Symmetry in Particle Physics, Problem Sheet 3 [SOLUTIONS]

1. Consider a Lie group G. Then, for any vector X in the Lie algebra of G, and for any $g \in G$, consider the map
$$D(g)X = gXg^{-1}\,,$$

   (a) Show that $D(g)X$ is an element of the Lie algebra of G.
   Hint: use the relation
$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots$$
   Each element $g \in G$ can be written as $\exp[Y]$, with Y an element of the Lie algebra of G. Therefore
$$e^Y X e^{-Y} = X + [Y, X] + \frac{1}{2!}[Y, [Y, X]] + \ldots$$
   Since the Lie algebra is closed under commutation, all commutators in the above equation are elements of the Lie algebra.

   (b) Show that $D(g)$ is a representation of G into the linear operators over the Lie algebra of G.
   First, $D(g)$ is linear due to the linearity properties of the commutator. Consider now two elements $g_1, g_2 \in G$. Then
$$D(g_1 g_2^{-1})X = (g_1 g_2^{-1})X(g_1 g_2)^{-1} = g_1(g_2^{-1} X g_2)g_1^{-1} = D(g_1)[D(g_2^{-1})X] = [D(g_1)D(g_2^{-1})]X\,.$$

   (c) Compute the generators of D and show that D is the adjoint representation.
   Let us consider a matrix $D(g)$ that is close to the identity. Then
$$e^Y X e^{-Y} \simeq X + [Y, X]\,.$$
   We now need to work out the commutator $[Y, X]$, using that $Y = i\alpha_a X_a$, and $X = \beta_a X_a$, where X_a are the generators of the Lie group:
$$[Y, X] = i\alpha_a\beta_b[X_a, X_b] = i\alpha_a\beta_b(if_{abc}X_c) = i\alpha_a\beta_b(\underbrace{-if_{acb}}_{=(T_a)_{cb}}X_c) = i\alpha_a[(T_a)_{bc}X_b]X_c\,,$$
   where $(T_a)_{bc}$ are precisely the generators of the adjoint representation.

2. Consider an infinitesimal Lorentz transformation
$$\Lambda^\mu_\nu = \eta^\mu_\nu + \omega^\mu_\nu\,, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}\,.$$

(a) Show that ω^μ_ν can be written in the form

$$\omega^\mu_\nu = \frac{i}{2}\omega^{\rho\sigma}(M_{\rho\sigma})^\mu_\nu ,$$

where $M_{\rho\sigma}$ are the generators of Lorentz transformations, given by

$$(M_{\rho\sigma})^\mu_\nu = i\left(\eta^\mu_\sigma \eta_{\nu\rho} - \eta^\mu_\rho \eta_{\nu\sigma}\right) .$$

From a direct computation

$$\omega^\mu_\nu = \eta^\mu_\rho \eta_{\nu\sigma}\omega^{\rho\sigma} = \frac{1}{2}\omega^{\rho\sigma}(\eta^\mu_\rho \eta_{\nu\sigma} - \eta^\mu_\sigma \eta_{\nu\rho}) = \frac{i}{2}\omega^{\rho\sigma}\left[i(\eta^\mu_\sigma \eta_{\nu\rho} - \eta^\mu_\rho \eta_{\nu\sigma})\right]$$

(b) Using the explicit form of $M_{\mu\nu}$, compute the commutation rules

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta\mu\sigma M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\right) .$$

From the definition of the commutator

$$[M_{\mu\nu}, M_{\rho\sigma}]^\alpha_\beta = (M_{\mu\nu})^\alpha_\gamma (M_{\rho\sigma})^\gamma_\beta - (M_{\rho\sigma})^\alpha_\gamma (M_{\mu\nu})^\gamma_\beta$$

We work out the first term of the commutator, keeping track of imaginary units

$$(M_{\mu\nu})^\alpha_\gamma (M_{\rho\sigma})^\gamma_\beta = i\left[i\left(\eta^\alpha_\nu \eta_{\gamma\mu} - \eta^\alpha_\mu \eta_{\gamma\nu}\right)\left(\eta^\gamma_\sigma \eta_{\beta\rho} - \eta^\gamma_\rho \eta_{\beta\sigma}\right)\right]$$

$$i\left[\eta_{\mu\sigma}(\underbrace{i\eta^\alpha_\nu \eta_{\beta\rho}}_{\to(-M_{\nu\rho})} + \eta_{\nu\sigma}(\underbrace{-i\eta^\alpha_\mu \eta_{\beta\rho}}_{\to M_{\mu\rho}}) + \eta_{\mu\rho}(\underbrace{-i\eta^\alpha_\nu \eta_{\beta\sigma}}_{\to M_{\nu\sigma}}) + \eta_{\nu\rho}(\underbrace{i\eta^\alpha_\mu \eta_{\beta\sigma}}_{\to(-M_{\mu\sigma})})\right]$$

where the remaining part of the generators are completed by the other half of the commutator.

