Cosmological Constant From Quantum Spacetime

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Quantum spacetime hypothesis:

Quantum Gravity??? ⇒ Quantum Riemannian geometry (NCRG) ⇒ Classical geometry

This is not surjective, not every classical geometry is `quantisable'!

① Illustrate this on Majid-Ruegg quantum spacetime $[x^i, t] = \imath\lambda_P x^i$ → Bertotti-Robinson exact soln w/ cosmological const

② Analyse in general at semiclassical level → Poisson-Riemannian geometry

I Review of Quantum Spacetime

SM & H. Ruegg PLB 334 (1994)

$$m:\ [x_i,t]=\imath\lambda_P\, x_i \qquad U(m)$$ noncommutative coordinate algebra

- Quantum Born reciprocity

SM Class. Quant. Gravity 5 (1988)

	Position	Momentum
Gravity	Curved	Noncommutative
Cogravity	Noncommutative	Curved
Quantum Gravity	Both	Both

Qua. Fou. Trans.

$\overline{U(m)} \longleftrightarrow C(M)$

$\swarrow$ Quantum spacetime hypothesis

$\searrow$ Curved momentum space hypothesis

$C(M){\blacktriangleright\!\!\triangleleft} U(so_{3,1})$ acting on $U(m)$ $\quad$ semidual'n $\longleftrightarrow$ $\quad U(so_{3,1}{\bowtie} m)$ acting on $C(M)$

$C(SU_2){\blacktriangleright\!\!\triangleleft} U(su_2)$ acting on $U(su_2)$ $\quad \longleftrightarrow \quad$ $U(su_2\oplus su_2)$ acting on $C(SU_2)$

bicrossproduct quantum group

factorising (quantum) group

See this in 3D QG

SM & B. Schroers J. Phys A 42 (2009) 425402

- quantum Poincare group relation

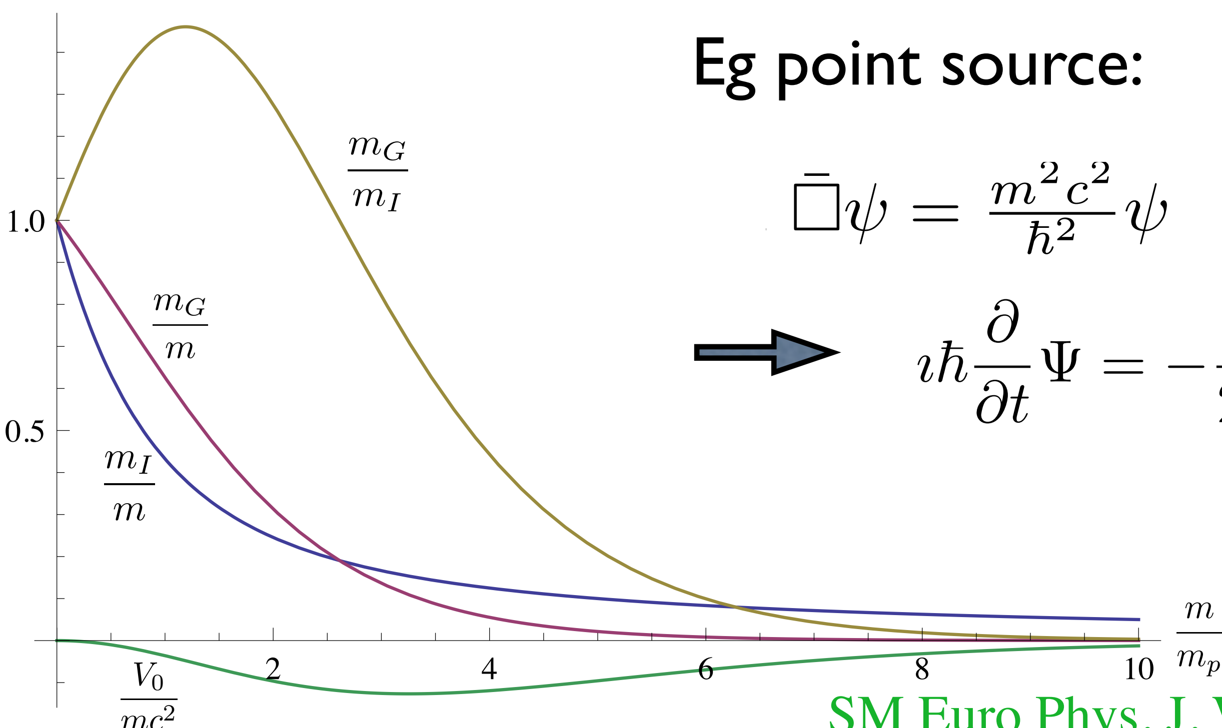
$$[p^i, N_j] = -\frac{\imath}{2}\delta^i_j\left(\frac{1-e^{-2\lambda p^0}}{\lambda} + \lambda\vec{p}^2\right) + \imath\lambda p^i p_j, \quad \Rightarrow \quad ||p||^2_\lambda = \vec{p}^2 e^{-\lambda p^0} - \frac{2}{\lambda^2}(\cosh(\lambda p^0) - 1)$$

Wave operator on nc plane waves $e^{i\vec{x}\cdot\vec{p}}e^{itp_0} \quad \Rightarrow \quad |\frac{\partial p^0}{\partial p^i}| = e^{-\lambda p^0}$

