Symmetry in Particle Physics, Problem Sheet 2 [SOLUTIONS]

1. Show that a representation D of a Lie group is unitary if and only if the generators X_a are hermitian.

   Let $D(\alpha) = \exp(i\alpha_a X_a)$. D is unitary if and only if

   $$D^{-1}(\alpha) = \exp(-i\alpha_a X_a) = D^\dagger(\alpha) = \exp(-i\alpha_a X_a^\dagger)\,.$$

   Since this holds for any value of α, it is possible if and only if $X_a = X_a^\dagger$.

2. Consider the elements of an algebra X_a, with commutator

   $$[X_a, X_b] = i\, f_{abc}\, X_c\,.$$

   Show that if this algebra generates a unitary representation, then the structure constants f_{abc} are real.

   From a direct computation

   $$([X_a, X_b])^\dagger = [X_b^\dagger, X_a^\dagger] = -i\, f_{abc}^*\, X_c^\dagger\,.$$

   However, if the X_a's generate a unitary representation, then $X_a^\dagger = X_a$. Using commutation rules, we have

   $$[X_b^\dagger, X_a^\dagger] = [X_b, X_a] = -[X_a, X_b] = -i\, f_{abc}\, X_c\,.$$

   Comparing with the result of the direct computation, we get

   $$[X_b^\dagger, X_a^\dagger] = -i\, f_{abc}\, X_c = -i\, f_{abc}^*\, X_c \implies f_{abc}^* = f_{abc}\,.$$

3. Analytic functions of operators (matrices) A are defined via their Taylor expansion about $A = 0$. Consider the function

   $$g(x) = \exp(xA)\, B\, \exp(-xA)\,,$$

   where x is real and A, B are operators.

   (a) Compute the derivatives $d^n g(x)/dx^n$ for integer n, and simplify the result using the convention $[A, B] = AB - BA$.

   From a direct computation we get

   $$\frac{d}{dx}\exp(xA) = A\,\exp(xA) = \exp(xA)\,A\,.$$

This suggests the following ansatz

$$\frac{d^n g(x)}{dx^n} = \exp(xA) \underbrace{[A, [A, \dots, [A}_{n \text{ times}}, B] \dots]] \exp(-xA) .$$

The latter equality can be shown by induction. For $n = 1$ we have

$$\frac{dg(x)}{dx} = \exp(xA)(AB - BA) \exp(-xA) = \exp(xA)[A, B] \exp(-xA) .$$

Assuming the statement is true for n, we show that it holds for $n+1$. In fact

$$\begin{aligned}
\frac{d^{n+1} g(x)}{dx^{n+1}} = \frac{d}{dx}\frac{d^n g(x)}{dx^n} = \frac{d}{dx} \exp(xA) \underbrace{[A, [A, \dots, [A}_{n \text{ times}}, B] \dots]] \exp(-xA) \\
= \exp(xA) A \underbrace{[A, [A, \dots, [A}_{n \text{ times}}, B] \dots]] \exp(-xA) \\
- \exp(xA) \underbrace{[A, [A, \dots, [A}_{n \text{ times}}, B] \dots]] A \exp(-xA) \\
= \exp(xA) \underbrace{[A, [A, \dots, [A}_{n+1 \text{ times}}, B] \dots]] \exp(-xA)
\end{aligned}$$

(b) Using the result of part (a), show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots$$

$$e^A B e^{-A} = g(1) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n g(x)}{dx^n} \right|_{x=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A, [A, \dots, [A}_{n \text{ times}}, B] \dots]] ,$$

which is exactly what we need to show.

4. We want to prove the Baker-Campbell-Hausdorff formula in a situation where the operators (matrices) A, B have the property $[A, [A, B]] = 0 = [B, [A, B]]$.

(a) Show that $[A^n, B] = nA^{n-1}[A, B]$ for integer $n \geq 1$.

We prove the statement by induction. It is true for $n = 1$. Assuming it is valid for n, we prove it for $n + 1$. In fact

$$[A^{n+1}, B] = A^{n+1}B - BA^{n+1} = A\,A^n B - BA^n\,A = A\,[A^n, B] + [A, B]\,A^n .$$

Since $[A, [A, B]] = 0$, we have $[A, B]\,A^n = A^n\,[A, B]$. This gives, using the induction hypothesis

$$\begin{aligned}
[A^{n+1}, B] &= A^{n+1}B - BA^{n+1} = A(A^n B - BA^n) + AB\,A^n - BA\,A^n \\
&= A[A^n, B] + [A, B]A^n = \left(A\,(nA^{n-1}) + A^n\right)[A, B] = (n+1)A^n[A, B] .
\end{aligned}$$

