The renormalization group and Weyl invariance

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FRGE I: DEFINITIONS

$$e^{-W_k[J]} = \int (D\phi) \exp(-(S + \Delta S_k + \int J\phi))$$

$$\Delta S_k(\phi) = \frac{1}{2} \int d^4 q \phi(-q) R_k(q^2) \phi(q)$$

with $R_k(q^2) \to 0$ for $k \to \infty$, $R_k(q^2) \to k^2$ for $k \to 0$

$$\Gamma_k[\phi] = W_k[J] - \int J\phi - \Delta S_k$$

note $\lim_{k\to 0} \Gamma_k = \Gamma$

FRGE II

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k \right)^{-1} \partial_t R_k$$

C. Wetterich, Phys. Lett. B **301** (1993) 90.

Trace is over momenta and all quantum numbers

FRGE III: One loop

$$\Gamma^{(1)} = S + \frac{1}{2}\ln\det\mathcal{O} \quad ; \qquad\qquad \mathcal{O} = \frac{\delta^2 S}{\delta\phi\delta\phi}$$

$$\Gamma_k^{(1)} = S + \Delta S_k + \frac{1}{2}\ln\det\mathcal{O}_k - \Delta S_k \quad ; \qquad \mathcal{O}_k = \frac{\delta^2(S + \Delta S_k)}{\delta\phi\delta\phi} = \mathcal{O} + R_k$$

$$\partial_t\Gamma_k^{(1)} = \frac{1}{2}(\det\mathcal{O}_k)^{-1}\partial_t\det\mathcal{O}_k = \frac{1}{2}\mathrm{Tr}\mathcal{O}_k^{-1}\partial_t\mathcal{O}_k$$

One–loop RG equation

$$\partial_t\Gamma_k^{(1)} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right)^{-1}\partial_t R_k$$

FRGE IV: Beta functions

$$\Gamma_k(\phi) = \sum_i g_i(k)\mathcal{O}_i(\phi)$$

$$\partial_t \Gamma_k = \sum_i \partial_t g_i \mathcal{O}_i = \sum_i \beta_{g_i} \mathcal{O}_i$$

compare with

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k \right)^{-1} \partial_t R_k$$

use to calculate Γ
or to investigate the UV properties of the theory

Dimensional analysis

Choose dimensionless coordinates.
Invariance under *global* Weyl transformations
$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, $\psi_a \to \Omega^{d_a}\psi_a$, $g_i \to \Omega^{d_i} g_i$

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{d_a}\psi_a, \Omega^{d_i} g_i)$$

fixes d_a, d_i. *ALWAYS* true.
Global scale transformations

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{d_a}\psi_a, g_i)$$

not always true, must be $d_i = 0$.

Weyl's idea

Interpret scale transformations as changes of the unit of length.

Allow choice of unit to depend on position.

"It is evident that two rods side by side, stationary with respect to each other, can be intercompared....this cannot be done for....rods with either a space- or time-like separation".

"A statement such as "a hydrogen atom on Sirius has the same diameter as one on the Earth" is either a definition or else meaningless" (Dicke 1962)

Physics must be formulated in a way that is invariant under local changes of units, i.e. under *local* rescalings of the metric.

Furthermore: allow parallel transport to affect norm of vectors.

Weyl gauging

Abelian gauge field $b_\mu \mapsto b_\mu + \Omega^{-1}\partial_\mu\Omega$
scalar field ϕ transforming as $\phi \to \Omega^d\phi$

$$D_\mu\phi = \partial_\mu\phi - db_\mu\phi$$

More generally

$$\hat{\Gamma}_\mu{}^\lambda{}_\nu = \Gamma_\mu{}^\lambda{}_\nu - \delta_\mu^\lambda b_\nu - \delta_\nu^\lambda b_\mu + g_{\mu\nu}b^\lambda$$

is invariant under local Weyl transformations, hence for a tensor of dimension d

$$D_\mu t = \hat{\nabla}_\mu t - db_\mu t$$

is diffeomorphism and Weyl covariant.

Weyl curvature

$$[D_\mu, D_\nu] v^\rho = \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma v^\sigma$$

$$\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} \quad = \quad & R_{\mu\nu\rho\sigma} + (d-1) F_{\mu\nu} g_{\rho\sigma} \\
& + g_{\mu\rho} \left(\nabla_\nu b_\sigma + b_\nu b_\sigma \right) - g_{\mu\sigma} \left(\nabla_\nu b_\rho + b_\nu b_\rho \right) \\
& - g_{\nu\rho} \left(\nabla_\mu b_\sigma + b_\mu b_\sigma \right) + g_{\nu\sigma} \left(\nabla_\mu b_\rho + b_\mu b_\rho \right) \\
& - \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) b^2
\end{aligned}$$

$$F_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$$

Weyl's failed unification idea

Weyl regarded b_μ as e.m. field.
Einstein's critique: spectral lines would be blurred.

