Renormalisation Group Flows in Four Dimensions and the 'a-theorem'

John Cardy

University of Oxford

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Renormalisation Group

The RG, as applied to fluctuating systems extended in space or space-time ('quantum or statistical field theories') is one of the great organising principles of modern physics:

- suppose the physics at a given length scale ℓ_0 (= inverse energy or momentum scale) is specified by dimensionless parameters $\{g_1(\ell_0), g_2(\ell_0), \ldots\}$ (= masses, coupling constants)
- then the physics at some other length scale ℓ is the same as if we stay at ℓ_0 but allow the couplings $\{g(\ell)\}$ to *flow* according to

$$\ell \frac{dg_j(\ell)}{d\ell} = -\beta_j(\{g(\ell)\})$$

- in particular, as $\ell \to \infty$ (IR limit) or as $\ell \to 0$ (but still $\gg$ any UV cut-off) (UV limit) we expect that $\{g\} \to \{g^*\}$ where $\beta_j(\{g^*\}) = 0$ – a RG fixed point

RG fixed points and and conformal field theories

- RG fixed points correspond to scale-invariant systems: e.g. massless QFTs or statistical models at a critical point
- when such systems are in addition Lorentz (rotationally) invariant scale invariance is enlarged to conformal symmetry: we have a conformal field theory (CFT)
- so all such systems are characterised by their possible fixed points (= CFTs) and the allowed flows between them

$$\mathrm{CFT}_{UV} \longrightarrow \mathrm{CFT}_{IR}$$

Is there a general principle constraining such flows?

Two dimensions: Zamolodchikov's c-theorem

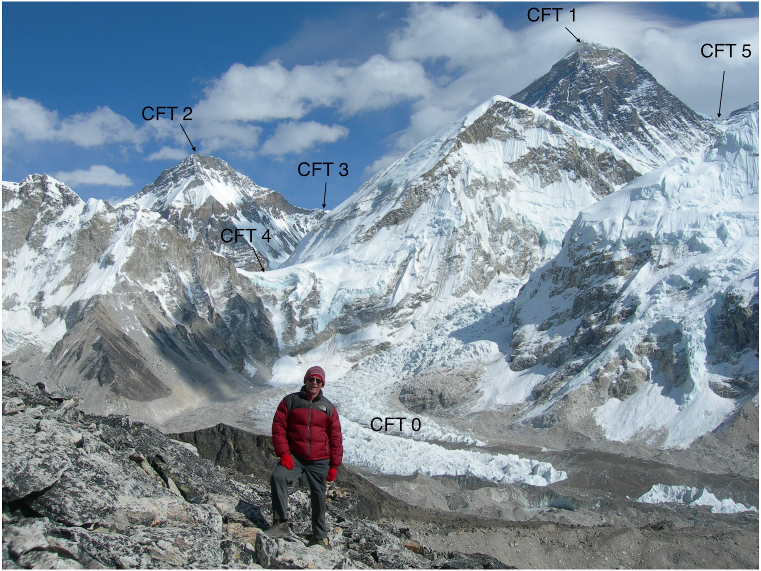
In $d = 2$, each CFT is characterised by its conformal anomaly number c (for a free scalar or Dirac fermion, $c = 1$, but in general c can be non-integer.)
In 1987 A. Zamolodchikov showed that:

- there exists a function $C(\{g\})$ on the space of all 2d QFTs which is
  - decreasing along RG flows
  - stationary at each RG fixed point (CFT), where its value is the appropriate c
- in particular this implies

$$c_{UV} > c_{IR}$$

- in 2d RG flows 'go downhill'

A landscape of CFTs



Higher Dimensional Generalisations

- in 1988 JC proposed a generalisation to all even dimensions d, which came to be known as the 'a-theorem':
  - there exists a pure number a characterising d-dimensional CFTs, such that, along RG flows

$$a_{UV} > a_{IR}$$

- this was shown to be true for 'weakly relevant flows' (when CFT_{IR} is close to CFT_{UV}) and seemed to be satisfied by all known examples
- for free field theories, a measures the *diversity of species* of massless particles: in four dimensions

$$a = \#\text{ scalars} + 11\,\#\text{ Dirac fermions} + 62\,\#\text{ gauge fields} + \cdots$$

Example

QCD with N_c colours and N_f massless fermions:

- asymptotic freedom implies that

$$a_{UV} = 11 N_c N_f + 62(N_c^2 - 1)$$

- in the IR we expect chiral symmetry breaking to leave $N_f^2 - 1$ Goldstone bosons, so

$$a_{IR} = N_f^2 - 1$$

- the conjectured a-theorem is therefore violated if

$$N_f > \tfrac{11}{2} N_c + \left[(\tfrac{11}{2})^2 N_c^2 + 62(N_c^2 - 1) + 1 \right]^{1/2}$$

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- however, asymptotic freedom is already lost if $N_f > \frac{11}{2} N_c$, so there is no contradiction

A little history of the 'a-theorem'

- Osborn (1989) showed that a decreases to all orders in perturbation theory
- as knowledge of strongly coupled gauge theories increased (especially because of Seiberg duality (1995)) the conjecture passed ever more rigorous tests
- for example, if the numbers $(11, 62, \ldots)$ are modified there exist counterexamples
- in supersymmetric theories it is related to R-symmetry and a version was proved (Intriligator and Wecht, 2003)
- holographic version proposed in 1999 (Freedman et al)
- related to entanglement entropy (Myers and Sinha, 2011)
- in 2008 Shapere and Tachikawa claimed a counterexample, however this was rebutted by Gaiotto, Seiberg and Tachikawa (2010)

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- in July 2011 Komargodski and Schwimmer posted arXiv:1107.3987 which provides a ‘proof’ of the a-theorem

Outline of the rest of the talk

- the role of the stress tensor in CFT
- Zamolodchikov's argument in 2d and why it doesn't work for $d > 2$
- the a-theorem proposal
- Komargodski and Schwimmer's argument
- open questions

The stress tensor

- suppose the action of a QFT with a set of fields $\{\phi\}$, in curved space with metric $g_{\mu\nu}$, is

$$S = \int d^d x \sqrt{g}\, \mathcal{L}(\{\phi\}, g_{\mu\nu})$$

- classically, the stress tensor (= stress-energy tensor, (improved) energy-momentum tensor) is

$$T^{\mu\nu}(x) = \frac{\delta S}{\delta g_{\mu\nu}(x)}$$

- this is what appears on the RHS of Einstein's equation
- in flat space, up to a total derivative it is the same as the Noether current corresponding to translational symmetry
- it is symmetric and conserved: $\partial_\nu T^{\mu\nu} = 0$
- scale invariance under $x^\mu \to e^b x^\mu$ implies $T^\mu_\mu = 0$

- in the quantum theory, $T_{\mu\nu}$ must be regularised, leading to possible anomalies. In general there is a trace anomaly in flat space:

$$T^\mu_\mu \propto \sum_j \beta_j(\{g\})\,\Phi_j$$

- so in a CFT, $T^\mu_\mu = 0$ in flat space
- however in curved space there are further c-number anomalies and $T^\mu_\mu \neq 0$

Two dimensions

In 2d CFT there is only one anomaly number called c, which plays various equivalent roles:

- the 2-point function of the stress tensor in flat space

$$\langle T_{\mu\nu}(x) T_{\lambda\sigma}(0)\rangle = \frac{c}{x^4} \times \text{index structure}$$

- the entropy density at finite temperature $s = \pi c T/3$
- the von Neumann (entanglement) entropy of an interval A of length L at zero temperature: $S_A \sim (c/3)\log L$
- the anomaly in curved space:

$$\langle T^\mu_\mu\rangle = -\frac{cR}{12}$$

where R is the scalar (gaussian) curvature

Zamolodchikov's argument in 2d

- in general $T_{\mu\nu}$ has a spin-2 traceless symmetric part and a spin-0 part (the trace), so in 2d there are only 3 independent components T, $\overline{T}$ and $\Theta = T^{\mu}_{\mu}$
- rotational invariance implies ($r^2 = z\bar{z}$)

$$\begin{aligned} \langle T(z,\bar{z})T(0)\rangle &= F(r)/z^4 \\ \langle T(z,\bar{z})\Theta(0)\rangle &= G(r)/z^3\bar{z}^2 \\ \langle \Theta(z,\bar{z})\Theta(0)\rangle &= H(r)/z^2\bar{z}^2 \end{aligned}$$

Conservation $\partial_\mu T^{\mu\nu} = 0$ then gives

$$r(d/dr)C = -\tfrac{3}{8}H \quad \text{where} \quad C \equiv F - \tfrac{1}{2}G - \tfrac{3}{16}H$$

But $H \propto \langle\Theta\Theta\rangle > 0$ by reflection positivity (= unitarity).