(b) Use the above to show that, for analytic functions $f(x)$,

$$[f(A), B] = f'(A)\,[A, B]\,.$$

Using the Taylor expansion of $f(A)$, we get

$$f(A) = \sum_{n=0}^{\infty} f_n\, A^n\,.$$

This, and the linearity of the commutator, implies

$$[f(A), B] = \sum_{n=0}^{\infty} f_n\,[A^n, B] = \sum_{n=0}^{\infty} f_n(nA^{n-1})\,[A, B] = f'(A)\,[A, B]\,.$$

Show then that, if x is a c-number (i.e. a real or complex number), we have

$$[B, \exp(-Ax)] = \exp(-Ax)[A, B]x\,.$$

From a direct computation, defining $f(A) \equiv \exp(-Ax)$, we have

$$[B, \exp(-Ax)] = -[\exp(-Ax), B] = -(-x)\exp(-Ax)\,[A, B] = \exp(-Ax)[A, B]x\,.$$

(c) Consider the function $f(x) = \exp(xA)\exp(xB)$ and, using the result of part (b), show that it obeys the differential equation

$$\frac{df(x)}{dx} = (A + B + [A, B]\,x)\,f(x)\,.$$

Differentiating $f(x)$ we get

$$\frac{df(x)}{dx} = Af(x) + f(x)B\,.$$

Let us concentrate on the latter term:

$$\begin{aligned} f(x)B &= \exp(xA)\exp(xB)B = \exp(xA)B\exp(-xA)\ \exp(xA)\ \exp(xB) \\ &= Bf(x) + \exp(xA)[B, \exp(-xA)]\,f(x)\,. \end{aligned}$$

Using the result of part (b) we get

$$[B, \exp(-xA)] = x\exp(-xA)[A, B]\,,$$

which leads immediately to the desired equation.

Compute $f(x)$ by solving the above equation with an appropriate initial condition, and use the result to deduce the Baker-Campbell-Hausdorff formula for this case, i.e.

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} .$$

Note: $f(x)$ is not in general invertible, so the equation has to be solved using an ansatz.

Let us take as initial condition $f(0) = 1$. Then

$$f(x) = \exp((A+B)x + \frac{x^2}{2}[A,B])$$

is the solution of the differential equation with the appropriate initial condition. This gives

$$f(1) = e^{A+B+\frac{1}{2}[A,B]} = e^{A+B} e^{\frac{1}{2}[A,B]} ,$$

where the last equality holds because $[A,B]$ commutes with both A and B.

5. Compute the dimension of the group $SU(N)$.

   The number of real parameter in any $N \times N$ complex matrix is $2N^2$.

   Each $M \in SU(N)$ satisfies $MM^\dagger = \mathbb{1}$. This involves N real constraints on the diagonal of the identity matrix (each corresponding to constraining the magnitude of a complex number) and $2N(N-1)/2 = N^2 - N$ real constraints from the off-diagonal terms (the remaining off-diagonal terms can be obtained by complex conjugation, so they do not give rise to addional constraints).

   The condition $\det M = 1$ gives another constraint. In total, the number of real parameters minus the number of constraints is $2N^2 - N^2 - 1 = N^2 - 1$.

6. Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

   (a) Using the conventions $[A,B] = AB - BA$, $\{A,B\} = AB + BA$, show that the matrices $\frac{\sigma_i}{2}$ are a two-dimensional representation of the $SU(2)$ algebra,

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\, \epsilon_{ijk} \frac{\sigma_k}{2} ,$$

   and $\epsilon_{123} = +1$. Show also that

$$\left\{\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right\} = \frac{\sigma_0}{2}\, \delta_{ij} ,$$

   with $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ denoting the two-dimensional identity matrix.

From a direct computation

$$\sigma_1\sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \sigma_2\sigma_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

This gives

$$[\sigma_1, \sigma_2] = 2i\sigma_3\,, \qquad \{\sigma_1, \sigma_2\} = 0\,.$$

Similar computations for the other matrices lead to

$$[\sigma_2, \sigma_3] = 2i\sigma_1\,, \qquad \{\sigma_2, \sigma_3\} = 0\,.$$
$$[\sigma_3, \sigma_1] = 2i\sigma_2\,, \qquad \{\sigma_3, \sigma_1\} = 0\,.$$

Since $\sigma_i^2 = 1$*, we have obtained all the requested relations.*

(b) Consider the $SU(2)$ group element $G = \exp(\frac{i}{2}\theta_3\sigma_3)$ with parameter θ_3. Show by explicit computation that

$$\exp\left(\frac{i\theta_3}{2}\sigma_3\right) = \sigma_0 \cdot \cos(\theta_3/2) + i\,\sigma_3\; \cdot \sin(\theta_3/2)$$