Weyl's theory still viable if b_μ interpreted as a piece of the spacetime connection and its curvature is sufficiently weak.
Assume curvature of b_μ is zero. Then

$$b_\mu = -\chi^{-1}\partial_\mu \chi$$

where the field χ transforming as $\chi \to \Omega^{-1}\chi$ is....

...the dilaton

Starting from any action $S(g_{\mu\nu}, \psi_a, g_i)$, replace $g_i \to \chi^{d_i} \hat{g}_i$, $\nabla \to D$ and all curvatures $R \to \mathcal{R}$.

Resulting action $\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i)$ is Weyl invariant and agrees with original action in the gauge where χ is constant

$$\hat{S}(g_{\mu\nu}, \chi_0, \psi_a, \hat{g}_i) = S(g_{\mu\nu}, \psi_a, g_i)$$

Dilaton acts as *classical* compensator field (Stückelberg)

Motivation: scalar-tensor theory

$$\int d^4x\sqrt{g}\,\left[\frac{1}{2}(\nabla\phi)^2 + V(\phi^2) + F(\phi^2)R\right]$$

$V(\phi^2) = \lambda_0 + \lambda_2\phi^2 + \lambda_4\phi^4 + \dots$
$F(\phi^2) = \xi_0 + \xi_2\phi^2 + \dots$

$$\int d^4x\sqrt{g}\,\left[\lambda_4\phi^4 + \frac{1}{2}(\nabla\phi)^2 + \xi_2\phi^2 R\right]$$

Weyl-invariant for $\xi_2 = \frac{1}{12}$.
Not preserved by RG flow

D. Perini and R.P. Phys. Rev. **D68** 044018 (2003) hep-th/0304222;

G. Narain and R.P. Cl. Q. Grav. **27** 075001 (2010) arXiv:0911.0386 [hep-th])

The conformal anomaly

Cutoff breaks scale invariance, but anomaly can be avoided in theories with dilaton

Englert, C. Truffin and R. Gastmans, “Conformal invariance in quantum gravity”, Nucl. Phys. B117, 407 (1976)
R.Floreanini and R. P., “Average effective potential for the conformal factor”, Nucl. Phys. B436, 141 (1995)

M. Shaposhnikov and I. Tkachev, “Quantum scale invariance on the lattice”, Phys. Lett. B675, 403 (2009)

Example: matter in background metric and dilaton

$$S(\phi, g_{\mu\nu}) = \frac{1}{2} \int d^4x \sqrt{g}\, \phi \Delta^{(1/6)} \phi\,, \quad \Delta^{(1/6)} = -\Box + \frac{R}{6}$$

$$S(\psi, g_{\mu\nu}) = \int d^4x \sqrt{g}\, \bar{\psi} D \psi$$

$$S(A, g_{\mu\nu}) = \frac{1}{4} \int d^4x \sqrt{g}\, F_{\mu\nu} F^{\mu\nu}$$

Trace anomaly

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta\Gamma}{\delta g^{\mu\nu}} = b\, C^2 + b'\, E$$

$$E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$
$$C^2 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$$

$$b = \frac{1}{120(4\pi)^2} \left(n_S + 6 n_D + 12 n_M \right)$$

$$b' = -\frac{1}{360(4\pi)^2} \left(n_S + 11 n_D + 62 n_M \right)$$

The scalar and fermion measure

$$(d\phi) = \prod_x \frac{d\phi(x)}{\mu}$$

$$\Gamma(g_{\mu\nu}) = \int (d\phi) e^{-\int d^4x \sqrt{g}\phi\Delta^{(1/6)}\phi} = \frac{1}{2} \ln \det \left(\frac{\Delta^{(1/6)}}{\mu^2} \right)$$

$$(d\psi) = \prod_x \frac{d\psi(x)}{\mu^{3/2}}$$

The Maxwell theory measure

$$(dA_\mu) = \prod_x dA_\mu(x)$$

$$S_{GF} = \frac{1}{2\alpha}\int d^4x\sqrt{g}(\nabla^\mu A_\mu)^2$$

$$S + S_{GF} = \frac{1}{2}\int d^4x\sqrt{|g|}\,A_\mu\left(-\nabla^2 g^{\mu\nu} + R^{\mu\nu}\right)A_\nu$$

$$S_{gh} = \int d^4x\sqrt{|g|}\,\bar{C}(-\nabla^2)C$$

$$\begin{aligned}\Gamma(g) &= \log\int(dAd\bar{C}dC)e^{-S_{em}(A,g)-S_{GF}(A,g)-S_{gh}(\bar{C},C,g)}\\ &= \frac{1}{2}\mathrm{Tr}\left(\frac{\log\Delta^{(1)}}{\mu^2}\right) - \mathrm{Tr}\log\left(\frac{\Delta^{(0)}}{\mu^2}\right)\end{aligned}$$

