Quantum Amplitudes in Black–Hole Evaporation

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1. Quantum theories, amplitudes and boundary data

This paper describes work done with Andrew Farley in his 1997-2002 Cambridge PhD [1] on quantum states in black–hole evaporation. This concerns the quantum–mechanical decay of a Schwarzschild–like black hole, formed by gravitational collapse, into almost–flat space and weak radiation at a very late time [2,3]. One has to make a realistic assumption about the Lagrangian for the combined gravitational and matter fields, which here (for the sake of definiteness) we take to include at least Einstein gravity with a massless scalar field.

If one works with Feynman–diagram amplitudes, based on particle–number in– and out–states, then, with such boundary data, almost all interacting 4–dimensional quantum field theories yield irremovable divergences or infinities. Alternatively, one may work with Dirac's Hamiltonian–based approach to the quantisation of theories with continuous symmetries — such as local co–ordinate transformations in general relativity or local gauge transformations in gauge theory [4]. The quantity naturally calculated in Dirac's approach is the quantum amplitude to go from initial 'co–ordinate' data, such as the intrinsic spatial metric $h_{ij}(x_I)$ $(i, j = 1, 2, 3)$ and scalar field $\phi(x_I)$ on the **space–like hypersurface**, Σ_I , to final 'co–ordinate' data, $(h_{ij}(x_F), \phi(x_F))$ on Σ_F . Here, the space–like hypersurfaces Σ_I and Σ_F are asymptotically flat, and $T > 0$ denotes the time–interval between Σ_I and Σ_F , measured at spatial infinity.

The quantum amplitudes which naturally arise from the Dirac constrained–quantisation approach are in a **field representation**. A more detailed investigation suggests that, in the present context, the Feynman–diagram particle representation and the present field representation may be **unitarily inequivalent** [5]; that is, one may have an example of Haag's theorem [6]. Given the above difficulty with infinities in Feynman diagrams, which are based on particle–number eigenstates, it would seem sensible to investigate the alternative Dirac canonical approach, with boundary data given on space–like hypersurfaces.

To find examples of 4–dimensional quantum field theories which are, at least, well behaved, with calculable quantum amplitudes, *via* the Dirac approach in the field representation, one naturally looks for models with the largest amount of local symmetry. These are locally–supersymmetric theories such as simple supergravity or gauge–invariant supergravity [7]. The latter consists of supermatter coupled to supergravity in such a way that the whole theory is invariant under local supersymmetry, local co–ordinate transformations and local gauge transformations ($SU(n)$, say). Remarkably, it turns out that quantum amplitudes in these locally–supersymmetric models are, in a certain sense, **semi–classical**, being of the form $\exp(-I_{\rm class})$ times a delta–functional of the fermionic supersymmetry constraints at the boundaries. This last factor enforces the classical supersymmetry constraints at $\Sigma_{I,F}$. Here, $I_{\rm class}$ denotes the Euclidean action, I , evaluated at the classical infilling solution which joins the initial data on Σ_I to the final data on Σ_F , again separated from Σ_I by a time–interval T , measured at spatial infinity. One can verify more directly that the above 'semi–classical' form for the quantum amplitude **does** hold, in a calculable but non–trivial locally–supersymmetric example: One starts from Witten's supersymmetric quantum mechanics [8], extended to local supersymmetry by Alvarez [9], and describes quantum amplitudes in terms of a boundary–value problem, by analogy with the field–theory version above. Working within this Dirac framework, one arrives at precisely the form of the amplitude described above, namely, Amplitude $=$ δ(supersymmetry constraints) $\exp(-I_{\rm class})$ [5].

