Symmetry in Particle Physics, Problem Sheet 3

1. Consider a Lie group G. Then, for any vector X in the Lie algebra of G, and for any $g \in G$, consider the map
$$D(g)X = gXg^{-1}\,,$$
   (a) Show that $D(g)X$ is an element of the Lie algebra of G.
   Hint: use the relation
$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots$$
   (b) Show that $D(g)$ is a representation of G into the linear operators over the Lie algebra of G.

   (c) Compute the generators of D and show that D is the adjoint representation.

2. Consider an infinitesimal Lorentz transformation
$$\Lambda^\mu_\nu = \eta^\mu_\nu + \omega^\mu_\nu\,, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}\,.$$
   (a) Show that ω^μ_ν can be written in the form
$$\omega^\mu_\nu = \frac{i}{2}\omega^{\rho\sigma}(M_{\rho\sigma})^\mu_\nu\,,$$
   where $M_{\rho\sigma}$ are the generators of Lorentz transformations, given by
$$(M_{\rho\sigma})^\mu_\nu = i\left(\eta^\mu_\sigma \eta_{\nu\rho} - \eta^\mu_\rho \eta_{\nu\sigma}\right)\,.$$
   (b) Using the explicit form of $M_{\mu\nu}$, compute the commutation rules
$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta\mu\sigma M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\right)\,.$$

3. The three 4×4 matrices $\vec{K}$ – the anti-hermitian boost generators – are defined as $K_k = M_{0k}$ with non-vanishing matrix elements $(K_j)_{0k} = (K_j)_{k0} = i\delta_{jk}$.

   (a) Show that $(iK_i)^2$ is a projector, and that $(iK_i)^{2n+1} = iK_i$.

   (b) Compute the Lorentz boost matrix $\exp(iu_i\,K_i)$ in terms of $(\hat{u}_i\,K_i)$, $(\hat{u}_i\,K_i)^2$. Here $\vec{u} = (u_1, u_2, u_3)$ is an arbitrary three-vector, $\hat{u} = \vec{u}/u$, where $u = |\vec{u}|$. [10]
   Compare your result to the case of a boost along the 1-direction in its standard form
$$\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$
   From that comparison, determine the relation between v and u.

4. Consider the 2×2 matrices $\sigma_\mu = (\sigma_0, \sigma_i)$, with σ_0 the 2×2 identity matrix and σ_i the Pauli matrices. For a space-time coordinate x^μ consider the matrix

$$\hat{x} = x^\mu \, \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} . \tag{1}$$

(a) Show that every complex hermitian 2×2 matrix M can be written in the form (1) for some real x^μ.

(b) Show that $\det \hat{x} = x^\mu \, x_\mu$, and that this implies

$$x^\mu \, y_\mu = \frac{1}{4} \left[\det(\hat{x} + \hat{y}) - \det(\hat{x} - \hat{y})\right] .$$

(c) Consider the matrices $\bar{\sigma}_\mu = (\sigma_0, -\sigma_i)$, and establish the identities

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\eta_{\mu\nu}\mathbb{1} \quad \text{and} \quad \mathrm{Tr}(\sigma_\mu \bar{\sigma}_\nu) = 2\eta_{\mu\nu} \, .$$

Hint. Recall the properties of the Pauli matrices

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \, , \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\sigma_0 \, .$$

(d) Show that, for a complex 2×2 matrix M with unit determinant, $M \in SL(2, \mathbb{C})$,

$$\hat{x}' = M \, \hat{x} \, M^\dagger$$

can be written in the form (1) with $x'^\mu = \Lambda^\mu_{\ \nu} \, x^\nu$, with

$$\Lambda^\mu_{\ \nu} = \frac{1}{2} \mathrm{Tr} \left(\bar{\sigma}^\mu \, M \, \sigma_\nu \, M^\dagger \right) , \tag{2}$$

and $\bar{\sigma}^\mu = \eta^{\mu\nu} \bar{\sigma}_\nu$.

(e) By considering the above relations for the matrices $\hat{x}' = M \, \hat{x} \, M^\dagger$ and $\hat{y}' = M \, \hat{y} \, M^\dagger$, show that

$$\eta_{\mu\nu} \, x'^\mu \, y'^\nu = \eta_{\mu\nu} \, x^\mu \, y^\nu \, ,$$

i.e. the matrix $\Lambda_\mu^{\ \nu}$ of part (d) corresponds to a Lorentz transformation. Then, show that it also corresponds to a proper orthochronous Lorentz transformation.

Remark. From Eq. (2) you can conclude that there is a *unique* Lorentz transformation matrix $\Lambda \in SO(3, 1)$ for every matrix $M \in SL(2, \mathbb{C})$. On the other side, M and $-M$ lead to the *same* matrix Λ. In fact, there is an isomorphism from $SL(2, \mathbb{C})/\mathbb{Z}_2$ (the set of complex 2×2 matrices with unit determinant, with M and $-M$ identified) to the orthochronous Lorentz group $L_+^\uparrow$ consisting of matrices that conserve the metric tensor with $\Lambda^0_{\ 0} > 0$ and $\det \Lambda = 1$.

5. Consider the matrices

$$\Omega_L = \exp\left[\frac{i}{2}(\alpha_i - i\beta_i)\sigma_i\right] , \qquad \Omega_R = \exp\left[\frac{i}{2}(\alpha_i + i\beta_i)\sigma_i\right] ,$$

where α_i and β_i are real parameters, and σ_i the three Pauli matrices.

(a) Show that

$$\Omega_L^{-1} = \Omega_R^\dagger\,, \qquad \Omega_R^{-1} = \Omega_L^\dagger$$

(b) Using the fact that

$$\sigma_2 \Omega_L \sigma_2 = \Omega_R^*\,.$$

show that

$$\sigma_2 \Omega^T \sigma_2 \Omega = \mathbb{1}\,,$$

and if ψ transforms according to some representation of $SL(2,\mathbb{C})$, then $\sigma_2\psi^*$ transforms according to the conjugate representation.

(c) Show that the generators of the $(1/2,0)$ and $(1/2,0)$ representations are

$$\Sigma^L_{\mu\nu} \equiv M^{(1/2,0)}_{\mu\nu} = \frac{i}{4}\left(\bar\sigma_\mu \sigma_\nu - \bar\sigma_\nu \sigma_\mu\right)\,,$$
$$\Sigma^R_{\mu\nu} \equiv M^{(0,1/2)}_{\mu\nu} = \frac{i}{4}\left(\sigma_\mu \bar\sigma_\nu - \sigma_\nu \bar\sigma_\mu\right)\,.$$