Symmetry in Particle Physics, Problem Sheet 10 [SOLUTIONS]

1. Consider the following Lagrangian for a real scalar field ϕ and a Dirac spinor ψ:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) + i\bar{\psi}\gamma^\mu\partial_\mu\psi - g\phi\bar{\psi}\psi\,, \qquad V(\phi) \equiv \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4\,, \quad \lambda > 0\,,$$

(a) Show that the Lagrangian is invariant under the transformation

$$\phi \to -\phi\,, \qquad \psi \to i\gamma^5\psi\,.$$

The part of the Lagrangian containing only the scalar field is invariant because it is an even function of ϕ.

For the part that contains the fermions, we need first the transformation for $\bar{\psi}$, which is

$$\bar{\psi} = \psi^\dagger\gamma^0 \to (i\gamma^5\psi)^\dagger\gamma^0 = -i\psi^\dagger\gamma^5\gamma^0 = i\psi^\dagger\gamma^0\gamma^5 = i\bar{\psi}\gamma^5\,.$$

Therefore

$$\bar{\psi}\gamma^\mu\psi \to (i\bar{\psi}\gamma^5)\gamma^\mu(i\gamma^5\psi) = -i^2\bar{\psi}(\gamma^5)^2\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi\,.$$

Similarly

$$\phi\bar{\psi}\psi \to (-\phi)(i\bar{\psi}\gamma^5)(i\gamma^5\psi) = \phi\bar{\psi}\psi\,.$$

(b) Ground state configurations for a scalar field are those who have the minimun energy. As such, they have no kinetic energy and minimise the potential. Find the ground state configurations ϕ_0 for $\mu^2 \geq 0$. What particles (spin and masses) does the Lagrangian describe?

The ground state configurations are those that minimise the potential. For $\mu^2 \geq 0$ the potential is positive definite, hence the only minimum is for $\phi_0 = 0$.

The scalar field describes a particle of spin 0. From the quartic part of the Lagrangian we read the mass squared of the particle which is μ^2.

The part containing the Dirac field describes one massless Dirac particle, or equivalently four massless particles, two with helicity 1/2, and two with helicity -1/2.

Assume that $\mu^2 < 0$ and show that the ground state configurations obey

$$\phi_0 = \pm v\,, \quad \text{with } v \equiv \sqrt{-\frac{m^2}{\lambda}}\,.$$

We now set to zero the first derivative of the potential:

$$V'(\phi_0) = m^2\phi_0 + \lambda\phi_0^3 = 0 \qquad \Leftrightarrow \qquad \phi_0 = 0, \pm\sqrt{-\frac{m^2}{\lambda}} \equiv \pm v\,.$$

Comparing the value of the potential for the two cases we get

$$V(v) = \frac{v^2}{2}\left(m^2 + \frac{\lambda}{2}v^2\right) = m^2\frac{v^2}{4} < 0 = V(0)\,.$$

Expand the scalar field ϕ around the vacuum configuration v as follows:

$$\phi(x) = v + h(x)\,,$$

where $h(x)$ is a new scalar field. Rewrite the Lagrangian in terms of $h(x)$. What particles (spin and masses) does the theory describe in this case?

Since v is a constant $\partial_\mu h = \partial_\mu \phi$, so the kinetic part of the Lagrangian for the scalar field is $\partial_\mu h \partial^\mu h/2$.

The potential becomes

$$V(v+h) = \frac{1}{2}\mu^2(v+h)^2 + \frac{\lambda}{4}(v+h)^4 = \underbrace{\frac{1}{2}\mu^2 v^2 + \frac{\lambda}{4}v^4}_{constant\,\to\,drop} + \underbrace{\left(\mu^2 + \lambda v\right)}_{=0} h$$
$$+\frac{1}{2}\underbrace{\left(\mu^2 + 3\lambda v^2\right)}_{\equiv m^2} h^2 + \lambda v h^3 + \frac{\lambda}{4}h^4 = const. + V(h)\,.$$

From the above expression we see that the Lagrangian describes a particle of spin zero, with a mass squared

$$m^2 = \mu^2 + 3\lambda v^2 = \mu^2 + 3\lambda\left(-\frac{\mu^2}{\lambda}\right) = -2\mu^2\,.$$

Consider now the part of the Lagrangian containing the Dirac field. Expanding the field ϕ around v we obtain

$$\mathcal{L} \supset -gv\bar{\psi}\psi$$

This a Dirac mass term with

$$m_D = gv\,.$$

Therefore, the Lagrangian describes a Dirac fermion with mass m_D.

Note that, in this case, the symmetry is spontaneously broken because the vacuum configuration is not invariant under the symmetry. Since the symmetry is discrete, no massless scalars are expected.

2. Consider the following Lagrangian for a complex scalar field ϕ and a Dirac field ψ:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\phi^*\partial^\mu\phi - V(\phi^*\phi) - g(\phi\,\bar{\psi}_R\,\psi_L + \phi^*\,\bar{\psi}_L\,\psi_R)\,,$$

where ψ_L and ψ_R are the left- and right-handed components of ψ, defined by

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi\,, \qquad \psi_R = \frac{1}{2}(1+\gamma^5)\psi\,,$$

and

$$V(\phi^*\phi) = \mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2$$

with $\lambda > 0$. The above Lagrangian is invariant under the global chiral transformation

$$\psi_R \to e^{i\alpha}\psi_R\,, \qquad \psi_L \to e^{-i\alpha}\psi_L\,, \qquad \phi \to e^{2i\alpha}\phi$$

(a) Consider the case $\mu^2 \geq 0$. Show that the ground state of the theory corresponds to the field configuration $\phi = 0$. What particles (masses and spin) does this theory describe?