From the anticommutation relations of part (a) we have

$$\sigma_3^{2n} = 1\,, \qquad \sigma_3^{2n+1} = \sigma_3\,.$$

It makes sense to separate even and odd powers of σ_3 *in the expansion of* G*, as follows*

$$\begin{aligned} G &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\theta_3\right)^n \sigma_3^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta_3}{2}\right)^{2n} \underbrace{\sigma_3^{2n}}_{=1} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\theta_3}{2}\right)^{2n+1} \underbrace{\sigma_3^{2n+1}}_{=\sigma_3} \\ &= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta_3}{2}\right)^{2n}}_{=\cos(\theta_3/2)} + i\sigma_3 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\theta_3}{2}\right)^{2n+1}}_{=\sin(\theta_3/2)} \\ &= \sigma_0 \cdot \cos(\theta_3/2) + i\,\sigma_3\; \cdot \sin(\theta_3/2)\,. \end{aligned}$$

(c) A general $SU(2)$ group element is written as $G = \exp(\frac{i}{2}\theta_k\sigma_k)$ with parameters θ_k. Show, either by explicit computation or with the help of part (b) and symmetry arguments, that

$$\exp\left(\frac{i\theta_k}{2}\sigma_k\right) = \sigma_0 \cdot \cos(\theta/2) + i\,(\hat{\theta}_k\sigma_k) \cdot \sin(\theta/2)\,.$$

Here, $\hat{\theta}_k = \theta_k/\theta$ is the unit vector in the θ_k-direction, and $\theta \equiv |\theta|$.

The proposed solution uses an explicit computation. Using the anticommutation relations of the Pauli matrices

$$(\theta_k \sigma_k)^2 = (\theta_i \sigma_i)(\theta_j \sigma_j) = \theta_i \theta_j \sigma_i \sigma_j = \frac{1}{2}\theta_i \theta_j \{\sigma_i, \sigma_j\} = \theta_i \theta_j \delta_{ij} = \theta^2 \,.$$

This gives

$$\exp\left(\frac{i\theta_k}{2}\sigma_k\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} \underbrace{(\hat{\theta}_k\sigma_k)^{2n}}_{=1} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\theta}{2}\right)^{2n+1} \underbrace{(\hat{\theta}_k\sigma_k)^{2n+1}}_{=\hat{\theta}_k\sigma_k}$$
$$= \sigma_0 \cdot \cos(\theta/2) + i\,(\hat{\theta}_k \sigma_k) \cdot \sin(\theta/2) \,.$$

Perform the above group transformation using $\theta = 2\pi$ and $\theta = 4\pi$, respectively. What does this tell us about the relation between $SU(2)$ and $SO(3)$?

From a direct computation

$$\exp\left(\frac{i(2\pi)\hat{\theta}_k}{2}\sigma_k\right) = \cos(\pi) + i\,(\hat{\theta}_k\sigma_k) \cdot \sin(\pi) = -\sigma_0 \,,$$
$$\exp\left(\frac{i(4\pi)\hat{\theta}_k}{2}\sigma_k\right) = \cos(2\pi) + i\,(\hat{\theta}_k\sigma_k) \cdot \sin(2\pi) = \sigma_0 \,.$$

This tells us that, in spite of the fact that the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic, the two groups cannot have the same representations. In fact, a rotation of 2π, which belongs to the fundamental representation of $SO(3)$, is the identity, and so has to be in any representation of $SO(3)$. We have just found a representation of $SU(2)$ for which this is not the case.

7. The spin-1 representation of $SU(2)$ with generators T_1, T_2, T_3 satisfying $[T_i, T_j] = i\,\epsilon_{ijk} T_k$ reads

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider the corresponding $SU(2)$ group element $\Omega(\theta_i) = \exp(i\theta_i T_i)$ with parameters θ_i $i = 1, 2, 3$. Here $\hat{\theta}_i = \theta_i/|\theta|$ is the unit vector in the θ_i-direction, and $\theta \equiv |\theta|$.

(a) Show with a direct computation, or with symmetry arguments, that the matrix $\hat{\theta}_i T_i$ must have eigenvalues ± 1 and 0.

Since the choice of the 3-axis is arbitrary, we can choose to set it along the direction $\hat{\theta}$, and therefore $\hat{\theta}_i T_i$ becomes the matrix T_3, hence its eigenvalues are ± 1 and 0.

(b) Use the result of part (a) to show that the square $(\hat{\theta}_i T_i)^2$ is a projection operator (i.e. $(\hat{\theta}_i T_i)^4 = (\hat{\theta}_i T_i)^2$) and

$$(\hat{\theta}_i T_i)^3 = \hat{\theta}_i T_i \,.$$

From part (a) we know that there exists a basis in which $\hat{\theta}_i T_i$ has the same form as T_3. Hence the two properties follow from a direct calculation in that basis.

Show then that $\Omega(\theta_i)$ is the 3×3 matrix

$$\Omega(\theta_i) = \mathbb{1} + i\,(\hat{\theta}_i T_i) \cdot \sin\theta + (\hat{\theta}_i T_i)^2 \cdot (\cos\theta - 1) \;,$$

where $\mathbb{1}$ is the 3-dimensional identity matrix.