G. Amelino-Camelia & S. M, Int. J. Mod. Phys. A 15 (2000)

VSL prediction
Gamma-ray bursts

- Freedom in extended differential structure = newtonian gravity



Eg point source:

$$\bar{\Box}\psi = \frac{m^2c^2}{\hbar^2}\psi \qquad \psi = \Psi(x,t)e^{-\imath\frac{mc^2}{\hbar}t}$$

$$\Rightarrow \quad \imath\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m_I}\bar{\Delta}^{flat}\Psi + (V_0 - \frac{GMm_G}{r})\Psi$$

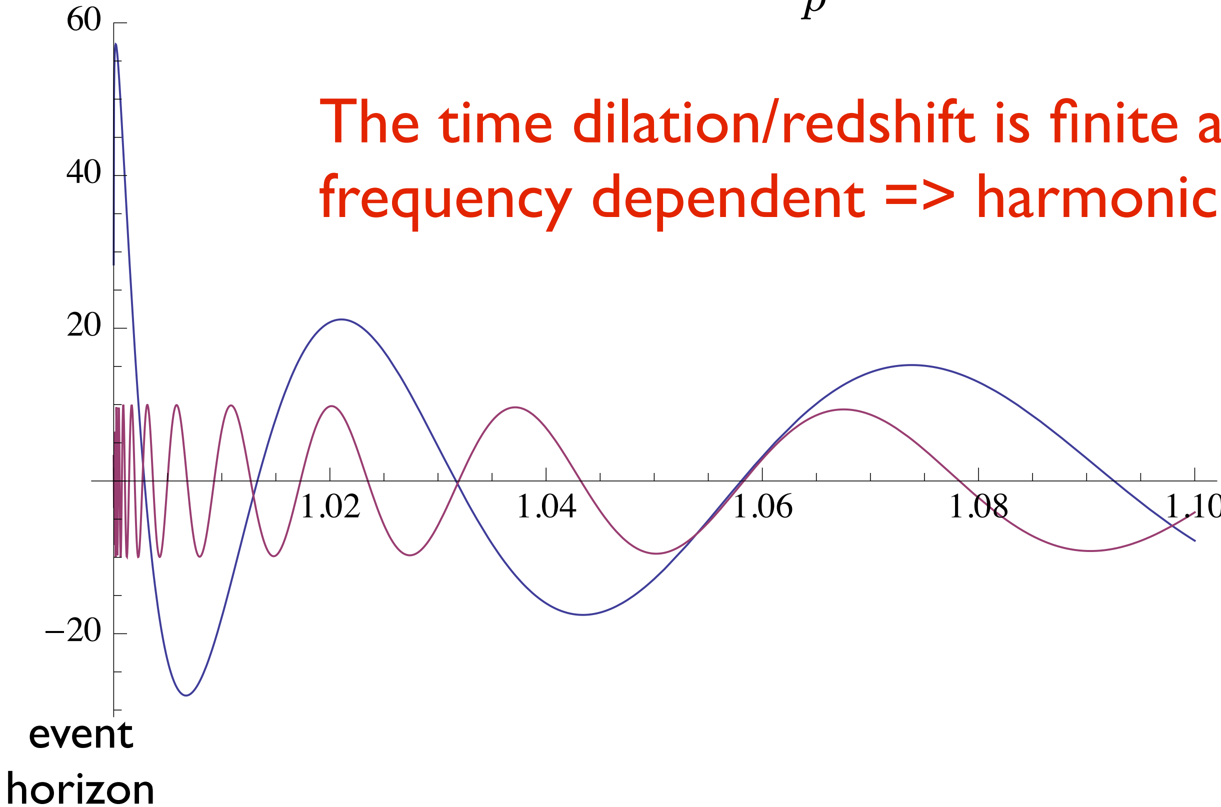
separation of intertial and gravl masses

SM Euro Phys. J. Web of Conferences, **70** (2014) 00082

- BH potential + minimal coupling => quantum Schw. black hole wave operator => FT of ∂_t^2 obeys

$$\lim_{r\to\gamma} D(\omega, r) = \frac{\sinh(\omega\lambda_p)}{\lambda_p^2}$$

S.M Commun. Math. Phys. **310** (2012)



- Continuum ⇒ ∞ zero point energy. Planck scale cut off still 10^{122}x obs.

Non-zero cosmological constant may be forced by quantum spacetime which would explain why its small compared to Planck scale

Quantum differentials on an algebra A

- space of 1-forms, i.e. `differentials dx'

Ω^1 a((db)c)=(a(db))c `bimodule'

$\mathrm{d}:A\to\Omega^1$ d(ab)=(da)b+a(db) `Leibniz rule'

$\{\sum a\mathrm{d}b\}=\Omega^1$ `surjectivity'

$\ker\mathrm{d}=k.1$ (`connected')

- require this to extend to a DGA $\Omega=T_A\Omega^1/\mathcal{I}=\oplus_n\Omega^n,\quad \mathrm{d}^2=0$

Thm. (SM+W.Tao) Let $A=U(\mathfrak{g})=T\mathfrak{g}/\langle xy-yx-[x,y]\rangle$

- bicovariant $\Omega^1(U(\mathfrak{g}))\quad\longleftrightarrow\quad$ surjective $\zeta\in Z^1(\mathfrak{g},\Lambda^1)$