2. Complex time–separation

From Sec.1, in a locally–supersymmetric theory, the **quantum amplitude** to go from initial to final purely–bosonic data on $\Sigma_I\,,\Sigma_F$, separated by a time–interval T at spatial infinity, is

$$\text{Amplitude} \;=\; \delta(\text{supersymmetry constraints on boundaries})\,\exp(-\,I_{\rm class}), \qquad (2.1)$$

in units with $\hbar = 1$. This holds provided that there exists a (unique) classical solution, joining the initial to final data in time T at infinity. Existence is precisely most difficult when T is a **real** Lorentzian time–interval. Conversely, existence results are known to hold in a range of cases for which $T = i\tau$ is purely imaginary, giving an elliptic boundary–value problem. In such a case, τ denotes the 'Euclidean time–interval' between Σ_I and Σ_F , as measured at spatial infinity. The classical infilling 4–geometry, $g_{\mu\nu}$, will typically be **Riemannian** (equivalently **positive–definite**). At the opposite extreme, consider, for example, the boundary–value problem for a massless scalar field ϕ in flat Minkowski space–time, with data posed on parallel flat space–like hypersurfaces, Σ_I and Σ_F , separated by a Lorentzian time–interval T . Such a Lorentzian boundary–value problem for the wave equation is badly posed, with neither existence nor uniqueness in general [10,11].

We study the intermediate case,

$$T \;=\; |T|\exp(-\,i\theta), \qquad\qquad 0 < \theta \leq \pi/2\,, \qquad (2.2)$$

with **complex** time–interval, as measured at spatial infinity. The limiting case $\theta = \pi/2$ gives the 'Riemannian boundary–value problem', above. The opposite limit, $\theta \longrightarrow 0_+$, approaches the badly–posed Lorentzian case, $\theta = 0$, but is expected to remain well-posed for $\theta > 0$. Feynman [12] has taught us that Lorentzian quantum amplitudes are given by taking this limit of the complex–θ amplitude; equivalently, in Feynman diagrams, one takes the '$+\,i\epsilon$ prescription'.

The complex case, $0 < \theta < \pi/2$, studied in this work, is expected to lead to a **strongly elliptic** boundary–value problem, which has good existence and uniqueness properties [13]. For example, in the simple case of a complexified flat metric, $g_{\mu\nu}$, one can solve the classical field equation (linear wave equation) for a typical boundary–value problem, provided $\theta > 0$. In this case, one can construct the classical $\phi(x)$ explicitly, and examine the singular behaviour as $\theta \longrightarrow 0_+$ [11].

This leads to a prescription for calculating quantum amplitudes in our locally–supersymmetric case: evaluate the action, $I_{\rm class}$, for a classical solution given (say) non–trivial initial and final weak–field scalar data, ϕ , with time–interval $T = |T| \exp(-i\theta)$ and $0 < \theta \leq \pi/2$, then take the limit of $\exp(-I_{\rm class})$ as $\theta \longrightarrow 0_+$.

3. Weak perturbations

Consider the approximate solution of the boundary–value problem, given our Lagrangian. One takes the classical background (bosonic) fields, the metric $g_{\mu\nu}$ and real scalar field ϕ , each to have a 'large' time–dependent spherically–symmetric part, $g^{(0)}_{\mu\nu}$ and $\Phi(t,r)$, plus a 'small' perturbative part, $h^{(1)}_{\mu\nu}$ and $\phi^{(1)}$. The 'large' or background Lorentzian space–time metric can be written in the form

$$ds^2 \;=\; -\, e^{b(t,r)}dt^2 + \,e^{a(t,r)}dr^2 + \,r^2(d\theta^2 + \,\sin^2\theta\, d\varphi^2). \tag{3.1}$$

The spherically–symmetric part of the classical Einstein and scalar–field equations will be as for exact spherical symmetry [2,3], except for an additional effective energy–momentum contribution, $T^{\rm EFF}_{\mu\nu}$, resulting from local space–time averaging of the contribution to the Einstein equations of perturbation terms quadratic and higher.

The perturbative part of (for example) the scalar field can be expanded in spherical harmonics,

$$\phi^{(1)}(t,r,\theta,\varphi) \;=\; \frac{1}{r}\sum\nolimits_{\ell=0}^{\infty}\;\sum\nolimits_{m=-\ell}^{\ell}\; Y_{\ell m}(\Omega)\,R_{\ell m}(t,r). \tag{3.2}$$

We are assuming that the scalar boundary data on the final surface, Σ_F , are 'generic', in that extremely high multipoles, labelled by (ℓ, m), are present in a stochastic manner, suitably weighted. One expects that the resulting infilling perturbative scalar solution, $\phi^{(1)}$, corresponds to a distribution stochastic in time, and that the effective quantity $T^{\rm EFF}_{\mu\nu}$ above will be spherically symmetric.