The Weyl-invariant scalar measure

$$(d\phi) = \prod_x d\left(\frac{\phi(x)}{\chi(x)}\right)$$

$$\Gamma(g_{\mu\nu}) = \int (d\phi) e^{-\int d^4x \sqrt{g}\phi\Delta^{(1/6)}\phi} = \frac{1}{2}\ln\det\left(\frac{1}{\chi^2}\Delta^{(1/6)}\right)$$

$$\frac{1}{\Omega^{-2}\chi^2}\,\Delta^{(1/6)}_{\Omega^2 g}(\Omega^{-1}\phi) = \Omega^{-1}\left(\frac{1}{\chi^2}\Delta^{(1/6)}\phi\right)$$

eigenvalues are Weyl invariant $\rightarrow$ det $\left(\frac{1}{\chi^2}\Delta^{(1/6)}\right)$ is Weyl invariant

Weyl-covariant gauge fixing

$$S_{GF} = \frac{1}{2\alpha} \int d^4x \sqrt{g} (D^\mu A_\mu)^2$$

$$S + S_{GF} = \frac{1}{2} \int d^4x \sqrt{g} \chi^2 g^{\mu\nu} A_\mu \frac{1}{\chi^2} \left(-D^2 \delta^\sigma_\nu + \mathcal{R}^\sigma_\nu \right) A_\sigma$$

eigenvalues of $\frac{1}{\chi^2}(-D^2 \delta^\mu_\nu + \mathcal{R}^\mu_\nu)$ are Weyl invariant

$\rightarrow$ det $\frac{1}{\chi^2}(-D^2 \delta^\mu_\nu + \mathcal{R}^\mu_\nu)$ is Weyl invariant

Weyl invariant quantization

dilaton acts as compensator (Stückelberg) field in the *quantum* effective action

μ has been promoted to a field

Stückelberg trick commutes with quantization

Another point of view

$$\Gamma^{\mathrm{I}}(g^{\chi}) - \Gamma^{\mathrm{I}}(g) = \Gamma_{WZ}(g, \chi)$$

Wess-Zumino consistency condition:

$$\Gamma_{WZ}(g^{\Omega}, \chi^{\Omega}) - \Gamma_{WZ}(g, \chi) = -\Gamma_{WZ}(g, \Omega)$$

where $g^{\Omega} = \Omega^2 g$, $\chi^{\Omega} = \Omega^{-1}\chi$
If we identify $\Gamma^{\mathrm{I}}(g)$ with $\Gamma^{\mathrm{II}}(g, \chi = \mu)$,

$$\Gamma^{\mathrm{II}}(g, \chi) = \Gamma^{\mathrm{I}}(g) + \Gamma_{WZ}(g, \chi)$$

Dynamical metric and dilaton

$$S = \int d^4x \sqrt{g} \left[\lambda Z^2 \chi^4 - \frac{1}{2} Z \left(\xi \chi^2 R + g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \right) \right]$$

for $\xi = 1/6$

$$S = \int d^4x \sqrt{g} \left[\lambda Z^2 \chi^4 - \frac{1}{12} Z \chi^2 \mathcal{R} \right]$$

can choose Weyl gauge (i.e. units) where

$$Z\chi^2 = \frac{12}{16\pi G} \; ; \qquad \lambda = \frac{2\pi}{9} G\Lambda$$

S reduces to Hilbert action.
For $\xi \neq 1/6$ scalar field is physical.

