Horizon wave-function and the quantum hoop conjecture

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Abstract

We address the issue of (quantum) black hole formation by particle collision in quantum physics. We will introduce the horizon wave-function for quantum mechanical states representing a single localised particle, from which we derive a Generalised Uncertainty Principle. For two highly boosted non-interacting particles that collide in a one-dimensional space, this wave-function determines a probability that the system becomes a black hole depending on the initial momenta and spatial separation between the particles. This probability allows us to extend the hoop conjecture to quantum mechanics and estimate corrections to its classical counterpart.
ArXiv:1305.3195, 1306.5298 [EPJ C], 1311.5698 [PLB]
Collaboration: A. Giugno, O. Micu, A. Orlandi, F. Scardigli, …

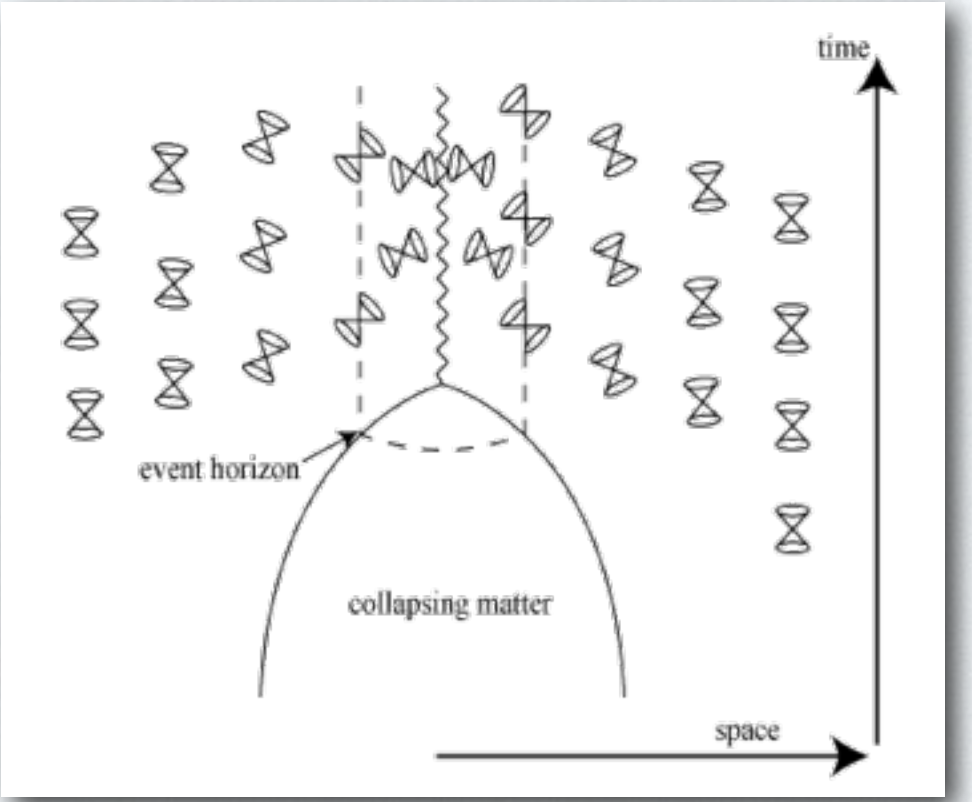
Plan of the talk

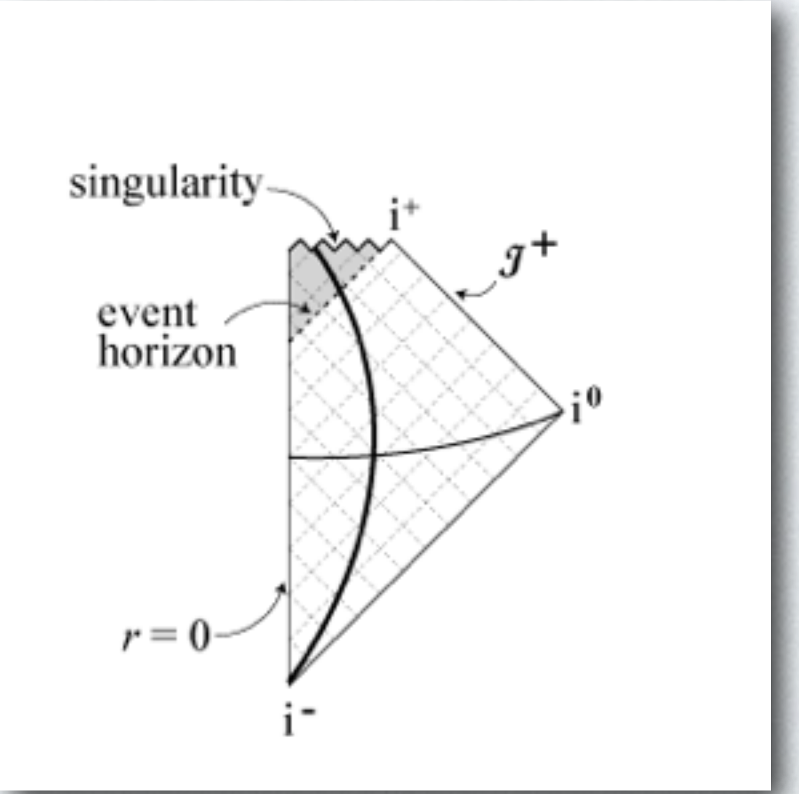
1. Physical system: gravitational collapse of quantum matter
2. Hawking radiation: lessons from the semiclassical picture
3. Hoop conjecture: Schwarzschild radius of a classical particle
4. Problem: Schwarzschild radius of a quantum particle?
5. Single particle: horizon wave-function and the GUP
6. 2-particle collision: horizon wave-function and the quantum hoop
7. Summary and outlook