3. The three 4×4 matrices $\vec{K}$ – the anti-hermitian boost generators – are defined as $K_k = M_{0k}$ with non-vanishing matrix elements $(K_j)_{0k} = (K_j)_{k0} = i\delta_{jk}$.

(a) Show that $(iK_i)^2$ is a projector, and that $(iK_i)^{2n+1} = iK_i$.

We perform an explicit calculation for iK_1, and the result holds for any $i = 1, 2, 3$.

$$iK_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow (iK_1)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

This implies that $[(iK_1)^2]^2 = (iK_1)^2$, hence $(iK_1)^2$ is a projector.

(b) Compute the Lorentz boost matrix $\exp(iu_i\,K_i)$ in terms of $(\hat{u}_i\,K_i)$, $(\hat{u}_i\,K_i)^2$. Here $\vec{u} = (u_1, u_2, u_3)$ is an arbitrary three-vector, $\hat{u} = \vec{u}/u$, where $u = |\vec{u}|$. [10]

Since the choice of the axes is arbitrary, we can set the 1-axis along $\hat{u}$. Therefore, we have that $(i\hat{u}_i\,K_i)^2$ is a projector and $(i\hat{u}_i\,K_i)^{2n+1} = i\hat{u}_i\,K_i$.

This gives

$$\begin{aligned}\exp(iu_i\,K_i) &= \sum_{n=0}^{\infty}\frac{u^n}{n!}(i\hat{u}_i\,K_i)^n \\ &= \mathbf{1} + \underbrace{\sum_{n=0}^{\infty}\frac{u^{2n+1}}{(2n+1)!}}_{=\sinh(u)}(i\hat{u}_i\,K_i) + \underbrace{\sum_{n=1}^{\infty}\frac{u^{2n}}{(2n)!}}_{=\cosh u-1}(i\hat{u}_i\,K_i)^2\end{aligned}$$

Compare your result to the case of a boost along the 1-direction in its standard form

$$\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

From that comparison, determine the relation between v and u.

For a boost along the 1-direction $\vec{u} = (u, 0, 0)$. Therefore

$$i\hat{u}_i K_i = iK_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This gives

$$\exp(iu\,K_1) = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Comparing with the given matrix we get

$$v = \frac{\gamma v}{\gamma} = \frac{\sinh u}{\cosh u} = \tanh u\,.$$

4. Consider the 2×2 matrices $\sigma_\mu = (\sigma_0, \sigma_i)$, with σ_0 the 2×2 identity matrix and σ_i the Pauli matrices. For a space-time coordinate x^μ consider the matrix

$$\hat{x} = x^\mu\,\sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{1}$$

(a) Show that every complex hermitian 2×2 matrix M can be written in the form (1) for some real x^μ.

The matrices σ_μ are a basis for complex hermitian 2×2 matrices. Hence, each matrix M can be written as $M = x^\mu \sigma_\mu$. Since both M and σ_μ are hermitian, we have

$$M^\dagger = (x^\mu)^* \sigma_\mu = M = x^\mu \sigma_\mu \implies (x^\mu)^* = x^\mu \, .$$

Hence the numbers x^μ are real.

(b) Show that $\det \hat{x} = x^\mu \, x_\mu$, and that this implies

$$x^\mu \, y_\mu = \frac{1}{4} \left[\det(\hat{x} + \hat{y}) - \det(\hat{x} - \hat{y}) \right] \, .$$

From a direct calculation

$$\det \hat{x} = (x^0 + x^3)(x^0 - x^3) - (x^1 - i x^2)(x^1 + i x^2) = (x^0)^2 - (x^3)^2 - (x^1)^2 - (x^2)^2 = x^\mu \, x_\mu \, .$$

We then compute

$$x^\mu \, y_\mu = \frac{1}{4} \left[(x+y)^2 - (x-y)^2 \right] = \frac{1}{4} \left[\det(\hat{x} + \hat{y}) - \det(\hat{x} - \hat{y}) \right] \, .$$

(c) Consider the matrices $\bar{\sigma}_\mu = (\sigma_0, -\sigma_i)$, and establish the identities

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2 \eta_{\mu\nu} \mathbb{1} \quad \text{and} \quad \mathrm{Tr}(\sigma_\mu \bar{\sigma}_\nu) = 2 \eta_{\mu\nu} \, .$$

Hint. Recall the properties of the Pauli matrices

$$[\sigma_i, \sigma_j] = 2 i \epsilon_{ijk} \sigma_k \, , \qquad \{\sigma_i, \sigma_j\} = 2 \delta_{ij} \sigma_0 \, .$$