In the series expansion of $\Omega(\theta_i)$, we divide odd and even powers of $\hat{\theta}_i T_i$ as follows

$$\begin{aligned} \Omega(\theta_i) &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} (\hat{\theta}_i T_i)^n \\ &= \mathbb{1} + \sum_{n=1}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} \underbrace{(\hat{\theta}_i T_i)^{2n}}_{=(\hat{\theta}_i T_i)^2} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \underbrace{(\hat{\theta}_i T_i)^{2n+1}}_{=\hat{\theta}_i T_i} \\ &= \mathbb{1} + \underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!}}_{=\cos\theta - 1} (\hat{\theta}_i T_i)^2 + i \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}}_{=\sin\theta} (\hat{\theta}_i T_i) \,, \end{aligned}$$

which is what we needed to show.

8. Given a three-dimensional vector $\vec{v} = (v_1, v_2, v_3)$, we construct the 2×2 matrix $\bar{v} = v_i \sigma_i$, with $\sigma_i, i = 1, 2, 3$ the three Pauli matrices, as follows

$$\bar{v} = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}$$

(a) Show that $\vec{v}^{\,2} = -\det(\bar{v})$. Then show that, for any two vectors $\vec{v}$ and $\vec{w}$,

$$\vec{v} \cdot \vec{w} = \frac{1}{4} \left[\det(\bar{v} - \bar{w}) - \det(\bar{v} + \bar{w}) \right] \,.$$

From a direct computation

$$-\det(\bar{v}) = v_3^2 + (v_1 - iv_2)(v_1 + iv_2) = v_1^2 + v_2^2 + v_3^2 = \vec{v}^{\,2} \,.$$

Also,

$$\vec{v} \cdot \vec{w} = \frac{1}{4} \left[(\vec{v} + \vec{w})^2 - (\vec{v} - \vec{w})^2 \right] = \frac{1}{4} \left[\det(\bar{v} - \bar{w}) - \det(\bar{v} + \bar{w}) \right] \,.$$

(b) Using the properties of Pauli matrices, show that, for any matrix $U \in SU(2)$, the matrix

$$\bar{v}' = U\,\bar{v}\,U^\dagger\,,$$

can be written in the form $\bar{v}' = v'_i \sigma_i$, where

$$v'_i = \Omega_{ij}\,v_j\,, \qquad \Omega_{ij} = \frac{1}{2}\mathrm{Tr}\left[\sigma_i U \sigma_j U^\dagger\right]\,.$$

Hint. Any 2×2 *complex matrix* M *can be written as* $M = M_0 \mathbb{1} + M_i \sigma_i$.

The 2×2 *matrix* $\bar{v}'$ *can be written in the form*

$$\bar{v}' = v'_0 \mathbb{1} + v'_i \sigma_i\,,$$

where

$$v'_0 = \frac{1}{2}\mathrm{Tr}(\bar{v}') = \frac{1}{2}\mathrm{Tr}(U\,\bar{v}\,U^\dagger) = \frac{1}{2}\mathrm{Tr}(U^\dagger\,U\,\bar{v}) = \frac{1}{2}\mathrm{Tr}(\bar{v}) = 0\,.$$

Then $\bar{v}' = v'_i \sigma_i$. *Also, since* $\mathrm{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$, *we have*

$$v'_i = \frac{1}{2}\mathrm{Tr}(\sigma_i\,U\,\bar{v}\,U^\dagger) = \frac{1}{2}\mathrm{Tr}(\sigma_i\,U\,\sigma_j\,U^\dagger)\,v_j = \Omega_{ij}\,v_j\,.$$

(c) Show that Ω is an orthogonal transformation, i.e. if $\vec{v}' = \Omega\vec{v}$ and $\vec{w}' = \Omega\vec{w}$, then $\vec{v}' \cdot \vec{w}' = \vec{v} \cdot \vec{w}$.

From a direct computation, for any matrix $\bar{u} = u_i \sigma_i$, *and* $\bar{u}' = U\,\bar{u}\,U^\dagger$, *we have*

$$\det(\bar{u}') = \det\left(U\,\bar{u}\,U^\dagger\right) = (\underbrace{\det U}_{=1})\,\det(\bar{u})(\underbrace{\det U^\dagger}_{=1}) = \det(\bar{u})\,.$$

This gives

$$\vec{v}' \cdot \vec{w}' = \frac{1}{4}\left[\det(\bar{v}' - \bar{w}') - \det(\bar{v}' + \bar{w}')\right] = \frac{1}{4}\left[\det(\bar{v} - \bar{w}) - \det(\bar{v} + \bar{w})\right] = \vec{v} \cdot \vec{w}\,.$$