$\mathrm{d}x=1\otimes\zeta(x),\ \ \Omega^1=U(\mathfrak{g})\otimes\Lambda^1$

- connected and of classical dim $\longleftrightarrow$ pre-Lie algebra for $\mathfrak{g}$

$\circ:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}\qquad [x,y]=x\circ y-y\circ x$

$(x\circ y)\circ z-(y\circ x)\circ z=x\circ(y\circ z)-y\circ(x\circ z)\qquad \Lambda^1\cong\mathfrak{g}\quad [x,\mathrm{d}y]=\lambda\mathrm{d}(x\circ y)$

- $\Rightarrow\ \Omega(U(\mathfrak{g}))$

Classification in 2D

Burde 1998

$$\mathfrak{g} : [r,t] = r \qquad A = U_\lambda(\mathfrak{g})$$

$i)\quad t\circ r = -r, \quad t\circ t = \alpha t$

$ii)\quad r\circ t = \beta r, \quad t\circ r = (\beta-1)r, \quad t\circ t = \beta t$

$iii)\quad t\circ r = -r, \quad t\circ t = r - t$

$iv)\quad r\circ r = t, \quad t\circ r = -r, \quad t\circ t = -2t$

$v)\quad r\circ t = r, \quad t\circ t = r + t$

=> Calculi in n-D on $[x^i, t] = \lambda x^i$ that are rotationally inv:

- $(i)\ [t, \mathrm{d}x^i] = -\lambda \mathrm{d}x^i, \quad [t, \mathrm{d}t] = \lambda\alpha \mathrm{d}t$ $\qquad \alpha$-calculus

- $(ii)\ [x^i, \mathrm{d}t] = \lambda\beta \mathrm{d}x^i, \quad [t, \mathrm{d}x^i] = \lambda(\beta-1)\mathrm{d}x^i, \quad [t, \mathrm{d}t] = \lambda\beta \mathrm{d}t$

β-calculus

Quantum metric tensor

$$g \in \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \wedge(g) = 0$$ `quantum symmetric'

invertible in the sense exists inverse: $(\ , \ ) : \Omega^1 \underset{A}{\otimes} \Omega^1 \to A$

$$((\ , \ ) \otimes \mathrm{id})(\omega \otimes g) = \omega = (\mathrm{id} \otimes (\ , \ ))(g \otimes \omega), \quad \forall \omega \in \Omega^1$$

$$a(\omega, \eta) = (a\omega, \eta), \quad (\omega, \eta)a = (\omega, \eta a)$$ `bimodule map (tensorial)'

need this to be able to contract/ `raise/lower' via metric, eg to have well defined contraction:

$$(\ , \ ) \otimes \mathrm{id} : \ \Omega^1 \underset{A}{\otimes} \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1$$ " $T_{\mu\nu\rho} \mapsto g^{\mu\nu} T_{\mu\nu\rho}$ "

but

$$(\omega, g^1)g^2 = \omega \qquad g = g^1 \underset{A}{\otimes} g^2$$

$$\Longrightarrow (\omega, g^1)g^2 a = \omega a = (\omega a, g^1)g^2 = (\omega, a g^1)g^2$$

$$\Longrightarrow \quad ag = ga, \quad \forall a \in A$$ need metric to be central

Work over $\mathbb{C}$ but specify real differential geometry via

- $*: A \to A$ antilinear involution `*-algebra'
- extends to graded-anti-algebra hom on $\Omega(A), \ [*, \mathrm{d}] = 0$
- metric hermitian in sense $(* \otimes *)(g) = \mathrm{flip}(g)$
- Our case: $x^{i*} = x^i, \quad t^* = t, \quad \lambda^* = -\lambda, \quad r^* = r$

$\beta = 1$ Calculus

Class. Quant. Gravity 31 (2014) 035020 (39pp)

Propn.: In 2D the quantum metric has the unique form

$$g = \mathrm{d}r \otimes \mathrm{d}r + b(v^* \otimes v + \lambda(\mathrm{d}r \otimes v - v^* \otimes \mathrm{d}r)) \qquad b \in \mathbb{R}$$

$$v = r\mathrm{d}t - t\mathrm{d}r, \qquad v^* = (\mathrm{d}t)r - t\mathrm{d}r \qquad b \neq 0$$