From a direct computation

$$\begin{aligned}
\sigma_0 \bar{\sigma}_0 + \sigma_0 \bar{\sigma}_0 &= 2 \sigma_0^2 = 2 \times \mathbb{1} = 2 \eta_{00} \mathbb{1} \, , \\
\sigma_0 \bar{\sigma}_i + \sigma_i \bar{\sigma}_0 &= -\sigma_i + \sigma_i = 0 = 2 \eta_{0i} \mathbb{1} \, , \\
\sigma_i \bar{\sigma}_j + \sigma_j \bar{\sigma}_i &= -\{\sigma_i, \sigma_j\} = -2 \delta_{ij} \mathbb{1} = 2 \eta_{ij} \mathbb{1} \, .
\end{aligned}$$

Similarly, from a direct computation we get

$$\begin{aligned}
\mathrm{Tr}(\sigma^0 \bar{\sigma}^0) &= \mathrm{Tr}(\sigma_0^2) = 2 = 2 \eta_{00} \\
\mathrm{Tr}(\sigma_0 \bar{\sigma}_i) &= -\mathrm{Tr}(\sigma_i) = 0 = 2 \eta_{0i} \\
\mathrm{Tr}(\sigma^i \bar{\sigma}^i) &= -\frac{1}{2} \mathrm{Tr}(\{\sigma_i, \sigma_j\}) = -2 = 2 \eta_{ij} \, .
\end{aligned}$$

(d) Show that, for a complex 2×2 matrix M with unit determinant, $M \in SL(2,\mathbb{C})$,

$$\hat{x}' = M\,\hat{x}\,M^\dagger$$

can be written in the form (1) with $x'^\mu = \Lambda^\mu_\nu\, x^\nu$, with

$$\Lambda^\mu_\nu = \frac{1}{2}\mathrm{Tr}\left(\bar{\sigma}^\mu\, M\,\sigma_\nu\, M^\dagger\right)\,, \tag{2}$$

and $\bar{\sigma}^\mu = \eta^{\mu\nu}\bar{\sigma}_\nu$.

First, we check that $\hat{x}'$ is hermitian. In fact

$$(\hat{x}')^\dagger = (M^\dagger)^\dagger\,\hat{x}^\dagger\,M = M\,\hat{x}\,M^\dagger = \hat{x}'\,.$$

As such it can be written as $\hat{x}' = x'^\mu\sigma_\mu$ with x'^μ real. Then, we can write

$$\hat{x}' = x'^\rho\sigma_\rho = \left(M\,\sigma_\nu\,M^\dagger\right)\,x^\nu\,.$$

We now multiply each term to the left by $\bar{\sigma}_\mu$ and take the trace, thus obtaining

$$x'^\rho\,\underbrace{\mathrm{Tr}(\bar{\sigma}^\mu\sigma_\rho)}_{=2\eta^\mu_\rho} = \mathrm{Tr}\left(\bar{\sigma}^\mu\,M\,\sigma_\nu\,M^\dagger\right)\,x^\nu\,.$$

This gives

$$x'_\mu = \frac{1}{2}\mathrm{Tr}\left(\bar{\sigma}^\mu\,M\,\sigma_\nu\,M^\dagger\right)x_\nu = \Lambda_\mu{}^\nu x_\nu\,.$$

(e) By considering the above relations for the matrices $\hat{x}' = M\,\hat{x}\,M^\dagger$ and $\hat{y}' = M\,\hat{y}\,M^\dagger$, show that

$$\eta_{\mu\nu}\,x'^\mu\,y'^\nu = \eta_{\mu\nu}\,x^\mu\,y^\nu\,,$$

i.e. the matrix $\Lambda_\mu{}^\nu$ of part (d) corresponds to a Lorentz transformation. Then, show that it also corresponds to a proper orthochronous Lorentz transformation.

From a direct computation

$$\begin{aligned}\eta_{\mu\nu}\,x'^\mu\,y'^\nu &= \frac{1}{4}\left[\det(\hat{x}'+\hat{y}') - \det(\hat{x}'-\hat{y}')\right]\\ &= \frac{1}{4}\left[\det(M(\hat{x}+\hat{y}))M^\dagger) - \det(M(\hat{x}-\hat{y}))M^\dagger)\right]\\ &= \frac{1}{4}\left[\det(\hat{x}+\hat{y}) - \det(\hat{x}-\hat{y})\right] = \eta_{\mu\nu}\,x^\mu\,y^\nu\,.\end{aligned}$$