1) Gravitational collapse

Standard classical picture: classical matter and "geometrical" space-time*

[N.B. Very little can be done without supercomputers…]





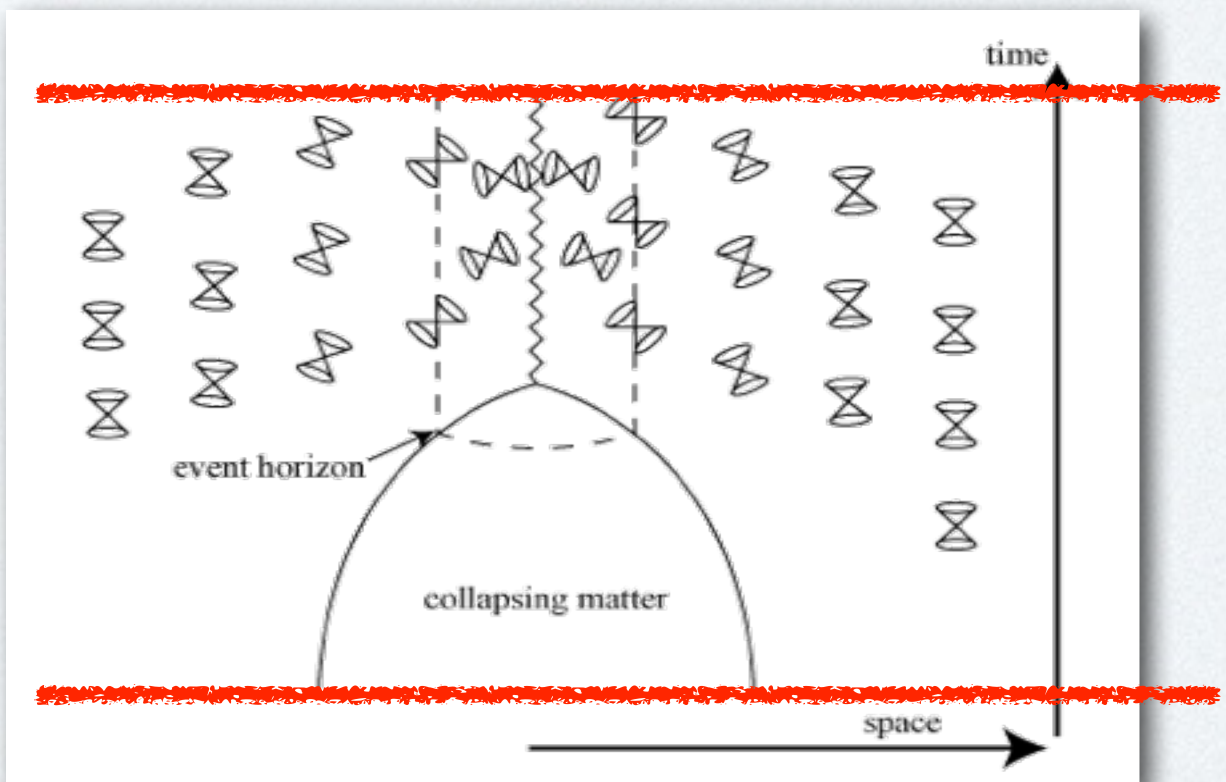
*Prototype background: $ds^2 = -\left(1-\frac{2\,M}{r}\right)dt^2 + \left(1-\frac{2\,M}{r}\right)^{-1} dr^2 + r^2\,d\Omega^2$

But matter is **quantum**…!

1) Gravitational collapse

Standard semiclassical picture: classical matter and "geometrical" space-time + foreground quantum particles

Prototype effect (with a few lessons to learn):