$\Longrightarrow$ in classical limit only

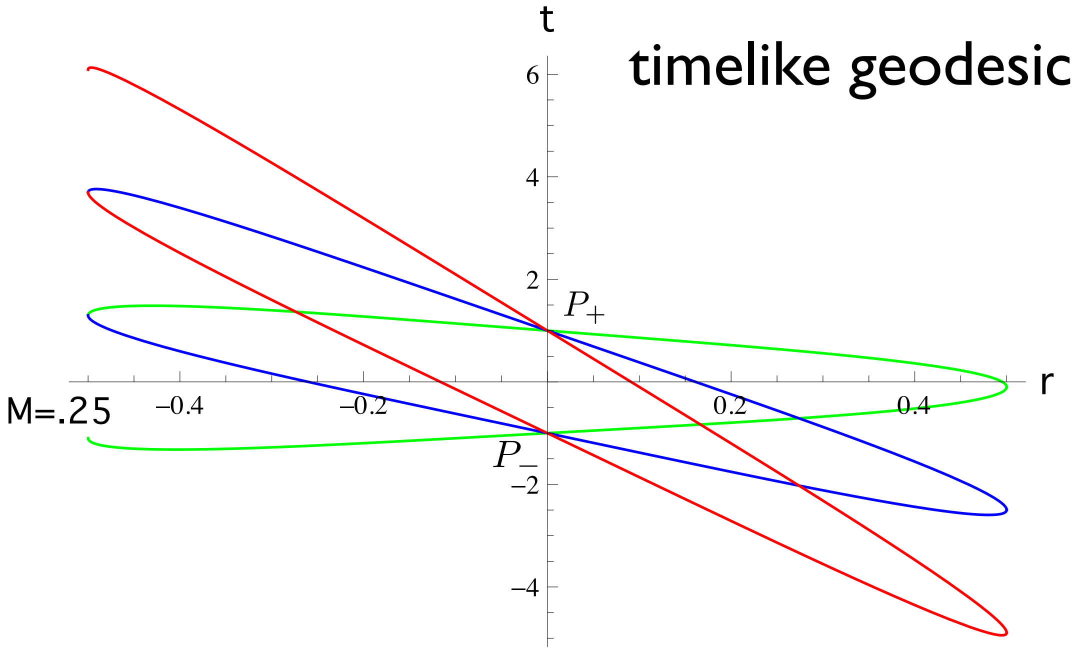
$$g = \mathrm{d}r \otimes \mathrm{d}r + bv \otimes v = (1 + bt^2)\mathrm{d}r^2 + br^2\mathrm{d}t^2 - 2brt\mathrm{d}r\,\mathrm{d}t$$

can emerge (i.e. be quantised)

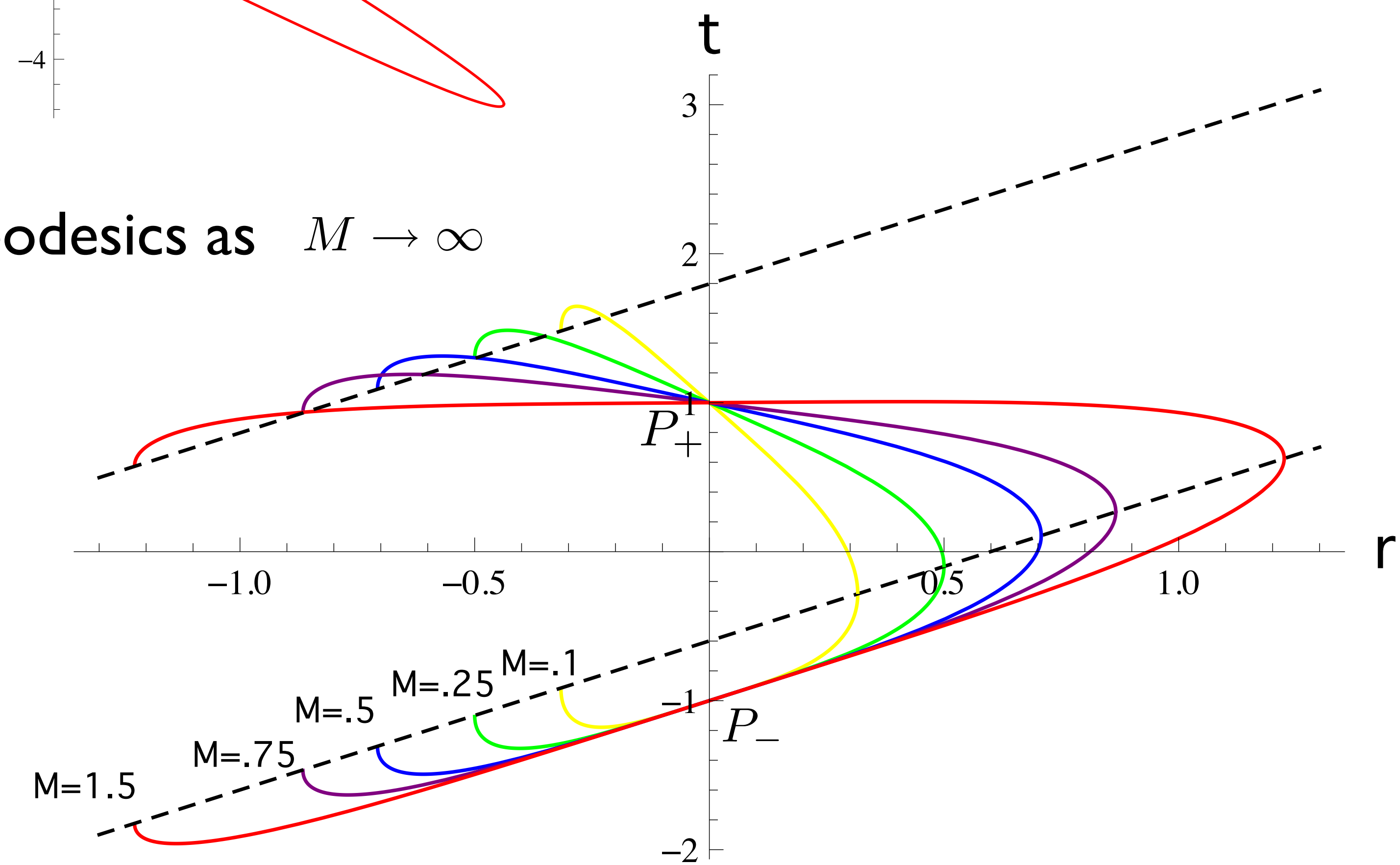
=> strong gravitational source/expanding universe