To show that Λ^μ_ν is a proper Lorentz transformation we need to show that $\det\Lambda = 1$. *Setting $M = \mathbb{1}$ leads to $\Lambda^\mu_\nu = \eta^\mu_\nu$. Any matrix $M \in SL(2,\mathbb{C})$ can be obtained from the identity by varying parameters continuously. Hence, also a generic Λ^μ_ν can be obtained from η^μ_ν by varying parameters continuously. Since* $\det\Lambda$ *is a continuous function of parameters, it can only assume the value*

+1, and hence $\Lambda^\mu_{\ \nu}$ is a proper Lorentz transformation. Similarly, Λ^0_0 is also a continuous function of the parameters, hence it has to stay in the same connected components as the identity, i.e. $\Lambda^0_0 \geq 1$, which implies $\Lambda^\mu_{\ \nu} \in L^\uparrow_+$.

Remark. From Eq. (2) you can conclude that there is a *unique* Lorentz transformation matrix $\Lambda \in SO(3,1)$ for every matrix $M \in SL(2,\mathbb{C})$. On the other side, M and $-M$ lead to the *same* matrix Λ. In fact, there is an isomorphism from $SL(2,\mathbb{C})/\mathbb{Z}_2$ (the set of complex 2×2 matrices with unit determinant, with M and $-M$ identified) to the orthochronous Lorentz group $L^\uparrow_+$ consisting of matrices that conserve the metric tensor with $\Lambda^0{}_0 > 0$ and $\det \Lambda = 1$.

5. Consider the matrices

$$\Omega_L = \exp\left[\frac{i}{2}(\alpha_i - i\beta_i)\sigma_i\right], \qquad \Omega_R = \exp\left[\frac{i}{2}(\alpha_i + i\beta_i)\sigma_i\right],$$

where α_i and β_i are real parameters, and σ_i the three Pauli matrices.

(a) Show that

$$\Omega_L^{-1} = \Omega_R^\dagger, \qquad \Omega_R^{-1} = \Omega_L^\dagger$$

From a direct computation

$$\Omega_L^{-1} = \exp\left[-\frac{i}{2}(\alpha_i - i\beta_i)\sigma_i\right] = \left(\exp\left[\frac{i}{2}(\alpha_i + i\beta_i)\sigma_i\right]\right)^\dagger = \Omega_R^\dagger .$$

Using the above, we have

$$\Omega_R^{-1} = \left((\Omega_R^{-1})^\dagger\right)^\dagger = \Omega_L^\dagger .$$

(b) Using the fact that

$$\sigma_2 \Omega_L \sigma_2 = \Omega_R^* .$$

show that

$$\sigma_2 \Omega^T \sigma_2 \Omega = \mathbb{1} ,$$

and if ψ transforms according to some representation of $SL(2,\mathbb{C})$, then $\sigma_2\psi^*$ transforms according to the conjugate representation.

Suppose $\Omega = \Omega_L$. Then

$$\sigma_2 \Omega_L^T \sigma_2 \Omega_L = (\Omega_R^*)^T \Omega_L = \Omega_R^\dagger \Omega_L = \Omega_L^{-1}\Omega_L = \mathbb{1} .$$

Consider now ψ that transforms as

$$\psi \to \Omega_L \psi \implies \sigma_2\psi^* \to \sigma_2 \Omega_L^* \psi^* = \sigma_2(\sigma_2 \Omega_R \sigma_2)\psi^* = \Omega_R(\sigma_2\psi^*) .$$

(c) Show that the generators of the $(1/2, 0)$ and $(1/2, 0)$ representations are

$$\Sigma^L_{\mu\nu} \equiv M^{(1/2,0)}_{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu\right) ,$$
$$\Sigma^R_{\mu\nu} \equiv M^{(0,1/2)}_{\mu\nu} = \frac{i}{4}\left(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu\right) .$$

For left-handed spinors

$$J_i = \frac{\sigma_i}{2}\,, \quad K_i = -i\frac{\sigma_i}{2}\,.$$

Similarly, for right-handed spinors

$$J_i = \frac{\sigma_i}{2}\,, \quad K_i = i\frac{\sigma_i}{2}$$

For boosts, we have

$$\Sigma^L_{0i} = \frac{i}{4}(-\sigma_i - \sigma_i) = -i\frac{\sigma_i}{2}\,,$$
$$\Sigma^R_{0i} = \frac{i}{4}(\sigma_i + \sigma_i) = i\frac{\sigma_i}{2}\,.$$

For rotations

$$\Sigma^L_{ij} = \Sigma^R_{ij} = \frac{i}{4}(\underbrace{-\sigma_i\sigma_j + \sigma_j\sigma_i}_{=-2i\epsilon_{ijk}}) = \epsilon_{ijk}\frac{\sigma_k}{2}\,.$$