Expansion of action

Let $\delta g_{\mu\nu} = h_{\mu\nu}$, $\delta\chi = \eta$.

$$\begin{aligned} S^{(2)} &= \frac{1}{2} Z \mathcal{H}((h,\eta),(h,\eta)) \\ &= \frac{1}{2} Z \int d^4x \sqrt{g} \left(h_{\mu\nu} \quad \eta \right) \begin{pmatrix} \mathcal{H}_{hh}^{\mu\nu\rho\sigma} & \mathcal{H}_{h\eta}^{\mu\nu} \\ \mathcal{H}_{\eta h}^{\rho\sigma} & \mathcal{H}_{\eta\eta} \end{pmatrix} \begin{pmatrix} h_{\rho\sigma} \\ \eta \end{pmatrix} \end{aligned}$$

For $\xi = 1/6$

$$\mathcal{H}_{hh}^{\mu\nu\rho\sigma} = \frac{1}{12}\chi^2 \Bigg[-\frac{1}{2}\mathbf{1}^{\mu\nu\rho\sigma} D^2 + g^{(\nu|\sigma} D^{|\mu)} D^\rho - \frac{1}{2} g^{\mu\nu} D^\rho D^\sigma - \frac{1}{2} g^{\rho\sigma} D^{(\mu} D^{\nu)}$$

$$+ \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} D^2 - \mathcal{R}^{\mu\rho\nu\sigma} - g^{(\mu|\rho}\mathcal{R}^{|\nu)\sigma} + \frac{1}{2}(g^{\mu\nu}\mathcal{R}^{\rho\sigma} + \mathcal{R}^{\mu\nu} g^{\rho\sigma})$$

$$+ \left(\mathcal{R} - 12\lambda Z\chi^2\right) K^{\mu\nu\rho\sigma} \Bigg]$$

$$\mathcal{H}_{h\eta}^{\mu\nu} = \mathcal{H}_{\eta h}^{\mu\nu} = \frac{1}{6}\chi \left(g^{\mu\nu} D^2 - D^\mu D^\nu + \mathcal{R}^{\mu\nu} - \frac{1}{2}\mathcal{R} g^{\mu\nu} \right) + 2\lambda Z \chi^3 g^{\mu\nu}$$

$$\mathcal{H}_{\eta\eta} = D^2 - \frac{1}{6}\mathcal{R} + 12\lambda Z\chi^2 \, .$$

background Weyl transformations

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu} \, ; \qquad \chi \mapsto \Omega^{-1}\chi \, ; \qquad h_{\mu\nu} \mapsto \Omega^2 h_{\mu\nu} \, ; \qquad \eta \mapsto \Omega^{-1}\eta$$

Wave operators

$$\mathcal{G}((h_1,\eta_1),(h_2,\eta_2)) = \int d^4x\sqrt{g}\left[\chi^4 h_{1\mu\nu} g^{\mu\rho} g^{\nu\sigma} h_{2\rho\sigma} + \chi^2 \eta_1\eta_2\right]$$

$$S^{(2)} = \frac{1}{2} Z \mathcal{H}(\theta,\theta) = \frac{1}{2} Z \mathcal{G}(\theta,\mathcal{O}\theta)$$

$$\begin{aligned}
(\mathcal{O}_{hh})_{\mu\nu}{}^{\rho\sigma} &= \chi^{-4} g_{\mu\alpha} g_{\nu\beta} \mathcal{H}_{hh}^{\alpha\beta\rho\sigma} \ , \\
(\mathcal{O}_{h\eta})_{\mu\nu} &= \chi^{-4} g_{\mu\alpha} g_{\nu\beta} \mathcal{H}_{h\eta}^{\alpha\beta} \ , \\
\mathcal{O}_{\eta h}^{\rho\sigma} &= \chi^{-2} \mathcal{H}_{\eta h}^{\rho\sigma} \ , \\
\mathcal{O}_{\eta\eta} &= \chi^{-2} \mathcal{H}_{\eta\eta} \ .
\end{aligned}$$

Weyl covariance of wave operators

$$(\mathcal{O}_{hh(\Omega^2 g_{\mu\nu},\Omega^{-1}\chi)})_{\mu\nu}{}^{\rho\sigma}(\Omega^2 h_{\rho\sigma}) = \Omega^2(\mathcal{O}_{hh(g_{\mu\nu},\chi)})_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \ ,$$
$$(\mathcal{O}_{h\eta(\Omega^2 g_{\mu\nu},\Omega^{-1}\chi)})_{\mu\nu}(\Omega^{-1}\eta) = \Omega^2(\mathcal{O}_{h\eta(g_{\mu\nu},\chi)})_{\mu\nu}\eta \ ,$$
$$(\mathcal{O}_{\eta h(\Omega^2 g_{\mu\nu},\Omega^{-1}\chi)})^{\rho\sigma}(\Omega^2 h_{\rho\sigma}) = \Omega^{-1}(\mathcal{O}_{\eta h(g_{\mu\nu},\chi)})^{\rho\sigma} h_{\rho\sigma} \ ,$$
$$\mathcal{O}_{\eta\eta(\Omega^2 g_{\mu\nu},\Omega^{-1}\chi)}(\Omega^{-1}\eta) = \Omega^{-1}\mathcal{O}_{\eta\eta(g_{\mu\nu},\chi)}\eta \ .$$