(a) $b < 0$ All geodesics pass through $P_{\pm} = (0, \pm\frac{1}{\sqrt{-b}})$

timelike geodesics bounce between $r = \pm(-bM^2)^{\frac{1}{4}}$



and become null geodesics as $M \to \infty$

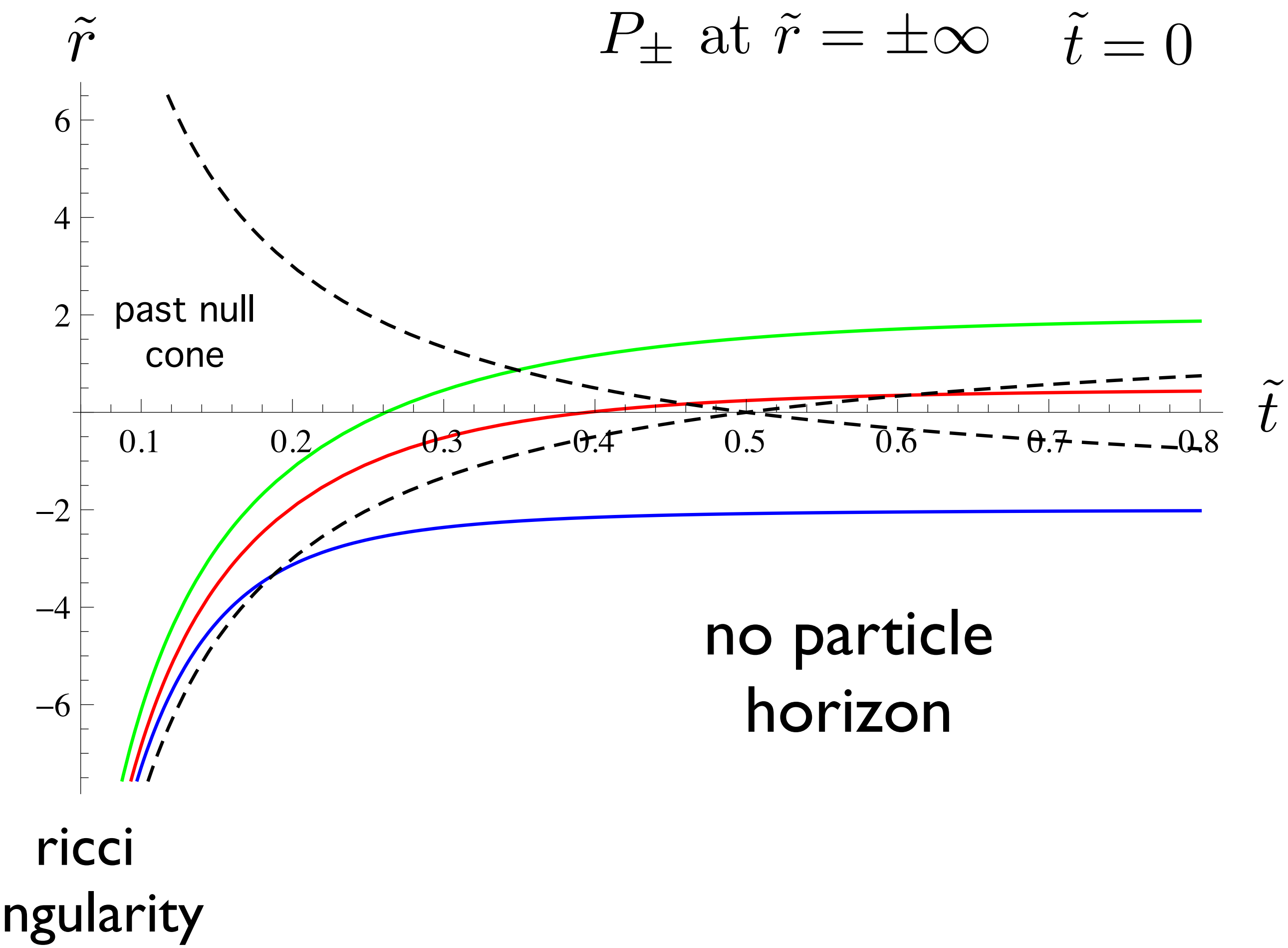


(b) $b > 0:$ use new FRW-like coordinates $\tilde{t} = r, \quad \tilde{r} = \frac{t}{r}$

$$g = -\mathrm{d}\tilde{t}^2 + R(\tilde{t})^2 \mathrm{d}\tilde{r}^2, \quad R(\tilde{t}) = \sqrt{b}\tilde{t}^2$$