At late times following the (assumed) gravitational collapse, the perturbed scalar equation is

$$\nabla^\mu\,\nabla_\mu\,\phi^{(1)} \;=\; 0\,. \tag{3.3}$$

When decomposed into the harmonics of Eq.(3.2), this gives the (ℓ, m) mode equation,

$$\big[e^{(b-a)/2}\partial_r\big]^2 R_{\ell m} - \ddot{R}_{\ell m} - \frac{1}{2}\,\big(\dot{a} - \dot{b}\big)\,\dot{R}_{\ell m} - V_\ell(t,r)R_{\ell m} \;=\; 0\,, \tag{3.4}$$

where

$$V_\ell(t,r) \;=\; \frac{e^{b(t,r)}}{r^2}\bigg(\ell(\ell+1) + \,\frac{2m(t,r)}{r}\bigg). \tag{3.5}$$

Here, $m(t,r)$ is defined by

$$\exp[-\,a(t,r)] \;=\; 1 - [2m(t,r)/r]. \tag{3.6}$$

For simplicity, consider the case in which the only non–zero perturbative boundary data are the spatial metric $h^{(1)}_{ij}$ and scalar field $\phi^{(1)}$ on Σ_F . The corresponding data on

Σ_I are taken to be zero; equivalently, the full initial data on Σ_I are taken to be exactly spherically symmetric. The Lorentzian action of the infilling classical solution is found, in Hamiltonian language, to be [2,3]

$$S_{\rm class}[h^{(1)}_{ij},\phi^{(1)}] \;=\; \frac{1}{32\pi}\int_{\Sigma_F} d^3x\;\pi^{(1)ij}\,h^{(1)}_{ij} \,+\, \frac{1}{2}\int_{\Sigma_F} d^3x\;\phi^{(1)}\,\pi_{\phi^{(1)}} - MT\,. \tag{3.7}$$

It is typical, for a theory not containing a mass, that $I_{\rm class}$ should take such a 'boundary' form. The linearised quantities, $\pi^{(1)ij}$ and

$$\pi_{\phi^{(1)}} \;=\; e^{-\,b(t,r)}\,\frac{\partial\phi^{(1)}}{\partial t}\;, \qquad .(3.8)$$

are the momenta canonically conjugate to $h^{(1)}_{ij}$ and $\phi^{(1)}$, and M is the ADM mass of the classical space–time.

The bosonic factor in the 'semi–classical' amplitude, Eq.(2.1), can thence be evaluated to give the quantum amplitude to go (say) from spherically–symmetric initial data on Σ_I to non–spherical final data, $h^{(1)}_{ij},\phi^{(1)}$, on Σ_F. The final scalar data, $\phi^{(1)}$ on Σ_F, can be expanded in spherical harmonics as in Eq.(3.2); correspondingly for the 'gravitational–wave data', $h^{(1)}_{ij}$ on Σ_F. The above quantum amplitude in our field representation can thus be related to quantum amplitudes in a particle description.

4. Quantum amplitude for weak scalar fields

We simplify the final boundary data of Sec.3 still further, allowing non–trivial scalar perturbations, $\phi^{(1)}\neq 0$, but no gravitational perturbations: $h^{(1)}_{ij} = 0$ on Σ_F. From Eqs.(3.2–6), the infilling linearised scalar field is a sum over angular modes, (ℓ,m), weighted by a solution, $R_\ell(t,r)$, of the radial wave equation (3.4,5) — note that the radial function, $R_{\ell m}(t,r)$, is, in fact, independent of the quantum number m.