- this fails for $d > 2$ because there are too many amplitudes

The 1988 proposal

- in 2d we also have $\langle T^\mu_\mu \rangle = -cR/12$
- note that by the Gauss-Bonnet theorem this implies

$$\int_{\mathcal{M}} \langle T^\mu_\mu \rangle \sqrt{g}\, d^2x = -(c/12) \int_{\mathcal{M}} R \sqrt{g}\, d^2x = -c\,\chi/12$$

  where χ is the Euler character of $\mathcal{M}$
- so let us define a candidate C-function for even $d \geq 2$

$$C = \alpha_d \int_{\mathcal{M}} \langle T^\mu_\mu \rangle \sqrt{g}\, d^dx$$

  where α_d is fixed by $C = 1$ for a free scalar boson
- for calculational purposes it seemed easiest to choose $\mathcal{M} =$ the sphere S^d
- the conjecture: C decreases along RG flows and $C_{UV} > C_{IR}$

- although this can be checked in perturbation theory (either ‘weakly relevant’ or Banks-Zaks flows), a general proof has up to now been absent
- one of the problems is that $\langle T^\mu_\mu \rangle$ contains quartic and quadratic diverges in 4d which must be subtracted, and these spoil naive positivity arguments
- one needs to relate the anomaly to something else physical and finite

Curved space anomalies in four dimensions

In fact in a general curved background there are two separate anomalies in a CFT in $d = 4$:

$$\langle T^\mu_\mu \rangle = -aE_4 + cW^2$$

where

$$\begin{aligned} E_4 &= R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \qquad \text{(Euler density)} \\ W^2 &= W_{\mu\nu\lambda\sigma}W^{\mu\nu\lambda\sigma} \\ &= R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2 \qquad \text{(Weyl tensor)}^2 \end{aligned}$$

- the first integrates up to be proportional to the Euler character, so the 1988 conjecture should properly be called the '*a*-theorem'
- in principle there could also be a *c*-theorem, but there are known counter-examples

Outline of Komargodski-Schwimmer's proof

- consider the UV CFT perturbed by relevant operators: in flat space

$$S = S_{CFT_{UV}} + \sum_j \lambda_j \int \Phi_j(x) d^4x$$

where Φ_j has dimension $\delta_j < 4$

- dimensionless coupling $g_j = \lambda_j \ell^{4-\delta_j}$, so $-\beta_j = (4-\delta_j)g_j$
- under the RG flow $g_j \to \infty$ and $S \to S_{CFT_{IR}}$

Is $a_{UV} > a_{IR}$?

Adding the dilaton

Consider a modified theory in which the fields are coupled to an additional scalar τ, known as the dilaton: in flat space

$$S = S_{CFT_{UV}} + \sum_j \lambda_j \int \Phi_j(x)\, e^{(\delta_j - 4)\tau} d^4x + f^2 \int e^{-2\tau} (\partial\tau)^2 d^4x$$

- under a scale transformation $x^\mu \to e^b x^\mu$, $\Phi_j \to e^{-b\delta_j}$, but $\tau \to \tau + b$, so the whole action is scale invariant
- in fact it is conformally invariant: $T^\mu_\mu|^{\text{total}} = 0$
- the last term is the action for a free scalar $\phi = 1 - e^{-\tau}$ in disguise
- f has the dimensions of mass: if we take $f \to \infty$ this picks out a VEV for τ (say $\tau = 0$) and we get back to the original theory
- the $O(\tau)$ term then couples to T^μ_μ of the original theory

- in practice, all we need is to take $f \gg$ any mass scale of the theory to see the UV$\rightarrow$IR crossover
- as this crossover happens, some of the degrees of freedom of CFT$_{UV}$ will become massive
- integrating these out will leave CFT$_{IR}$ plus an effective low-energy theory S_{dilaton} for the dilaton, which decouples at large f
- since the total theory is conformally invariant

$$a_{CFT_{UV}} = a_{UV}^{\mathrm{total}} = a_{IR}^{\mathrm{total}} = a_{CFT_{IR}} + a_{\mathrm{dilaton}}$$

so we need to argue that $a_{\mathrm{dilaton}} > 0$.