$$|0; t = +\infty\rangle$$

$$e^{-\frac{i}{\hbar} \int \hat{H}\, dt}$$

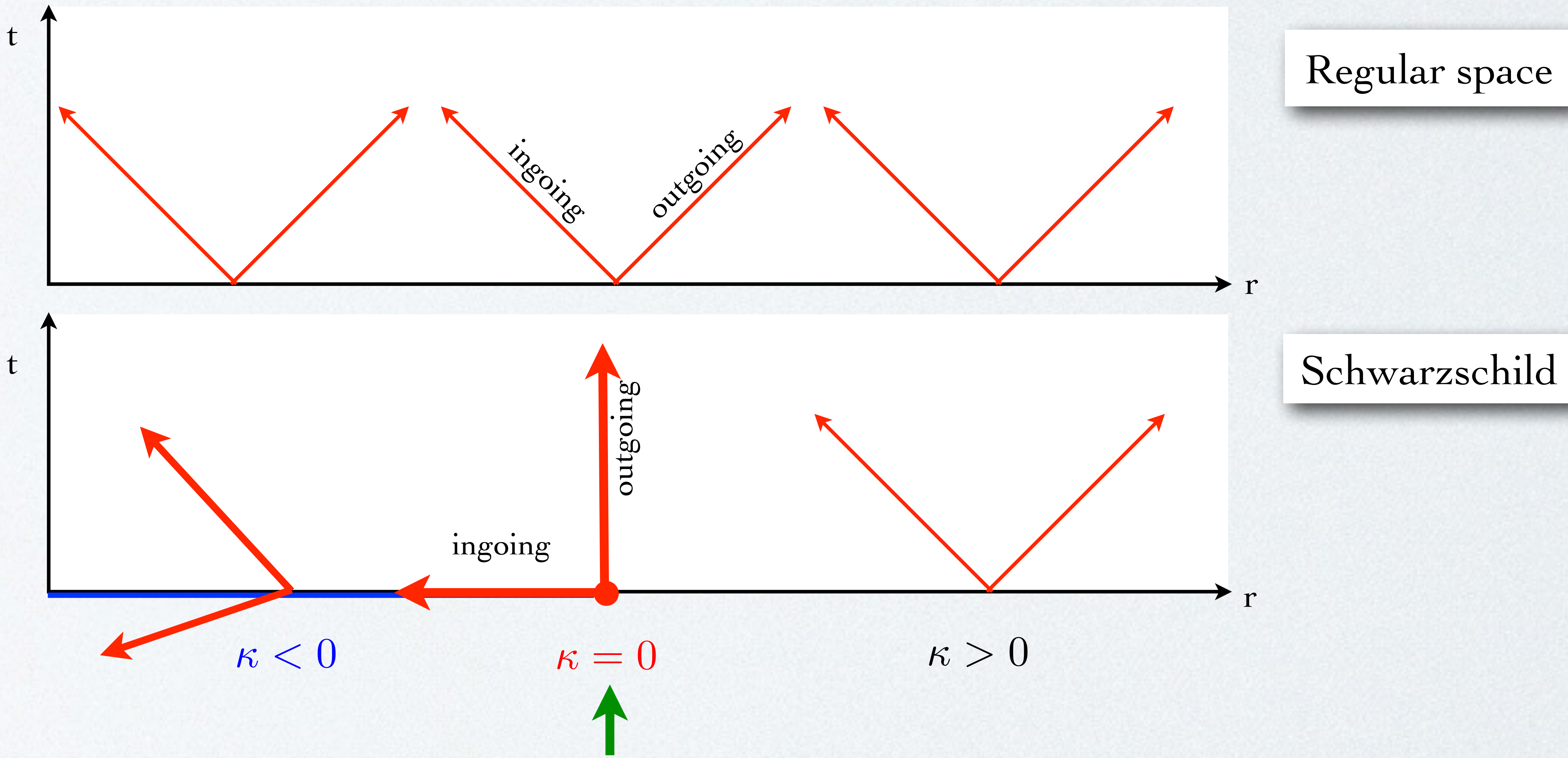
$$|0; t = -\infty\rangle$$

$$|0; t = +\infty\rangle = \sum \text{excitations} = \text{Hawking radiation}$$

Basic ingredients: QFT particles near horizons

1.1) Horizons

Naive concept: escape velocity = speed of light



Trapping surface: local concept = naive definition

$$-\theta_+ \theta_- = \kappa = \gamma^{ij}\,\partial_i R\,\partial_j R \qquad (\kappa \sim \dot{R}^2 - 1 \sim R - 2\,M) \qquad (\dot{R} = \partial_\tau R)$$

(Expansion of null geodesics)

Areal radius

$$ds^2 = \gamma_{ij}\,dx^i\,dx^j + R^2(x^i)\,d\Omega^2$$

Proper time of freely falling observer

1.2) Particles

Quantum field theory in a nutshell

1) Solve (classical) wave equation:

$$\Box\Phi = 0$$

2) Organise solutions into vector (formal Hilbert) space:

$$\Phi = \sum_{\vec{k}} \left[\underbrace{a_{\vec{k}}\, \phi_{\vec{k}}}_{\text{positive}} + \underbrace{a^{\dagger}_{\vec{k}}\, \phi^{*}_{\vec{k}}}_{\text{negative}} \right] \qquad (\phi_{\vec{k}} | \phi_{\vec{k}'}) = \delta_{\vec{k}\,\vec{k}'}$$

3) Lift (normal mode) solutions (excitations) to operators:

$$a_{\vec{k}} \mapsto \hat{a}_{\vec{k}} \qquad a^{\dagger}_{\vec{k}} \mapsto \hat{a}^{\dagger}_{\vec{k}}$$

4) Build (probabilistic Hilbert) Fock space of quantum states:

$$|\vec{k}; n\rangle \propto \left(\hat{a}^{\dagger}_{\vec{k}}\right)^{n} |\vec{k}, 0\rangle \qquad \underbrace{\hat{a}_{\vec{k}}}_{\text{positive}} |\vec{k}, 0\rangle = 0$$

So particles = field excitations or...?

