Symmetry in Particle Physics, Problem Sheet 2

1. Show that a representation D of a Lie group is unitary if and only if the generators X_a are hermitian.

2. Consider the elements of an algebra X_a, with commutator
$$[X_a, X_b] = i\, f_{abc}\, X_c\,.$$
Show that if this algebra generates a unitary representation, then the structure constants f_{abc} are real.

3. Analytic functions of operators (matrices) A are defined via their Taylor expansion about $A = 0$. Consider the function
$$g(x) = \exp(xA)\, B\, \exp(-xA)\,,$$
where x is real and A, B are operators.

   (a) Compute the derivatives $d^n g(x)/dx^n$ for integer n, and simplify the result using the convention $[A, B] = AB - BA$.

   (b) Using the result of part (a), show that
$$e^A\, B\, e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots$$

4. We want to prove the Baker-Campbell-Hausdorff formula in a situation where the operators (matrices) A, B have the property $[A, [A, B]] = 0 = [B, [A, B]]$.

   (a) Show that $[A^n, B] = nA^{n-1}[A, B]$ for integer $n \geq 1$.

   (b) Use the above to show that, for analytic functions $f(x)$,
$$[f(A), B] = f'(A)\, [A, B]\,.$$
Show then that, if x is a c-number (i.e. a real or complex number), we have
$$[B, \exp(-Ax)] = \exp(-Ax)[A, B]x\,.$$

   (c) Consider the function $f(x) = \exp(xA) \exp(xB)$ and, using the result of part (b), show that it obeys the differential equation
$$\frac{df(x)}{dx} = (A + B + [A, B]\, x)\, f(x)\,.$$
Compute $f(x)$ by solving the above equation with an appropriate initial condition, and use the result to deduce the Baker-Campbell-Hausdorff formula for this case, i.e.
$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}\,.$$
Note: $f(x)$ is not in general invertible, so the equation has to be solved using an ansatz.

5. Compute the dimension of the group $SU(N)$.

6. Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) Using the conventions $[A, B] = AB - BA$, $\{A, B\} = AB + BA$, show that the matrices $\frac{\sigma_i}{2}$ are a two-dimensional representation of the $SU(2)$ algebra,

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\,\epsilon_{ijk}\,\frac{\sigma_k}{2},$$

   and $\epsilon_{123} = +1$. Show also that

$$\left\{\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right\} = \frac{\sigma_0}{2}\,\delta_{ij},$$

   with $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ denoting the two-dimensional identity matrix.

   (b) Consider the $SU(2)$ group element $G = \exp(\frac{i}{2}\theta_3\sigma_3)$ with parameter θ_3. Show by explicit computation that

$$\exp\left(\frac{i\theta_3}{2}\sigma_3\right) = \sigma_0 \cdot \cos(\theta_3/2) + i\,\sigma_3\, \cdot \sin(\theta_3/2)$$

   (c) A general $SU(2)$ group element is written as $G = \exp(\frac{i}{2}\theta_k\sigma_k)$ with parameters θ_k. Show, either by explicit computation or with the help of part (b) and symmetry arguments, that

$$\exp\left(\frac{i\theta_k}{2}\sigma_k\right) = \sigma_0 \cdot \cos(\theta/2) + i\,(\hat{\theta}_k\sigma_k) \cdot \sin(\theta/2)\,.$$

   Here, $\hat{\theta}_k = \theta_k/\theta$ is the unit vector in the θ_k-direction, and $\theta \equiv |\theta|$.
   Perform the above group transformation using $\theta = 2\pi$ and $\theta = 4\pi$, respectively. What does this tell us about the relation between $SU(2)$ and $SO(3)$?

7. The spin-1 representation of $SU(2)$ with generators T_1, T_2, T_3 satisfying $[T_i, T_j] = i\,\epsilon_{ijk}T_k$ reads

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

   Consider the corresponding $SU(2)$ group element $\Omega(\theta_i) = \exp(i\theta_i T_i)$ with parameters θ_i $i = 1, 2, 3$. Here $\hat{\theta}_i = \theta_i/|\theta|$ is the unit vector in the θ_i-direction, and $\theta \equiv |\theta|$.

(a) Show with a direct computation, or with symmetry arguments, that the matrix $\hat{\theta}_i T_i$ must have eigenvalues ± 1 and 0.

(b) Use the result of part (a) to show that the square $(\hat{\theta}_i T_i)^2$ is a projection operator (i.e. $(\hat{\theta}_i T_i)^4 = (\hat{\theta}_i T_i)^2$) and

$$(\hat{\theta}_i T_i)^3 = \hat{\theta}_i T_i \,.$$

Show then that $\Omega(\theta_i)$ is the 3×3 matrix

$$\Omega(\theta_i) = \mathbb{1} + i\,(\hat{\theta}_i T_i) \cdot \sin\theta + (\hat{\theta}_i T_i)^2 \cdot (\cos\theta - 1) \,,$$

where $\mathbb{1}$ is the 3-dimensional identity matrix.

8. Given a three-dimensional vector $\vec{v} = (v_1, v_2, v_3)$, we construct the 2×2 matrix $\bar{v} = v_i \sigma_i$, with $\sigma_i, i = 1, 2, 3$ the three Pauli matrices, as follows

$$\bar{v} = \begin{pmatrix} v_3 & v_1 - i v_2 \\ v_1 + i v_2 & -v_3 \end{pmatrix}$$

(a) Show that $\vec{v}^2 = -\det(\bar{v})$. Then show that, for any two vectors $\vec{v}$ and $\vec{w}$,

$$\vec{v} \cdot \vec{w} = \frac{1}{4} \left[\det(\bar{v} - \bar{w}) - \det(\bar{v} + \bar{w}) \right] \,.$$

(b) Using the properties of Pauli matrices, show that, for any matrix $U \in SU(2)$, the matrix

$$\bar{v}' = U\, \bar{v}\, U^\dagger \,,$$

can be written in the form $\bar{v}' = v'_i \sigma_i$, where

$$v'_i = \Omega_{ij}\, v_j \,, \qquad \Omega_{ij} = \frac{1}{2} \mathrm{Tr} \left[\sigma_i U \sigma_j U^\dagger \right] \,.$$

Hint. Any 2×2 *complex matrix* M *can be written as* $M = M_0 \mathbb{1} + M_i \sigma_i$.

(c) Show that Ω is an orthogonal transformation, i.e. if $\vec{v}' = \Omega \vec{v}$ and $\vec{w}' = \Omega \vec{w}$, then $\vec{v}' \cdot \vec{w}' = \vec{v} \cdot \vec{w}$.