Symmetry in Particle Physics, Problem Sheet 1, Solutions

1. Consider a group G. Then show the following:

   (a) for any $a, b, c \in G$, if $ab = ac$, then $b = c$ (cancellation rule);

   $$b = 1 \cdot b = a^{-1}(ab) = a^{-1}(ac) = c\,.$$

   (b) the unit element is unique;
   Suppose there exists another unit e'. Then, for any $g \in G$, we have

   $$g = eg = e'g \implies e = e'\,.$$

   (c) the inverse of any group element is unique;
   Consider an element g, with two inverses g^{-1} and $(g^{-1})'$. Then

   $$e = gg^{-1} = g(g^{-1})' \implies g^{-1} = (g^{-1})'\,.$$

   (d) $(g^{-1})^{-1} = g$.

   $$e = (g^{-1})^{-1}g^{-1} = gg^{-1} \implies (g^{-1})^{-1} = g\,.$$

2. Consider a field $(\mathbb{K}, +, \cdot)$ and let 0 be the unit of the $+$ operation. Then show the following:

   (a) for any $x \in \mathbb{K}$, we have $0x = 0$;
   Using the fact that 0 is a unit, the distributive property and the cancellation rule, we get

   $$0x + 0 = 0x = (0 + 0)x = 0x + 0x \implies 0x = 0\,.$$

   (b) for any $x, y \in \mathbb{K}$, $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$;
   Suppose $xy = 0$. Then, since both x and y have inverses, we have

   $$0 = (x^{-1}y^{-1})(xy) = 1\,.$$

   But this is not possible because by construction $1 \neq 0$. Hence it must be $xy \neq 0$.

(c) for any $x, y \in \mathbb{K}$, $x(-y) = (-x)y = -(xy)$.

We need to show that $x(-y) + (xy) = 0$. In fact

$$x(-y) + (xy) = x(-y + y) = x0 = 0\,.$$

Similarly

$$(-x)y + (xy) = (-x + x)y = 0y = 0\,.$$

(d) $(-x)(-y) = xy$.

Using the result of part (c), we find

$$(-x)(-y) = -[x(-y)] = -[-(xy)] = xy\,.$$

3. Show that D is a representation of a group G into a vector space V if and only if $D(g_1 g_2^{-1}) = D(g_1)D(g_2)^{-1}$.

   If D is a representation of a group G, then $D(g^{-1}) = D(g)^{-1}$ for any $g \in G$. Also, for any $g_1, g_2 \in G$

   $$D(g_1 g_2^{-1}) = D(g_1)D(g_2^{-1}) = D(g_1)D(g_2)^{-1}\,.$$

   Now we deal with the reversed implication. For any $g \in G$, we have

   $$D(e) = D(gg^{-1}) = D(g)D(g)^{-1} = \mathbb{1}\,.$$

   Also, for any $g_1, g_2 \in G$, we have

   $$D(g_1 g_2) = D\left(g_1 (g_2^{-1})^{-1}\right) = D(g_1)D(g_2^{-1})^{-1} = D(g_1)\left(D(g_2)^{-1}\right)^{-1} = D(g_1)D(g_2)\,.$$

4. Consider the following map $D : \mathbb{Z}_3 \to GL(3, \mathbb{C})$ given by

   $$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \qquad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\,, \qquad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\,.$$

   Show that D is a representation of $\mathbb{Z}_3$.

   By construction $D(e) = \mathbb{1}$. An explicit calculation gives $D(a)D(b) = \mathbb{1}$, which gives the multiplication table of $\mathbb{Z}_3$.

5. Let V be a real vector space and g a scalar product. Let $\{e_i\}_{i=1,2,\dots,n}$ be an orthonormal basis and let us define the matrix $g_{ij} = g(e_i, e_j) = \pm\delta_{ij}$. By construction, the matrix g is its own inverse, in fact $g^2 = \mathbb{1}$.

(a) Consider a vector $u = u_i e_i \in V$. Show that $u_i = g_{ij} g(e_j, u)$.

$$u = u_i e_i \implies g(e_j, u) = u_k g_{kj} \implies g_{ij} g(e_j, u) = u_k g_{ij} g_{jk} = u_k \delta_{ik} = u_i \,.$$

(b) Let M be an orthogonal operator, and let us define the matrix $M_{ij} = g(e_i, Me_j)$. Show that

$$M^T g M = g \,.$$

where g is the matrix whose components are the g_{ij}.

The statatement to prove is better understood by writing indexes separately.

$$g_{kl} M_{ki} M_{lj} = g_{ij} \,,$$

Since M is orthogonal, $g(Me_i, Me_j) = g(e_i, e_j) = g_{ij}$. Using the result of part (a), we have

$$Me_i = (Me_i)_l e_l = g_{kl} g(e_k, Me_i) e_l \,.$$

This gives, using the bilinearity of the scalar product,

$$g(Me_i, Me_j) = g_{kl} g(e_k, Me_i) g(e_l, Me_j) = g_{kl} M_{ki} M_{lj} = g_{ij} \,,$$

which is what we needed to show.

6. Let $(A^\dagger)_{ij}$ be the matrix associated to the adjoint of the operator A. Show that $(A^\dagger)_{ij} = A^*_{ij}$.

$$(A^\dagger)_{ij} \equiv \langle e_i | A^\dagger e_j \rangle = \langle A e_i | e_j \rangle = \langle e_j | A e_i \rangle^* = A^*_{ij} \,.$$

7. Let U be an anti-unitary operator. Show that $U^\dagger = U^{-1}$.

For any v, w in a Hilbert space, the unitarity of U implies

$$\langle Uv | Uw \rangle = \langle v | w \rangle^* \,.$$

From the definition of the adjoint of an anti-unitary operator, we have

$$\langle v | w \rangle^* = \langle Uv | Uw \rangle = \langle U^\dagger U v | w \rangle^* \implies U^\dagger U = \mathbb{1} \,.$$