1.2) Particles

Naive concept: particle = localised object

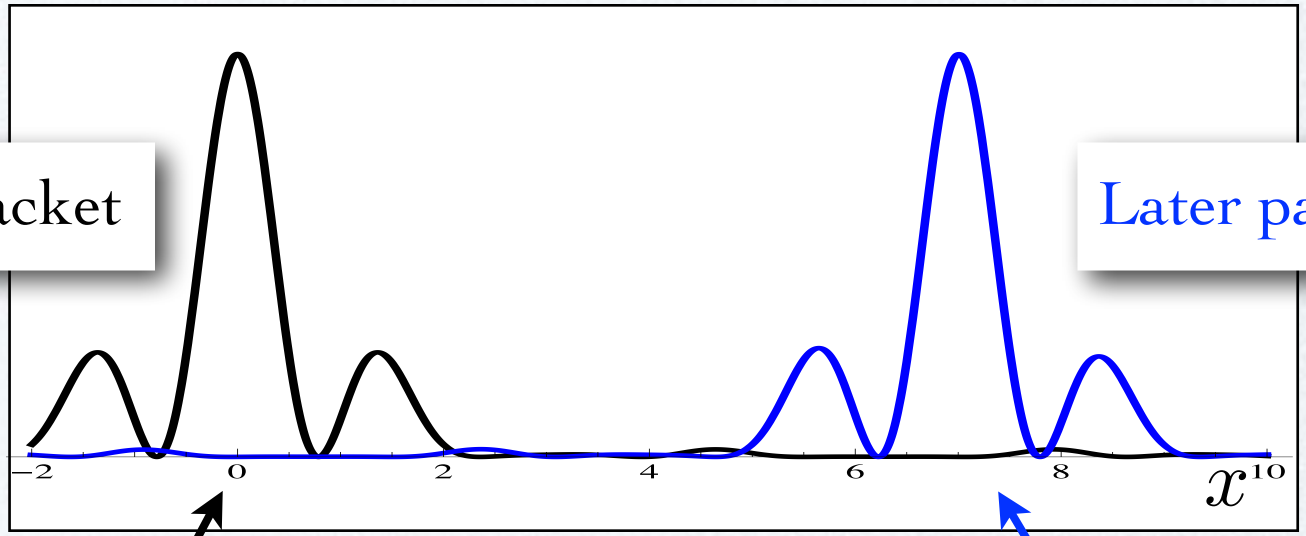
Example: free scalar field in 1+1 (Fourier transform in "formal solutions")

$$\phi_k = e^{-i\,\omega\,t + i\,k\,x} \qquad \omega^2 = k^2 + m^2$$

Time evolution preserves (physically sensible) "packets":

Initial packet

Later packet!