All geodesics start/end on

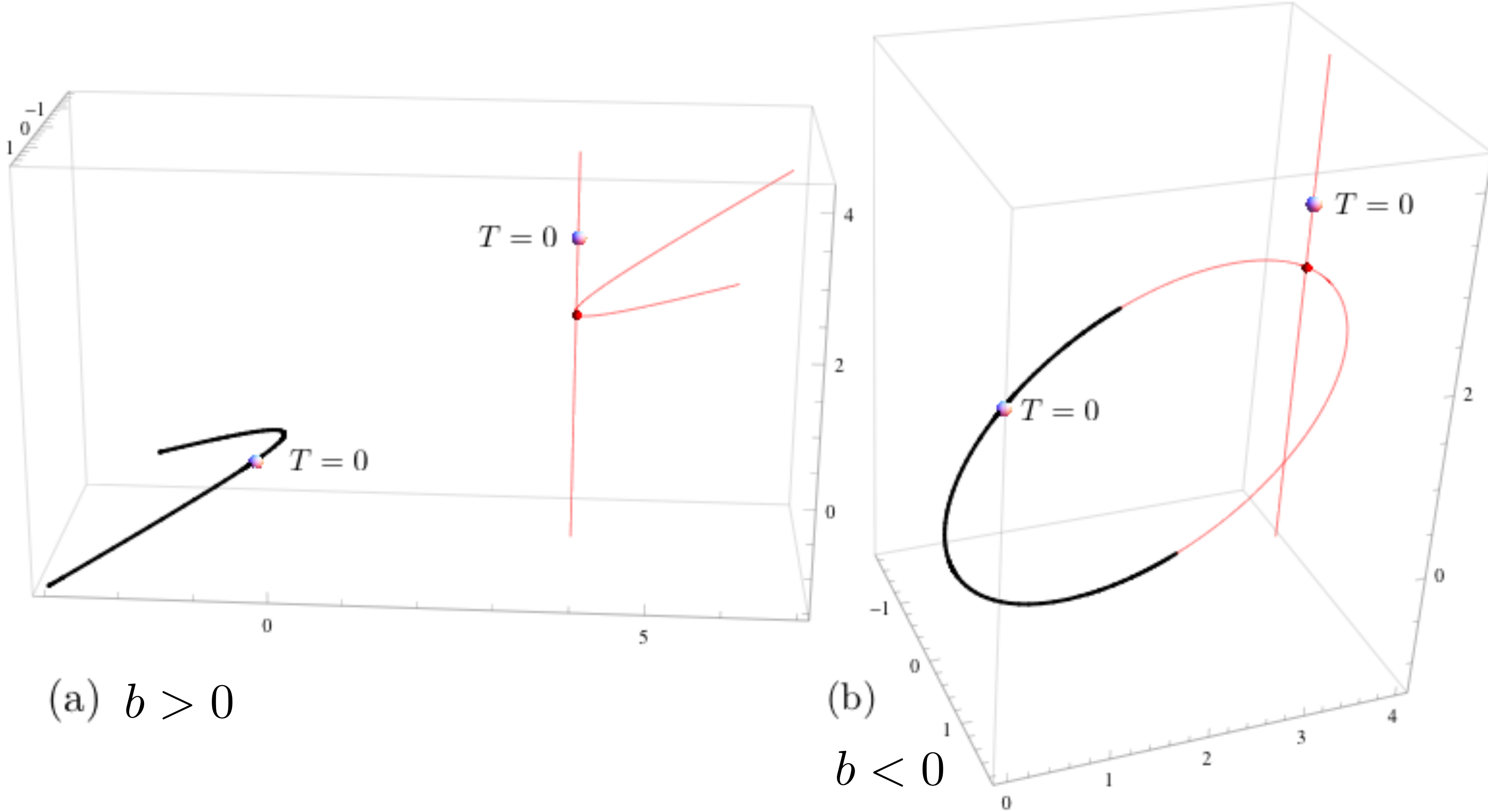
$P_{\pm}$ at $\tilde{r} = \pm\infty$ $\quad \tilde{t} = 0$



$\tilde{t} = 0$

Thm: no central metrics exist for the β calculus for n>2

<u>2D Ex:</u> Moduli of quantum metric-compat ∇ form a line + conic



- black parts have classical limit as $\lambda \to 0$
- red parts blow up as $\lambda \to 0$ so not visible classically
- in each case a unique `Levi-Civita point' where torsion T=0

α Calculus

arXiv:1412.2285 (gr-qc) 13pp

$$g=\sum_{i,j}^{n-1}a_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j+\sum_{i}^{n-1}b_i(\mathrm{d}x^i\otimes\mathrm{d}t+\mathrm{d}t\otimes\mathrm{d}x^i)+c\mathrm{d}t\otimes\mathrm{d}t$$

$[f,g]=0,\forall f$ => a_{ij},b_i,c of degree $-2,\alpha-1,2\alpha$

add spherical symmetry =>

$$g=\delta^{-1}\mathrm{d}\Omega^2+ar^{-2}\mathrm{d}r\otimes\mathrm{d}r+br^{\alpha-1}(\mathrm{d}r\otimes\mathrm{d}t+\mathrm{d}t\otimes\mathrm{d}r)+cr^{2\alpha}\mathrm{d}t\otimes\mathrm{d}t$$

$$\bar{\delta}=\frac{c\alpha^2}{b^2-ac}\qquad a,b,c\in\mathbb{R},\ \delta>0\qquad b^2-ac>0$$

$$G=-\frac{(n-2)(n-3)}{2}\delta g+((n-3)\delta-\bar{\delta})\mathrm{d}\Omega^2$$

- solves Einst Eqn with Maxwell field and cosmological constant

$$F=q\sqrt{b^2-ac}\,r^{\alpha-1}\mathrm{d}t\wedge\mathrm{d}r\qquad\Lambda=\frac{(n-2)(n-3)}{2}\delta-q^2G_N,\qquad q^2G_N=\frac{1}{2}\left((n-3)\delta-\bar{\delta}\right)$$

- This is the Bertotti-Robinson metric. We are forced to it!

