# Quantum Non-commutative Geometry

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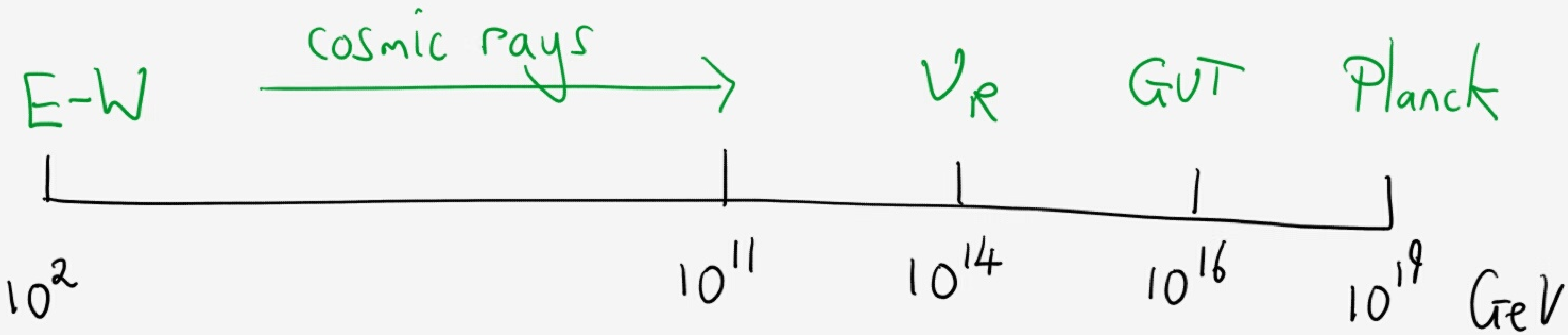
# 1. The Planck scale



## 2. The project

Dirac operator $D$ : finite-dimensional matrix

## 3. Dirac operator in physics

$$D = D_{grav} + A + MH + NJ + \text{conj}$$

$$D_{grav} = \gamma^a e_a^\mu(x) \nabla_\mu$$

$$A = \gamma^a e_a^\mu(x) A_\mu(x)$$

$$H = \text{Higgs}$$

$M$ = Dirac masses

$N$ = Majorana masses

$J\psi = \bar{\psi}$ charge conj.

Fermion action $\bar{\psi} D \psi$

Bosonic action $S(D)$

# 4. Non-commutative geometry

NC "manifold" coordinates : algebra $\mathcal{A}$

e.g. in S.M. (Connes)

$$\mathcal{A} = C^\infty(\mathbb{R}^4) \otimes_{\mathbb{R}} \left( \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \right)$$

C spacetime

NC "internal space"

## 5. $C \to NC$ : new axioms

$\mathcal{H}$: "Hilbert" space $\qquad \mathcal{A}$: $*$-$\mathbb{R}$-algebra

Operators $D$, $J$ (antilinear), $\Gamma$ $(\Gamma^2 = 1)$

C $\qquad\qquad$ NC $\qquad$ "bimodule"

$a, b \in \mathcal{A}$: $\quad J a J^{-1} = a^* \quad \rightsquigarrow \quad [a, J b J^{-1}] = 0$

$a \triangleright \psi \triangleleft b = a J b^* J^{-1} \psi$

$$[[D, a], b] = 0 \quad \rightsquigarrow \quad [[D, a], J b J^{-1}] = 0$$

"first-order"

# 6. Quantum Geometry ?

Fix fermion data : $\mathcal{H}, \mathcal{A}, J, \Gamma$.

Variable $D \in \mathcal{G}$, space of NC geometries

$$\int_{D \in \mathcal{G}} \ldots\ldots \, dD \qquad ?$$

Problems ($\mathcal{H}$ $\infty$-dim'l) :

- $D$ unbounded (domains etc)
- UV divergence/cutoff ?

# 7. Finite NC geometries

$\mathcal{H}$ : finite-dimensional

Axioms for $D$ linear

$\Rightarrow$ $\mathcal{G}$ is a finite-dim'l vector space!

$$\int_{D \in \mathcal{G}} \cdots\cdots \, dD \quad \text{defined (up to scaling)}$$

## 8. Example : Fuzzy sphere

Fix $n>0$.

$$\mathcal{H} = \mathbb{C}^4 \otimes M_n(\mathbb{C}) \ni \psi = \sum v \otimes m$$

$$\mathcal{A} = M_n(\mathbb{C}) \quad , \quad a \triangleright (v \otimes m) \triangleleft b = v \otimes (amb)$$

$\mathbb{C}^4$ has action of $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ $\quad + - - - = (1,3)$!