$$\Phi = \sum_{\text{a few}} \phi_k$$

$t = 0$

localised object

$t = 1$

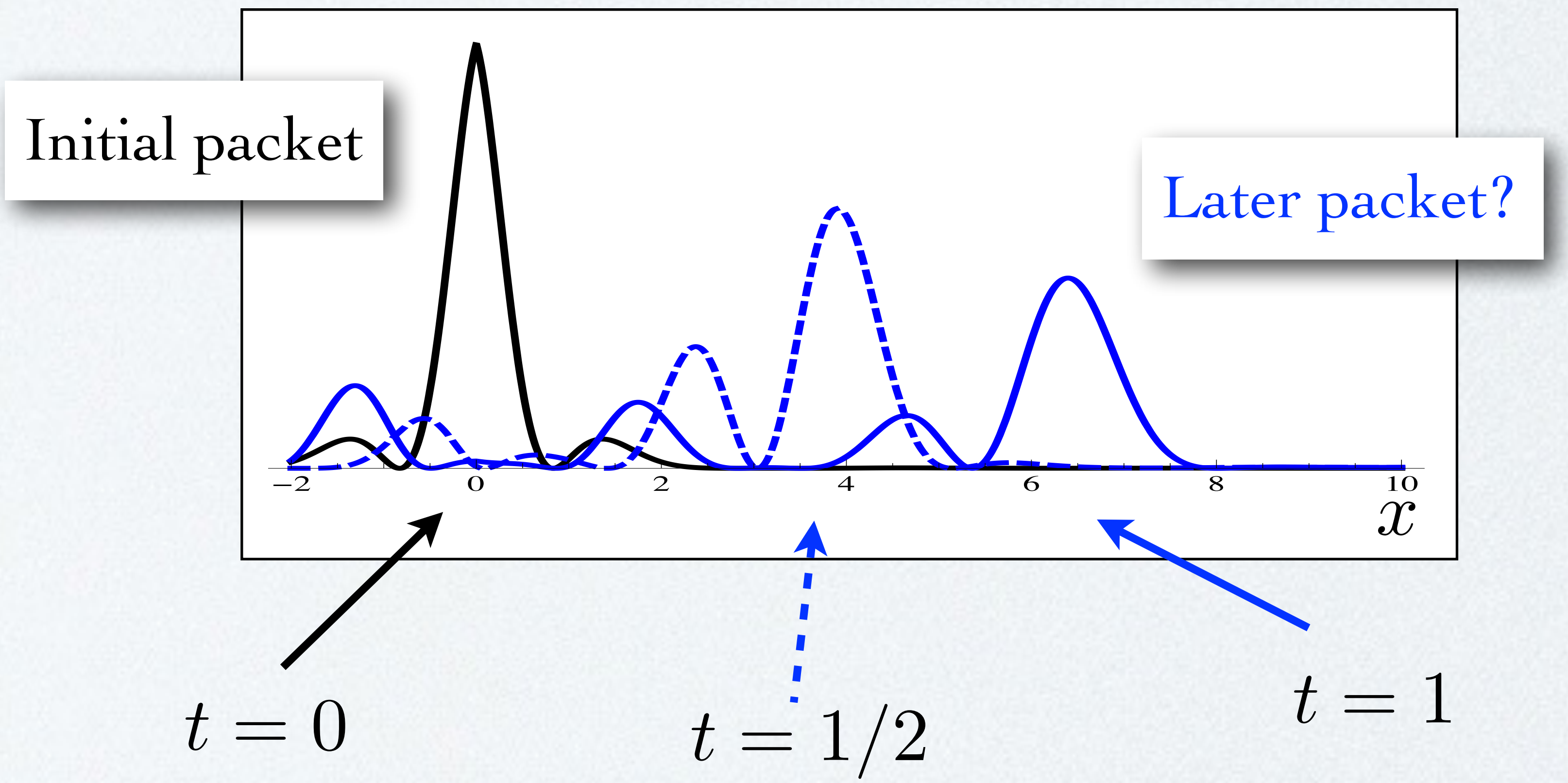
localised object

1.2) Particles

Counter-example: toy scalar field in 1+1

$$\phi_k = e^{-i\,\omega\,t + i\,k\,x - k*|x|} \qquad \omega^2 = k^2 + m^2$$

Time evolution does not preserve “packets”:



$$\Phi = \sum_{\text{a few}} \phi_k$$

5) In curved space-time, normal modes are not usually plane waves

↓

Packets not preserved (tidal effects induce Hawking emission)

2) Hawking radiation

<u>Backreaction</u>: (almost) never important in the standard (global) viewpoint

(Quantum stress tensor)

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\,\pi\, G_{\rm N} \left(T_{\mu\nu} + \langle \hat{T}_{\mu\nu} \rangle \right)$$

In Schwarzschild $\langle 0_{\rm H} | \hat{T}_{\mu\nu} | 0_{\rm H} \rangle \sim \dfrac{1}{M^2}$ ~ energy flux "small" down to horizon*

(in Unruh vacuum = with radiation)

$T_{\mu\nu} \sim 0$ ⟶ Small = globally $\int d^3x \, \langle \hat{T}_{\mu\nu} \rangle \ll M$

perturbatively $\langle \hat{T}_{\mu\nu} \rangle \ll M_{\rm p}^{-2}$

Backreaction large when $M \not\gg M_{\rm p}$

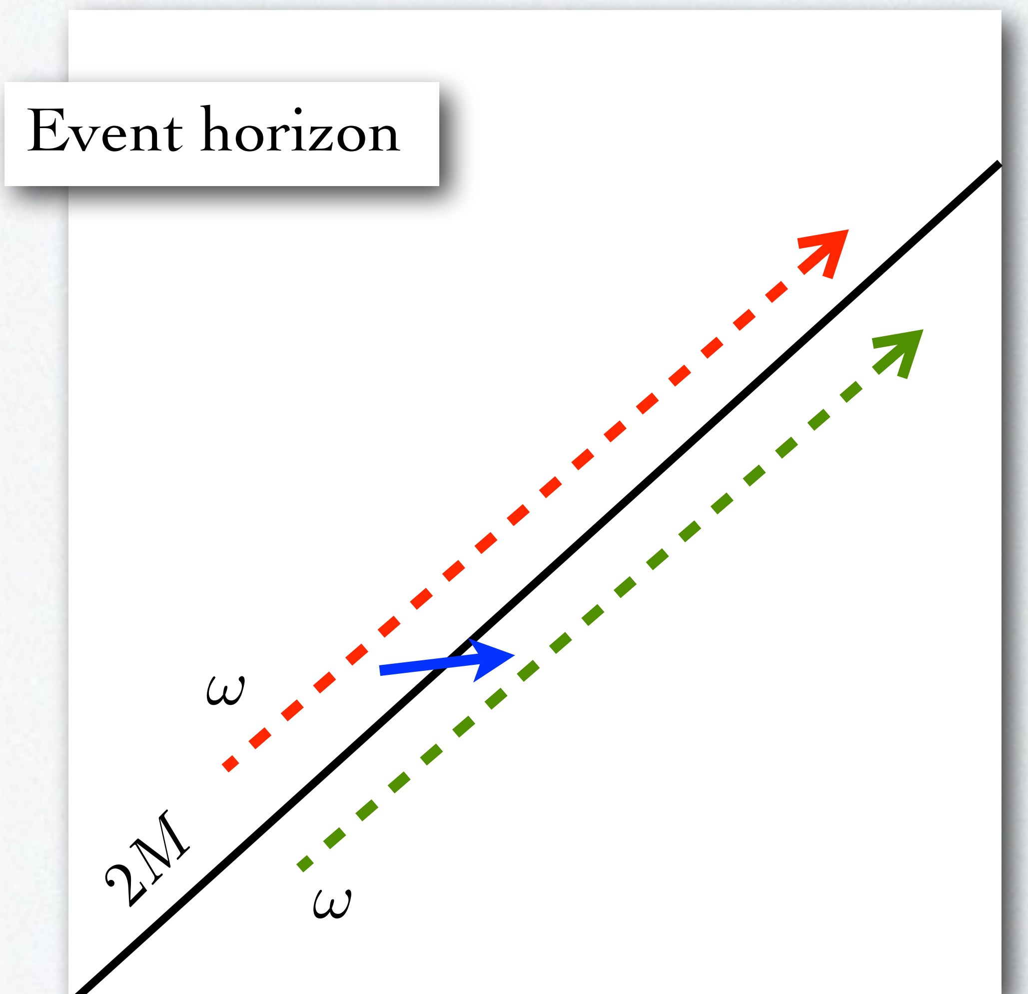
*Around a static star $\langle 0_{\rm B} | \hat{T}_{\mu\nu} | 0_{\rm B} \rangle \sim \dfrac{1}{r - 2\,M}$ (= radiation better than nothing)