On or near the final surface, Σ_F, long after the black hole has evaporated, one again expects a nearly–Schwarzschild background, more or less static when compared with the oscillations in the scalar field. In this case, the radial wave functions, $R_\ell(t,r)$, can be decomposed harmonically with respect to t. Given a suitable normalisation, one can write [3], at least near Σ_F,

$$\phi^{(1)} \;=\; \frac{1}{r}\sum\nolimits_{\ell m}\int_0^\infty dk\;a_{k\ell m}\,R_{k\ell}(t,r)\,Y_{\ell m}(\Omega), \tag{4.1}$$

where the $a_{k\ell m}$ are real quantities. The harmonic decomposition (4.1) should become more and more accurate near spatial infinity ($r\longrightarrow\infty$), where the space–time metric tends to the flat Minkowski metric. Hence, the eigenfrequencies k are discrete, of the form

$$k \;=\; k_n \;=\; \frac{n\pi}{|T|} \qquad\qquad (n=1,2,3,\dots\,). \tag{4.2}$$

One studies the radial wave equation for $R_{k\ell}(r)$ near $r=0$ and as $r\longrightarrow\infty$. One finds that

$$R_{k\ell}(r) \;\sim\; z_{k\ell}\,e^{ikr^*_S} + \bar{z}_{k\ell}\,e^{-ikr^*_S} \tag{4.3}$$

as $r \longrightarrow \infty$, where, for large r ,

$$r_S^* \;\sim\; r \,+\, 2M \log\Big(\big(r/2M\big) - 1\Big) \tag{4.4}$$

is the Regge–Wheeler 'tortoise' co–ordinate in the Schwarzschild geometry [14]. The $z_{k\ell}$ are dimensionless complex coefficients, determined by regularity at $r = 0$. These $z_{k\ell}$ are related to the Bogoliubov coefficients, thus making contact with the original formulation of black–hole evaporation [15,16,17].

Given the perturbative final scalar data, $\phi^{(1)}$, specified by the mode quantities $a_{n\ell m}$ above, one computes the (classical scalar contribution to the) Lorentzian action, $S_{\rm class}[\phi^{(1)}]$ of Eq.(3.7), in the case that the time–interval $T = \tau \exp(-\,i\theta)$ [Eq.(2.2)] at spatial infinity is rotated slightly into the complex, with $\theta > 0$. This classical scalar contribution to $S_{\rm class}$ will in general be complex. The Euclidean and Lorentzian actions are related by

$$I_{\rm class} \;=\; -\,i\,S_{\rm class}\;. \tag{4.5}$$

As described in Sec.2, if one is working within a locally–supersymmetric theory, one has a semi–classical amplitude, given by Eq.(2.1), to go from $\phi^{(1)} \;=\; 0$ on Σ_I to the prescribed non–zero $\phi^{(1)}$ above on Σ_F .

This amplitude, a function of the angle θ , where one eventually takes the limit $\theta \longrightarrow 0_+$, is exponential in form. It has a (computable) oscillating part in each mode, multiplied by a product of Gaussians,

$$|{\rm Amplitude}| \;\propto\; \exp\bigg(-\,\frac{4\pi^3}{|T|^2}\,\sum_{n=1}^{\infty}\,\sum_{\ell=0}^{\infty}\,\sum_{m=-\ell}^{\ell}\, n\,|z_{n\ell}|^2\,|a_{n\ell m}|^2\bigg). \tag{4.6}$$

in terms of the co–ordinates $\{a_{n\ell m}\}$ of Eqs.(4.1,2) for the perturbed final data. The corresponding late–time scalar radiation can be interpreted in terms of thermal Hawking radiation.

5. Comments

By following Dirac's approach for constrained Hamiltonian systems and working with locally–supersymmetric models, we have seen how quantum amplitudes (not just probabilities) may be found for transitions involving black holes to go from initial data on a space–like hypersurface, Σ_I , to final data on Σ_F . The semi–classical result (2.1) only applies in the case of a locally–supersymmetric Lagrangian. Further, this result was derived in the field representation, with data given on Σ_I and Σ_F . A basis for states in quantum field thory, natural to the above field representation, may well be **unitarily inequivalent** to a basis of the kind typically taken for particle scattering in particle physics, involving incoming or outgoing particle–number states.

Among bosonic fields, results analogous to Eq.(4.6) have also been derived for spin–1 Maxwell theory and for the spin–2 linearised gravitational field [18,19]. The corresponding quantum amplitude for the fermionic spin–$\frac{1}{2}$ field has also been derived [20].

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