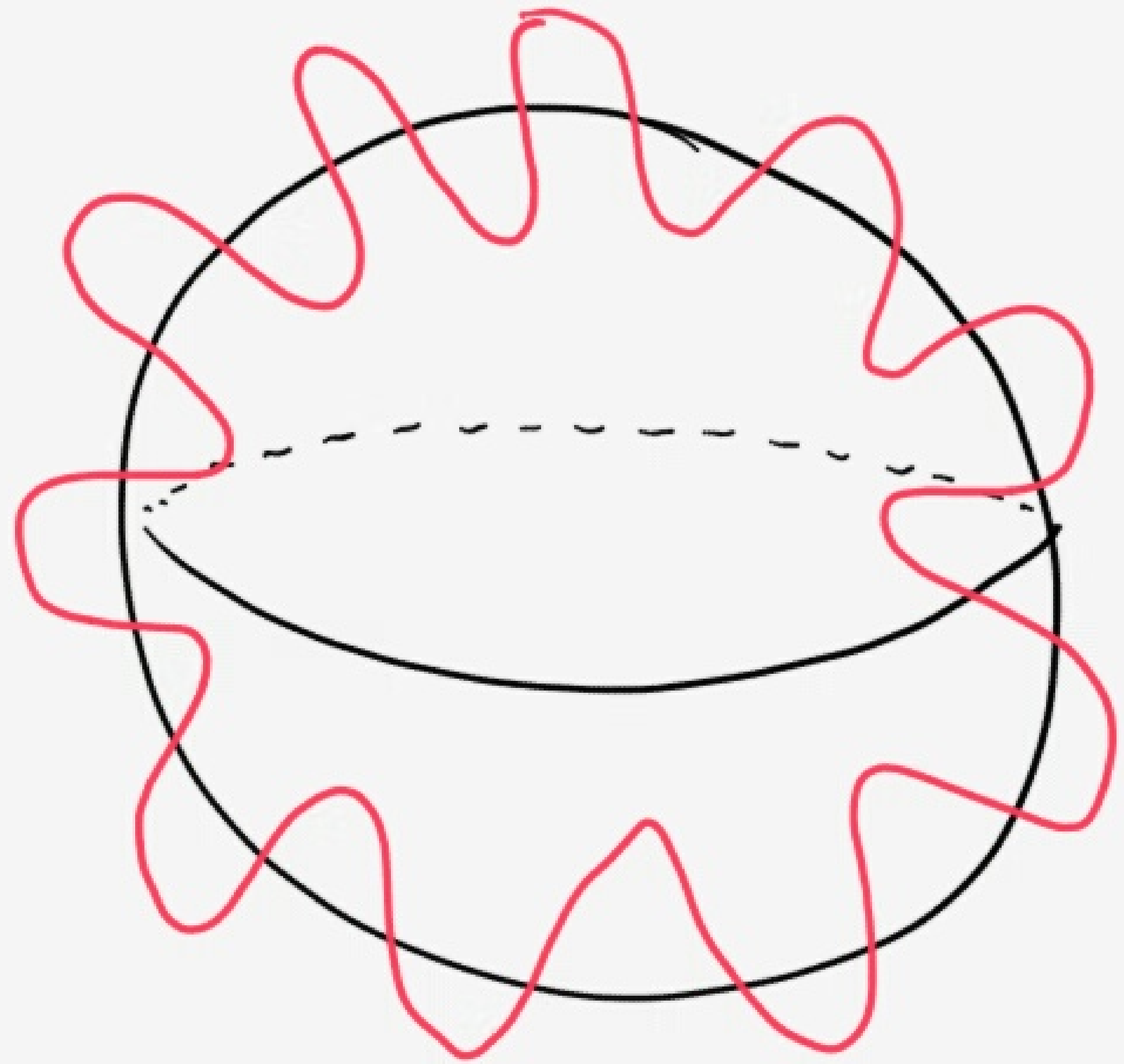
$$D = \gamma^0 + \gamma^0\gamma^1\gamma^1 \otimes [L_{12}, \cdot] + \gamma^0\gamma^2\gamma^3 \otimes [L_{23}, \cdot]$$
$$+ \gamma^0\gamma^1\gamma^3 \otimes [L_{13}, \cdot]$$

$L_{ij}$ generate $su(2)$

$$\text{spectrum}(D) = \left.\begin{matrix} \{\pm 1, \pm 2, \ldots, \pm(n-1), \pm n \\ \{\pm 1, \pm 2, \ldots, \pm(n-1) \end{matrix}\right\}$$

2 Dirac fields on $S^2$, cutoff "Planck scale"

Picture of fuzzy sphere



$S^2$

Minimum wavelength

Maximum momentum

# 9. Commutative limit

$n \to \infty$

$L_{ij} \to x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ on $S^2 \subset \mathbb{R}^3$

$$D = \gamma^0 + \gamma^0 \sum_{i<j} \gamma^i \gamma^j [L_{ij}, \cdot]$$

$$\to \gamma^0 + \gamma^0 \sum_{i<j} \gamma^i \gamma^j \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right) \text{ on } S^2.$$

$=$ 2 Dirac fields on $S^2$.

# 10. Fuzzy coadjoint orbit

Generalise : $SU(2) \longrightarrow G$, semisimple compact

$L_{ij} \longrightarrow$ generators of Lie$(G)$, rep $\rho$

$S^2 \longrightarrow$ Coadjoint orbit $(\rho) \subset \text{Lie}(G)^*$

symplectic, Riemannian

dim even

## 11. Fuzzy spaces

$\mathcal{H} = V \otimes M_n(\mathbb{C})$

$V$ : spinors, signature $(p,q)$

$\mathcal{A} = M_n(\mathbb{C})$

e.g.

$\mathcal{G}(1,3)$

• fuzzy $S^2$

$$D = \sum_a \gamma^a \otimes [K_a, \cdot]_\pm + \sum_{abc} \gamma^a \gamma^b \gamma^c \otimes [K_{abc}, \cdot]_\pm + \dots$$

$+$ : anticommutator, $\gamma^a\gamma^b\dots$ and $K_{ab\dots}$ hermitian

$-$ : commutator, $\gamma^a\gamma^b\dots$ and $K_{ab\dots}$ antihermitian

$K_\bullet \in M_n(\mathbb{C})$ free data

12. Quantum/Random NCG + Lisa Glaser

$$f, S : \mathcal{G} \to \mathbb{R}$$

$$\langle f \rangle = \frac{\int_{\mathcal{G}} f(D)\, e^{iS(D)}\, dD}{\int_{\mathcal{G}} e^{iS(D)}\, dD}$$

$i \to -1$

# 13. Postscript: 2d state sum model