Change of coordinates

If $\bar{\delta} > 0$ $\qquad t' = \frac{\alpha}{\sqrt{\bar{\delta}}} \ln r, \quad r' = \sqrt{c}t - \frac{\sqrt{a + \frac{\alpha^2}{\bar{\delta}}}}{\alpha r^\alpha}$

$$g = \delta^{-1}\mathrm{d}\Omega^2 + e^{2t'\sqrt{\bar{\delta}}}\mathrm{d}r'^2 - \mathrm{d}t'^2.$$

=> $S^{n-2} \times dS_2 \quad \bar{\delta} > 0$

similarly: $S^{n-2} \times AdS_2 \quad \bar{\delta} < 0$

Quantum algebra

$[t', r'] = \lambda' = \lambda\sqrt{b^2 - ac} \qquad [r', \mathrm{d}r'] = \lambda'\sqrt{\bar{\delta}}\mathrm{d}r'$

$$[r', \mathrm{d}t'] = [t', \mathrm{d}r'] = [t', \mathrm{d}t'] = 0$$

$$\nabla \mathrm{d}r' = -\sqrt{\bar{\delta}}(\mathrm{d}r' \otimes \mathrm{d}t' + \mathrm{d}t' \otimes \mathrm{d}r')$$

$$\nabla \mathrm{d}t' = -\sqrt{\bar{\delta}}\, e^{2t'\sqrt{\bar{\delta}}}(\mathrm{d}r' \otimes \mathrm{d}t' + \mathrm{d}t' \otimes \mathrm{d}r')$$

quantum
Levi-Civita

- Same change of variables that diagonalised metric also gives canonical commutation relations

Semiquantisation

arXiv:1403.4231(math.QA) 57pp

$A_0 = C^\infty(M)$ quantisation at order λ means a Poisson bracket

$$a.b - b.a = \lambda\{a,b\} + O(\lambda^2)$$

$\{\ ,\ \} \leftrightarrow \omega^{ij}$ Poisson tensor

Similarly, quantization of $\Omega^1(M)$ at order λ requires

$$a.\mathrm{d}b - (\mathrm{d}b).a = \lambda\nabla_{\hat{a}}\mathrm{d}b + O(\lambda^2)$$

$\Longrightarrow$ ∇ a Poisson pre-connection along Hamiltonian vec. fields $\hat{a} = \{a,\ \}$

1) $\nabla_{\hat{a}}(b\mathrm{d}c) = \{a,b\}\mathrm{d}c + b\nabla_{\hat{a}}\mathrm{d}c$

2) $\mathrm{d}\{a,b\} = \nabla_{\hat{a}}\mathrm{d}b - \nabla_{\hat{b}}\mathrm{d}a$

At order λ^2 the bimodule associativity is $(\nabla_{\hat{a}}\nabla_{\hat{b}} - \nabla_{\hat{b}}\nabla_{\hat{a}} - \nabla_{\widehat{\{a,b\}}})\mathrm{d}c = 0$

(just consider $[a,[b,\mathrm{d}c]] + [b,[\mathrm{d}c,a]] + [\mathrm{d}c,[a,b]] = 0$)

- non-flat connection => nonassociativity at $O(\lambda^2)$

Thm: suppose (ω,∇) Poisson compat and metric g, Levi-Civita conn. $\widehat\nabla$
Exists quantum metric at order λ <=> $\nabla g=0$

`quant metric' $g_1:=q^{-1}(g-\frac{\lambda}{4}g_{ij}\omega^{is}(T^j_{nm;s}-R^j{}_{nms}+R^j{}_{mns})\mathrm{d}x^m\otimes_0\mathrm{d}x^n)$

$$\nabla g=0 \quad \text{<=>} \quad \widehat\nabla=\nabla+S \qquad S^a_{bc}=\tfrac{1}{2}g^{ad}(T_{dbc}-T_{bcd}-T_{cbd})$$

(ω,∇) compat <=> $(\widehat\nabla_k\omega)^{ij}+\omega^{ir}S^j_{rk}-\omega^{jr}S^i_{rk}=0$

- Conditions on Riemann curvature for integrability

Thm: Exists best possible quantum Levi-Civita ∇_1: torsion free and symmetric part of $\nabla_1 g_1=0$.

- ∇_1 fully quantum Levi-Civita iff

$$\widehat\nabla\mathcal{R}+\omega^{ij}g_{rs}S^s_{jn}(R^r{}_{mki}+S^r_{km;i})\,\mathrm{d}x^k\otimes\mathrm{d}x^m\wedge\mathrm{d}x^n=0$$

`generalised Ricci form'

$$\mathcal{R}=g_{ij}\omega^{is}(T^j_{nm;s}-2R^j{}_{nms})\mathrm{d}x^m\wedge\mathrm{d}x^n$$

E.g. Schwarzschild black hole

Rotationally invariant t-indept Poisson bivector =>

$$\omega^{01} = -\omega^{10} = k(r) \;\; and \; \omega^{23} = -\omega^{32} = f(r)/\sin\theta$$
$$k(r)\, f'(r) = 0$$

T rotationally invariant & Poisson-compatibility => $k(r) = 0, \;\; f(r) = 1$

$$T_{001} = f_1(r) \qquad T_{101} = f_2(r) \qquad T_{203} = -T_{302} = -f_3(r)\sin\theta$$
$$T_{212} = r \qquad T_{313} = r\,\sin^2(\theta) \quad T_{213} = -T_{312} = -f_4(r)\sin\theta$$