Determining the dilaton effective action

- in curved space the coupling to the dilaton takes the form

$$\sum_j \lambda_j \int \Phi_j(x)\, e^{(\delta_j - 4)\tau} \sqrt{g}\, d^4 x$$

- the scale invariance in flat space now shows up as invariance under Weyl transformations of the metric:

$$g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}\,, \qquad \tau \to \tau + \sigma$$

  so the effective action should respect this, up to the anomaly
- KS (based on earlier work by Schwimmer and Theisen) determined the effective action $S_{\rm dilaton}$ for the dilaton up to four derivatives

Anomalous terms in S_{dilaton}

We need to construct an action S_{anomaly} such that its Weyl variation takes the form

$$\delta S_{\text{anomaly}}/\delta\sigma = c_{\text{dil}} W^2 - a_{\text{dil}} E_4$$

The result, up to 4 derivatives, is

$$S_{\text{anomaly}} = \int \tau(c_{\text{dil}} W^2 - a_{\text{dil}} E_4)\sqrt{g} d^4 x - a_{\text{dil}} \int \left[4(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R)\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\tau + 2(\partial\tau)^4\right]\sqrt{g} d^4 x$$

- a_{dil} couples linearly to the Euler density as expected but also to terms which survive in flat space
- there are also non-anomalous, Weyl-invariant, terms in S_{dilaton} at this order, but in flat space they vanish by the equation of motion $\Box\tau = (\partial\tau)^2$

Dilaton-dilaton scattering

- the terms proportional to $a_{\rm dilaton}$ which survive in flat space, after using the equation of motion, are

$$S_{\rm anomaly} \to 2a_{\rm dilaton} \int (\partial\tau)^4 d^4x$$

- so $a_{\rm dilaton}$ determines the on-shell low-energy elastic dilaton-dilaton scattering amplitude:

$$\mathcal{A}(s,t,u) = \frac{a_{\rm dilaton}}{f^4}(s^2+t^2+u^2) + \cdots$$

- going to the forward direction $t=0$, $u=-s$ we can write a dispersion relation for $\mathcal{A}(s)/s^3$

$$a_{\rm dilaton} = \frac{f^4}{\pi}\int \frac{\sigma^{\rm tot}(s')}{s'^2}ds'$$

  where $\sigma^{\rm tot}$ is the total cross-section for dilaton+dilaton$\to$ heavy particles
- since this is > 0, QED.

Comments and open questions

- relating $a_{UV} - a_{IR}$ to something physical (dilaton scattering) neatly sidesteps all the problems about subtractions, etc. (which are in fact buried in the non-universal, non-anomalous terms)
- the ‘proof’ uses classic and commonly accepted ideas of quantum field theory: anomaly matching, dispersion relations, etc., but is not as clean as Zamolodchikov’s in 2d: perhaps it can be more directly related to the $\langle TTTT \rangle$ 4-point function in flat space
- are RG flows gradients of an interpolating function $A(\{g\})$?
- the proof extends to all even d but something else is needed for odd dimensions – important for condensed matter applications

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- are RG flows gradients of an interpolating function $A(\{g\})$?
- the proof extends to all even d but something else is needed for odd dimensions – important for condensed matter applications
- but in the interesting case $d = 4$ we have a new principle governing all QFTs which might be used, for example, to constrain strongly coupled physics at the TeV scale

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References:

- A B Zamolodchikov, *"Irreversibility of the Flux of the RG in a 2d Field Theory,"* JETP Lett. **43**, 730 (1987)
- JC, *"Is there a c-theorem in Four Dimensions?"* Phys. Lett. **B215**, 749 (1988)
- Z Komargodski and A Schwimmer, *"On RG Flows in Four Dimensions",* arXiv:1107.3987; JHEP 1112 (2011) 099
- Z Komargodski, *"The Constraints of Conformal Symmetry on RG Flows".* arXiv:1112.4538
- *"Proof found for unifying quantum principle,"* Nature News 14 Nov 2011, `www.nature.com/news/proof-found-for-unifying-quantum-principle-1.9352`