\omega)\Lambda^{-1},\Lambda\epsilon-\Lambda\omega\Lambda^{-1}a)\\&=\mathbb{1}+\frac{i}{2}(\Lambda\omega\Lambda^{-1})^{\mu\nu}J_{\mu\nu}+i(\Lambda\epsilon-\Lambda\omega\Lambda^{-1}a)^{\mu}P_{\mu}\,.\end{aligned}$$

Comparing the terms containing ω and ϵ in the two expansions, we get

$$\begin{aligned}&\omega^{\mu\nu}\Omega(\Lambda,a)\,J_{\mu\nu}\,\Omega^{-1}(\Lambda,a)=(\Lambda\omega\Lambda^{-1})^{\mu\nu}J_{\mu\nu}-2(\Lambda\omega\Lambda^{-1}a)^{\mu}P_{\mu}\,,\\&\epsilon^{\mu}\Omega(\Lambda,a)\,P_{\mu}\,\Omega^{-1}(\Lambda,a)=(\Lambda\epsilon)^{\mu}P_{\mu}\,.\end{aligned}$$

We deal first with the second of these equations:

$$\epsilon^{\mu}\Omega(\Lambda,a)\,P_{\mu}\,\Omega^{-1}(\Lambda,a)=(\Lambda\epsilon)^{\mu}P_{\mu}=\Lambda^{\nu}{}_{\mu}\epsilon^{\mu}P_{\nu}\implies\Omega(\Lambda,a)\,P_{\mu}\,\Omega^{-1}(\Lambda,a)=\Lambda^{\nu}{}_{\mu}P_{\nu}\,.$$

4. Consider the operators

$$M_{\mu\nu}=-i(x_{\mu}\partial_{\nu}-x_{\nu}\partial_{\mu})\,,\quad P_{\mu}=-i\partial_{\mu}\,,$$

acting on smooth functions $f(x)$. Note that $[x_{\mu},P_{\nu}]=i\eta_{\mu\nu}$.

(a) Show that $[P_{\mu},P_{\nu}]=0$. Hence show that the Pauli-Ljubanski vector $W_{\mu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^{\nu}M^{\rho\sigma}=0$ vanishes in this representation, i.e. $W_{\mu}=0$.
Two derivatives always commute, hence $[P_{\mu},P_{\nu}]=0$. Then, we observe that

$$M_{\mu\nu}=x_{\mu}P_{\nu}-x_{\nu}P_{\mu}\,.$$

This gives

$$\begin{aligned}W_{\mu}&=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^{\nu}M^{\rho\sigma}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left(-P^{\nu}x^{\rho}P^{\sigma}+P^{\nu}P^{\rho}x^{\sigma}\right)\\&=\epsilon_{\mu\rho\nu\sigma}x^{\rho}[P^{\nu},P^{\sigma}]+\epsilon_{\mu\nu\rho\sigma}[P^{\nu},P^{\rho}]x^{\sigma}\,.\end{aligned}$$

But this is zero because P_{μ} commute.

(b) Compute the remaining commutators $[M_{\mu\nu},M_{\tau\sigma}]$, $[M_{\mu\nu},P_{\rho}]$ and $[P_{\mu},P_{\nu}]$ and compare the result with the definition of the Poincaré algebra given in the lectures. Conclude that $M_{\mu\nu}$ and P_{ν} give a representation thereof.
From a direct computation

$$[M_{\mu\nu},P_{\rho}]=[x_{\mu},P_{\rho}]P_{\nu}-[x_{\nu},P_{\rho}]P_{\mu}=i(\eta_{\mu\rho}P_{\nu}-\eta_{\nu\rho}P_{\mu})\,.$$

Similarly

$$[M_{\mu\nu}, x_\rho] = i(\eta_{\mu\rho} x_\nu - \eta_{\nu\rho} x_\mu)\,.$$

Therefore, using the fundamental properties of the commutator, we get

$$[M_{\mu\nu}, x_\tau P_\sigma] = [M_{\mu\nu}, x_\tau] P_\sigma + x_\tau [M_{\mu\nu}, P_\sigma] = i(\eta_{\mu\tau} x_\nu - \eta_{\nu\tau} x_\mu) P_\sigma + i x_\tau (\eta_{\mu\sigma} P_\nu - \eta_{\nu\sigma} P_\mu)\,.$$

This gives

$$\begin{aligned}
[M_{\mu\nu}, M_{\tau\sigma}] &= [M_{\mu\nu}, x_\tau P_\sigma] - [M_{\mu\nu}, x_\sigma P_\tau] \\
&= i(\eta_{\mu\tau} x_\nu - \eta_{\nu\tau} x_\mu) P_\sigma + i x_\tau (\eta_{\mu\sigma} P_\nu - \eta_{\nu\sigma} P_\mu) \\
&- i(\eta_{\mu\sigma} x_\nu - \eta_{\nu\sigma} x_\mu) P_\tau - i x_\sigma (\eta_{\mu\tau} P_\nu - \eta_{\nu\tau} P_\mu) \\
&= i(\eta_{\mu\tau} M_{\nu\sigma} - \eta_{\nu\tau} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\tau} - \eta_{\nu\sigma} M_{\mu\tau})\,,
\end{aligned}$$

which are the commutation rules for the generators of the Lorentz group.