2) Hawking radiation

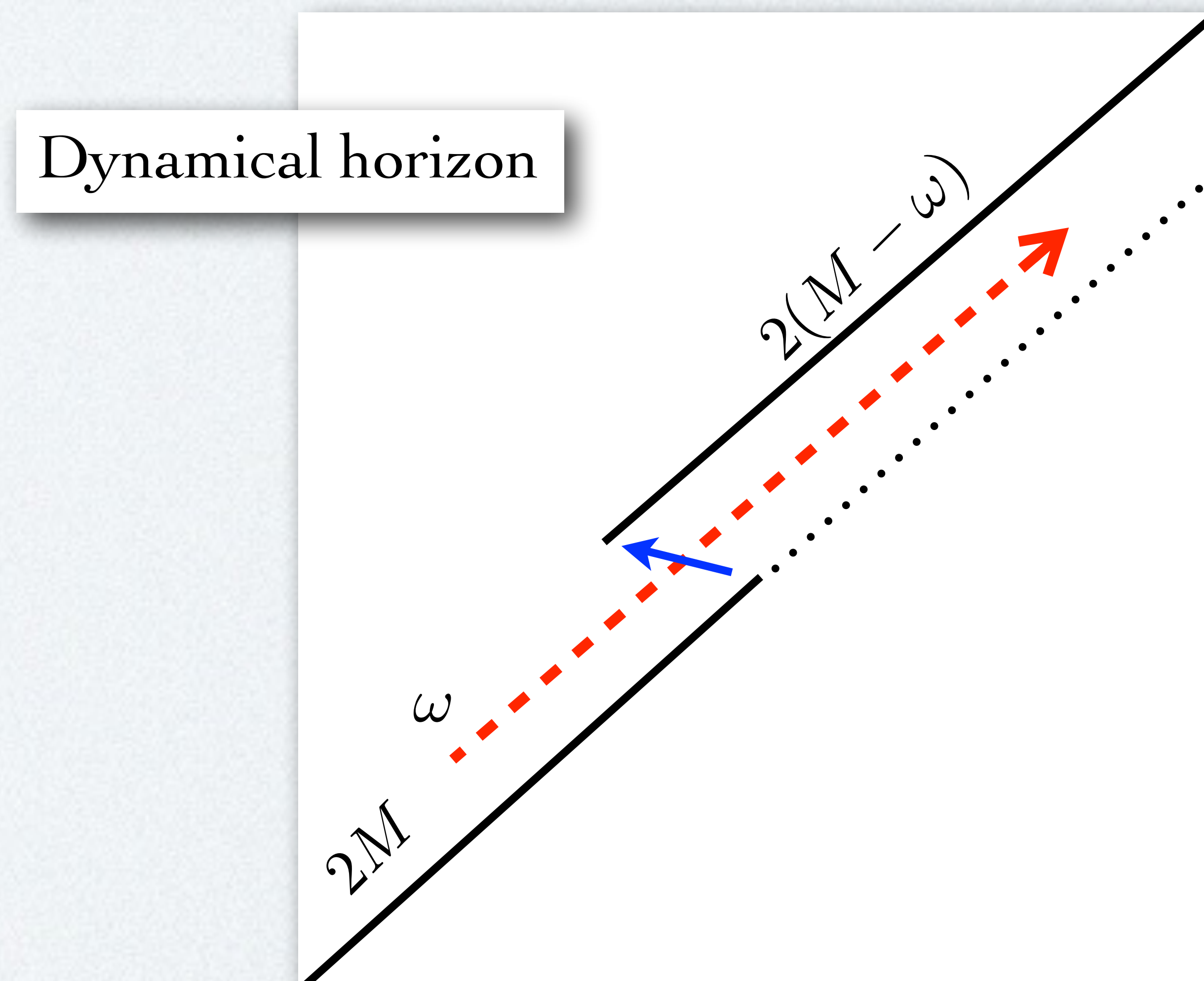
[Parikh, Wilczek, PRL 85 (2000) 5042]

Backreaction: (always) important in the local viewpoint

Quantum mechanical tunnelling across static horizon → Horizon quantum tunnelling (or particle self-tunnelling)



Without backreaction



With backreaction

("Particle opens its own exit door")

Transmission probability: $P \sim e^{-\beta_{\mathrm{H}}\,\omega}$ $\quad \beta_{\mathrm{H}} = 8\,\pi\,M$

2) Hawking radiation

Standard semiclassical picture: classical background geometry & matter + quantum foreground

$$R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} = 8\,\pi\, G_{\rm N} \left(T_{\mu\nu} + \langle \hat{T}_{\mu\nu} \rangle \right)$$

?