Our obstruction to full $\nabla_1 g_1 = 0$ is

$$= \begin{cases} -r\,\sin\theta & (k,m,n) = (2,3,1)\ \&\ (k,m,n) = (3,2,1) \\ r\,\sin\theta & (k,m,n) = (2,1,3)\ \&\ (k,m,n) = (3,1,2) \\ 0 & \text{otherwise} \end{cases}$$

=> antisymmetic obstruction to qua. LC

$$R^1{}_{010} = R^0{}_{110} = -\frac{f_1'(r) + c^2\, r_s\, r^{-3}}{c^2\,(1 - r_s/r)} \qquad R^2{}_{310} = \sin\theta\,(2\,f_3(r) - r\,f_3'(r))\,r^{-3}$$
$$R^3{}_{210} = -\csc\theta\,(2\,f_3(r) - r\,f_3'(r))\,r^{-3} \qquad R^3{}_{223} = -1 \qquad R^2{}_{323} = \sin^2\theta.$$

=> any quantization will have to be nonassociative.

II Why is there Riemannian Structure?

SM arXiv: 1307.2778 (math.QA)

$\Omega_{\theta'} = \mathbb{C} \oplus \mathbb{C}\theta, \quad \theta'^2 = 0, \quad \mathrm{d}\theta' = 0$ DGA of a `point'

Defn: a central extension of a DGA $\Omega(A)$ means

$$\Omega_{\theta'} \hookrightarrow \tilde{\Omega}(A) \twoheadrightarrow \Omega(A)$$

$\tilde{\Omega}(A) = \Omega(A) \otimes \Omega_{\theta'}$ as vector space, θ' graded-commutes

- *cleft* if the projection is a left A-module map.
- *flat* if equivalent to a central extension where d is undeformed

Extn <=> $\tilde{\mathrm{d}}\omega = \mathrm{d}\omega - \frac{\lambda}{2}\theta'\Delta\omega, \quad \omega\tilde{\wedge}\eta = \omega \wedge \eta - \frac{\lambda}{2}\theta'[\![\omega, \eta]\!]$

$$[\![\omega\eta, \zeta]\!] + [\![\omega, \eta]\!]\zeta = [\![\omega, \eta\zeta]\!] + (-1)^{|\omega|}\omega[\![\eta, \zeta]\!]$$

$$L_\Delta(\omega, \eta) = \mathrm{d}[\![\omega, \eta]\!] + [\![\mathrm{d}\omega, \eta]\!] + (-1)^{|\omega|}[\![\omega, \mathrm{d}\eta]\!]$$

$$[\Delta, \mathrm{d}] = 0$$

call $(\Delta, [\![\ ,\ ]\!])$ a `2-cocycle' (cf group central extensions). Here

$$L_B(\omega, \eta) := B(\omega\eta) - (B\omega)\eta - (-1)^{b|\omega|}\omega B\eta$$ `leibnizator'

Thm 1: Let M be a classical manifold. Associated to a cleft central extension $(\tilde{\Omega}(M),\tilde{\mathrm{d}})$ is a possibly degenerate metric and covariant derivative

$$(\omega,\mathrm{d}a)=\frac{1}{2}[\![\omega,a]\!],\quad \nabla_\omega\eta=\frac{1}{2}[\![\omega,\eta]\!],\quad \forall a\in C^\infty(M),\ \omega,\eta\in\Omega^1(M)$$

obeying $g^{ij}{}_{;m}=g^{ki}T^j_{km}+g^{jk}T^i_{km}$ (T the torsion of ∇)

- origin of metric, connection and weak metric compatibility.

Thm 2: The cleft extension is flat if $\Delta=\mathrm{d}\delta+\delta\mathrm{d}$ for some degree -1 map δ, which holds iff T=0.

- origin of torsion-freeness and form of the Hodge laplacian
- If δ \`symmetric' get a new formula for Levi-Civita and metric:

$$\nabla_\omega\eta=\frac{1}{2}\Big(\delta(\omega\eta)-(\delta\omega)\eta+\omega\delta\eta+\mathrm{i}_\omega\mathrm{d}\eta+\mathrm{i}_\eta\mathrm{d}\omega+\mathrm{d}(\omega,\eta)\Big)$$

$$(\omega,\mathrm{d}a)=\delta(a\omega)-a\delta(\omega)$$

- If (,) also nondegenerate, get BV identity

$$\delta(\omega\eta\zeta) = (\delta(\omega\eta))\zeta + (-1)^{|\omega|}\omega\delta(\eta\zeta) + (-1)^{(|\omega|-1)|\eta|}\eta\delta(\omega\zeta)$$

$$-(\delta\omega)\eta\zeta - (-1)^{|\omega|}\omega(\delta\eta)\zeta - (-1)^{|\omega|+|\eta|}\omega\eta\delta\zeta$$

If $\delta^2 = 0$ get usual codifferential/divergence and Riemannian structure becomes equiv to a type of Batalin-Vilkovisky algebra

- $$\boxed{\mathrm{Ricci} = -\frac{1}{2}\Delta g}$$

  Einstein's eqn becomes something like a wave equation for g and Λ a `mass' $10^{-33}\,\mathrm{ev}$

Classical Riemannian geometry starts to make sense!

Thank You