Symmetry in Particle Physics, Problem Sheet 1

1. Consider a group G. Then show the following:

   (a) for any $a, b, c \in G$, if $ab = ac$, then $b = c$ (cancellation rule);

   (b) the unit element is unique;

   (c) the inverse of any group element is unique;

   (d) $(g^{-1})^{-1} = g$.

2. Consider a field $(\mathbb{K}, +, \cdot)$ and let 0 be the unit of the + operation. Then show the following:

   (a) for any $x \in \mathbb{K}$, we have $0x = 0$;

   (b) for any $x, y \in \mathbb{K}$, $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$;

   (c) for any $x, y \in \mathbb{K}$, $x(-y) = (-x)y = -(xy)$.

   (d) $(-x)(-y) = xy$.

3. Show that D is a representation of a group G into a vector space V if and only if $D(g_1 g_2^{-1}) = D(g_1)D(g_2)^{-1}$.

4. Consider the following map $D : \mathbb{Z}_3 \to GL(3, \mathbb{C})$ given by

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

   Show that D is a representation of $\mathbb{Z}_3$.

5. Let V be a real vector space and g a scalar product. Let $\{e_i\}_{i=1,2,\ldots,n}$ be an orthonomal basis and let us define the matrix $g_{ij} = g(e_i, e_j) = \pm\delta_{ij}$. By construction, the matrix g is its own inverse, in fact $g^2 = \mathbb{1}$.

   (a) Consider a vector $u = u_i e_i \in V$. Show that $u_i = g_{ij} g(e_j, u)$.

   (b) Let M be an orthogonal operator, and let us define the matrix $M_{ij} = g(e_i, Me_j)$. Show that

$$M^T g M = g\,.$$

   where g is the matrix whose components are the g_{ij}.

6. Let $(A^\dagger)_{ij}$ be the matrix associated to the adjoint of the operator A. Show that $(A^\dagger)_{ij} = A^*_{ij}$.

7. Let U be an anti-unitary operator. Show that $U^\dagger = U^{-1}$.