Gauge fixing

$$S_{GF} = \frac{1}{2\alpha} \int d^4x \sqrt{g}\, \frac{1}{2} Z \xi \chi^2 F_\mu \bar{g}^{\mu\nu} F_\nu \ ,$$

where

$$F_\nu = D_\mu h^\mu{}_\nu - \frac{\beta+1}{4} D_\nu h$$

$$S_{gh} = \ \mathcal{G}_{gh}\left(\bar{C}, \mathcal{O}_{gh} C\right) \ .$$

$$\mathcal{G}_{gh}(A, B) = \int d^4x \sqrt{g}\, \chi^2 A_\mu g^{\mu\nu} B_\nu$$

$$(\mathcal{O}_{gh})_\mu^\nu = -\frac{1}{\chi^2}\left(\delta_\mu^\nu D^2 + \frac{1-\beta}{2} D_\mu D^\nu + \mathcal{R}_\mu{}^\nu\right)$$

Weyl-gauge-fixing $\eta = 0$

Weyl invariance of one loop effective action

$$\Gamma^{(1)}(g_{\mu\nu}, \chi) = S(g_{\mu\nu}, \chi) + \frac{1}{2}\mathrm{Tr}\log\mathcal{O}_{hh} - \mathrm{Tr}\log\mathcal{O}_{gh}$$

If $\Delta_{(g_{\mu\nu},\chi)} h_{\mu\nu} = \lambda h_{\mu\nu}$ then

$$\Delta_{(\Omega^2 g_{\mu\nu}, \Omega^{-1}\chi)} \left(\Omega^2 h_{\mu\nu}\right) = \Omega^2 \Delta_{(g_{\mu\nu},\chi)} h_{\mu\nu} = \lambda \Omega^2 h_{\mu\nu}$$

The spectrum of Δ is Weyl invariant, so $\Gamma^{(1)}$ is Weyl invariant.

FRGE

$$\Delta S_k = \frac{1}{2} Z \mathcal{G}\left(h, \frac{1}{12}\frac{1}{\chi^2} R_k(\chi^2 \mathcal{O}_{hh}) h\right) + \mathcal{G}_{gh}\left(\bar{C}, \frac{1}{\chi^2} R_k(\chi^2 \mathcal{O}_{gh}) C\right)$$

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr} F(\mathcal{O}_{hh}) - \mathrm{Tr} F(\mathcal{O}_{gh})$$

r.h.s. of FRGE is Weyl invariant if initial point is Weyl invariant
also Γ is Weyl invariant

Beta functions

Example:

$$\Gamma_k = \int d^4x \sqrt{g} \left[\lambda Z^2 \chi^4 - \frac{1}{12} Z \chi^2 \mathcal{R} \right]$$

$$\begin{aligned} u \frac{dZ}{du} &= \frac{23}{4\pi^2} u^2 \\ u \frac{d\lambda}{du} &= \frac{u^2}{16\pi^2 Z^2} \left(u^2 - 184 Z \lambda \right) \end{aligned}$$

where $u = k/\chi$ is assumed constant

Solutions

$$Z(u) = Z_0 + \frac{23}{8\pi^2}u^2 \to \frac{23}{8\pi^2}u^2$$

$$\lambda(u) = \frac{\pi^2(u^4 + 64\pi^2 Z_0^2 \lambda_0)}{(8\pi^2 Z_0 + 23u^2)^2} \to \frac{\pi^2}{529}$$

note: Z is redundant coupling and does not reach a FP,
λ is essential coupling and reaches a FP.

$$\tilde{m}_P^2 = Z\frac{\chi^2}{k^2} = \frac{Z}{u^2} \to \frac{23}{8\pi^2}$$

Summary

- in presence of a dilaton, it is possible to quantize theory preserving Weyl invariance.
- RG flow also preserves Weyl invariance.
- trace anomaly still present
- $\xi = 1/6$ is a fixed point of RG flow.
- fixed point of the Einstein-Hilbert formulation appears in different guise
- since all couplings are dimensionless, situation similar to usual renormalizable QFT

Further work

- cutoff and renormalization scale can depend on position. Theory with nonconstant cutoff equivalent to theory with constant cutoff but conformally related metric.
- relation between $f(R)$ and scalar-tensor theories
- generalize to nonflat Weyl connection