The ground state corresponds to the configurations with the smallest energy, hence those constant configurations such that the potential is at a minimum.

Since $\lambda > 0$ and $\mu^2 \geq 0$, the potential is positive definite, hence its only minimum is for $\phi = 0$.

Looking at the quadratic part of the Lagrangian for the scalar field

$$\mathcal{L} \supset (\partial_\mu\phi^*)(\partial^\mu\phi) - \mu^2\phi^*\phi\,,,$$

we see that it describes a particle of spin-0 and (tree-level) mass μ, and its antiparticle.

The Lagrangian for the Dirac field describes a left-handed and a right-handed massless particle of spin 1/2, as well as their antiparticles. Since the masses are zero in both cases, the Lagrangian in fact describes two massless particles with helicity 1/2, and two massless particles with helicity -1/2.

(b) Consider now the case $\mu^2 < 0$. What are the field configurations corresponding to the ground state?

Taking the derivative of the potential with respect to ϕ we obtain

$$\frac{d}{d\phi}V(\phi^*\phi) = \phi^*\left(\mu^2 + \lambda(\phi^*\phi)^2\right)\,.$$

This is zero for

$$\phi = 0\,, \quad |\phi| = \sqrt{\frac{-\mu^2}{\lambda}}\,.$$

However, $\phi = 0$ is not the minimum of the potential, in fact

$$V(0) = 0 > V\left(\frac{-\mu^2}{\lambda}\right) = -\frac{\mu^4}{2\lambda}\,.$$

Therefore, there exists infinitely many vacuum configurations, those satisfying $|\phi| = \sqrt{-\mu^2/\lambda}$.

(c) What particles (spin and masses) does the theory describe in the case $\mu^2 < 0$? <u>Hint</u>. Write vacuum configurations ϕ_0 in the form $\phi_0 = e^{i\alpha_0} v$, with $v \geq 0$. Choose one of them, and expand the field around ϕ_0, as follows

$$\phi(x) = \frac{e^{i\alpha_0}}{\sqrt{2}}(v + h(x) + i\chi(x)) .$$

We choose a vacuum configuration ϕ_0 and expand the Lagrangian around that configuration to find its quadratic part. For instance, choosing

$$\phi_0 \frac{1}{\sqrt{2}}(v + h(x) + i\chi(x)) = \sqrt{-\mu^2/\lambda} \equiv \frac{v}{\sqrt{2}} \implies \phi(x) = \frac{1}{\sqrt{2}}(v + h(x) + i\chi(x))$$

we obtain, for the part containing ϕ,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu h - i\partial_\mu \chi)(\partial^\mu h + i\partial^\mu \chi) - \frac{\mu^2}{2}\left((v+h)^2 + \chi^2\right) - \frac{\lambda}{8}\left((v+h)^2 + \chi^2\right)^2 \\ &= \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - v\underbrace{\left(\mu^2 + \frac{\lambda v^2}{2}\right)}_{=\mu^2-\mu^2=0} h - \frac{1}{2}\underbrace{\left(\mu^2 + \frac{3\lambda^2 v^2}{2}\right)}_{\mu^2-3\mu^2=-2\mu^2} h^2 - \frac{1}{2}\underbrace{\left(\mu^2 + \frac{\lambda v^2}{2}\right)}_{=\mu^2-\mu^2=0}\chi^2 + \dots\end{aligned}$$

The above Lagrangian describes two particles of spin zero, one of mass $m = \sqrt{-2\mu^2}$ and the other of mass zero.
We now consider the part of the Lagrangian containing the interaction between the Dirac and the scalar field. Expanding the scalar field around its vacuum configuration, we obtain

$$\mathcal{L} \supset -g\frac{v}{\sqrt{2}}(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) = -g\frac{v}{\sqrt{2}}\bar{\psi}\psi .$$

This is a Dirac mass term with mass

$$m_D = g\frac{v}{\sqrt{2}} .$$

Therefore, the Lagrangian describes a massive Dirac fermion with mass m_D.
Note that, in this case, chiral symmetry is broken because a chiral transformation transforms one vacuum configuration into another vacuum configuration. Also, the fact that we have a massless boson of spin zero is expected in view of Goldstone theorem, which states that for each generator of a spontaneously broken continuous symmetry there exists a massless spin-0 boson.

3. Consider the following Lagrangian for two spinor fields ψ_1, ψ_2, and four scalar fields $\sigma, \pi_1, \pi_2, \pi_3$:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi + \frac{1}{2}(\partial^\mu \sigma)(\partial_\mu \sigma) + \frac{1}{2}(\partial^\mu \vec{\pi})(\partial_\mu \vec{\pi}) - g\bar{\psi}\left(\sigma - i(\vec{\pi}\cdot\vec{\sigma})\gamma^5\right)\psi - V(\sigma^2 + \vec{\pi}^2) ,$$

where we have introduced the doublet $\psi \equiv (\psi_1, \psi_2)^T$ of spinor fields and the triplet $\vec{\pi} \equiv (\pi_1, \pi_2, \pi_3)^T$ of scalar fields, and $\vec{\sigma}$ is a vector containing the three Pauli matrices.

(a) Introduce the matrix

$$\Sigma \equiv \sigma + i\vec{\pi}\cdot\vec{\sigma} = \begin{pmatrix} \sigma + i\pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & \sigma - i\pi_3 \end{pmatrix}$$