<-- iterate -->

$$\Box \Phi = 0$$

!

Semiclassical gravity: classical background geometry + quantum matter

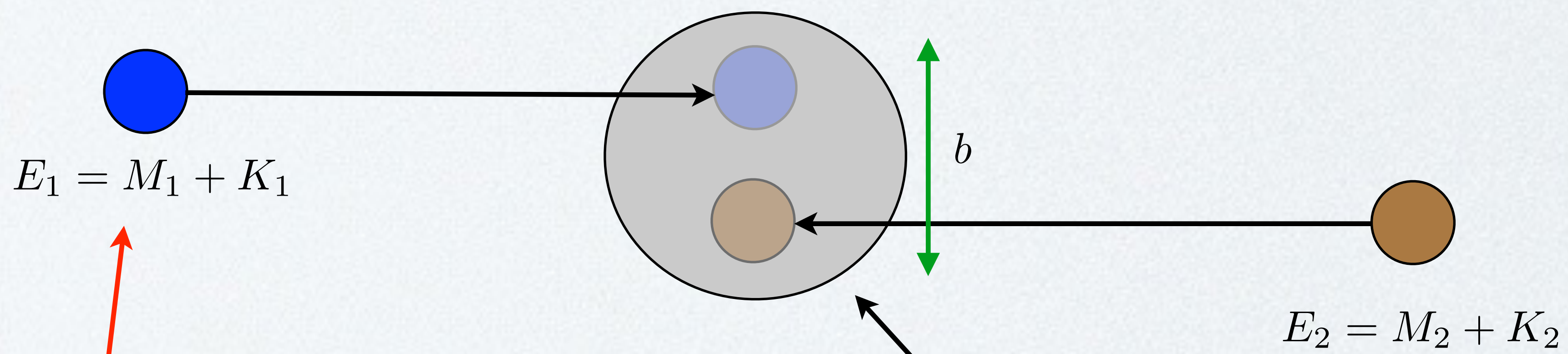
$$R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} = 8\,\pi\, G_{\rm N} \langle \hat{T}_{\mu\nu} \rangle$$

Let us take a step back to fully classical first…

3) Hoop conjecture

Thorne's *hoop conjecture* (1972):

A black hole forms when the impact parameter *b* of two colliding objects (of negligible spatial extension) is shorter than the radius of the would-be-horizon (Schwarzschild radius, for negligible angular momentum) corresponding to the total energy *E*



Quantum mechanical particle

?

$$b \lesssim 2\,\ell_{\rm p}\,\frac{E}{m_{\rm p}} \equiv R_{\rm H}$$

Classical geometry

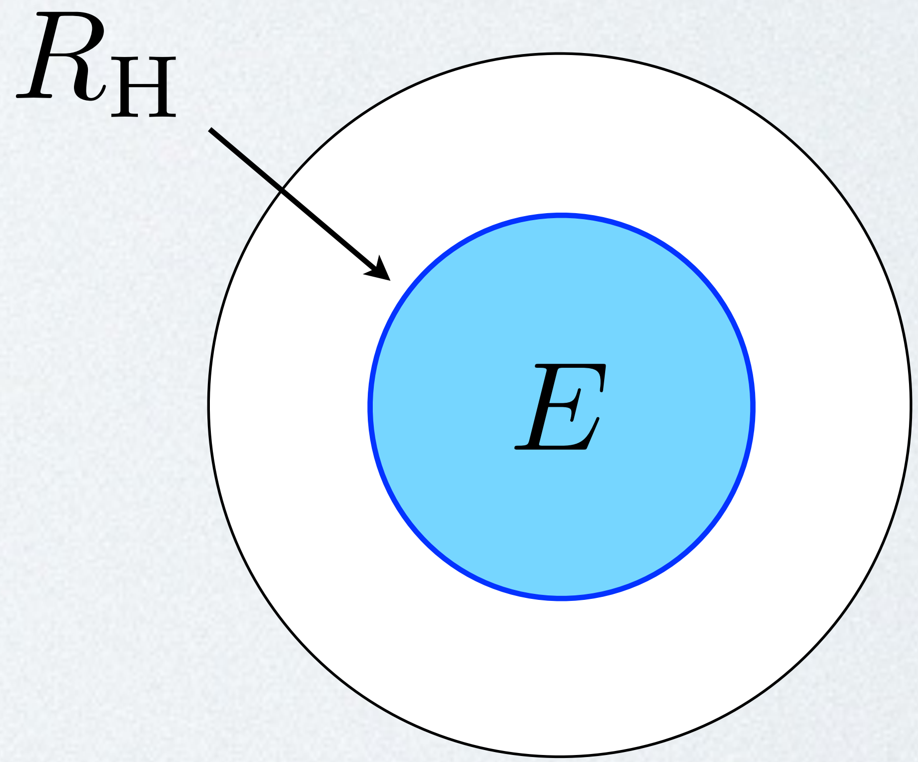
3) Hoop conjecture

Classical spherically symmetric system:

$$E(r) = \frac{4}{3}\pi \int_0^r \rho(t, r')\, r'^2\, dr'$$

Misner-Sharp mass

$$R_{\mathrm{H}} = 2\,\ell_{\mathrm{p}}\,\frac{E}{m_{\mathrm{p}}}$$



Surface is a "Horizon" if: $4\,\pi\,R_{\mathrm{H}}^2 = 4\,\pi\,r^2$

Schwarzschild radius

Areal radius

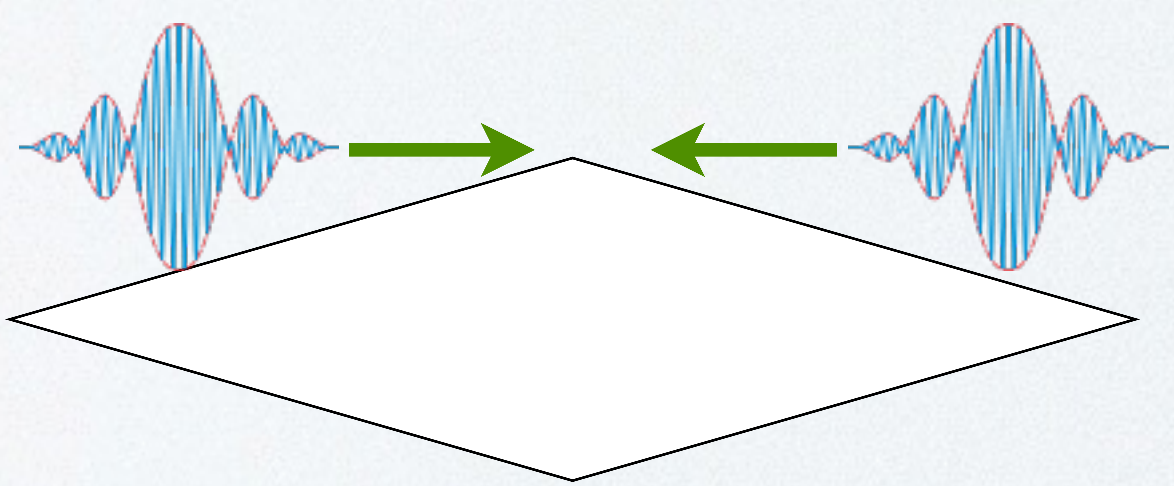
4) Horizon of QM particle

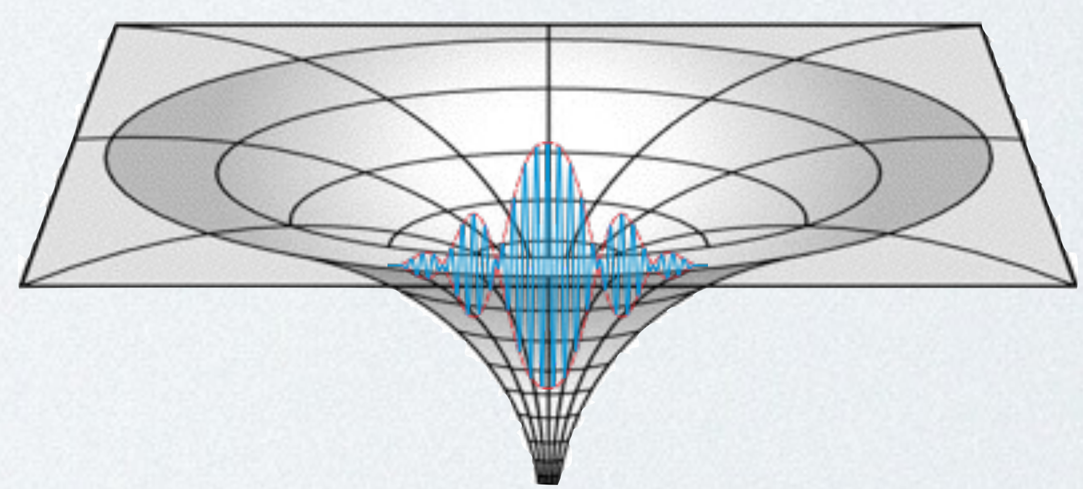
What is the Schwarzschild radius of QM particles?

From *Generalized uncertainty principles* (GUPs): $\Delta x \gtrsim \ell_{\rm p} \dfrac{m_{\rm p}}{\Delta p} + \alpha\, \ell_{\rm p} \dfrac{\Delta p}{m_{\rm p}}$

To Dvali's *classicalization* (2010):

At high (~Planckian) energy, quantum particle scatterings lead to formation of "classicalons" and quantum degrees of freedom disappear (*no UV divergences*).
For gravity, "classicalons" = black holes = BEC of gravitons

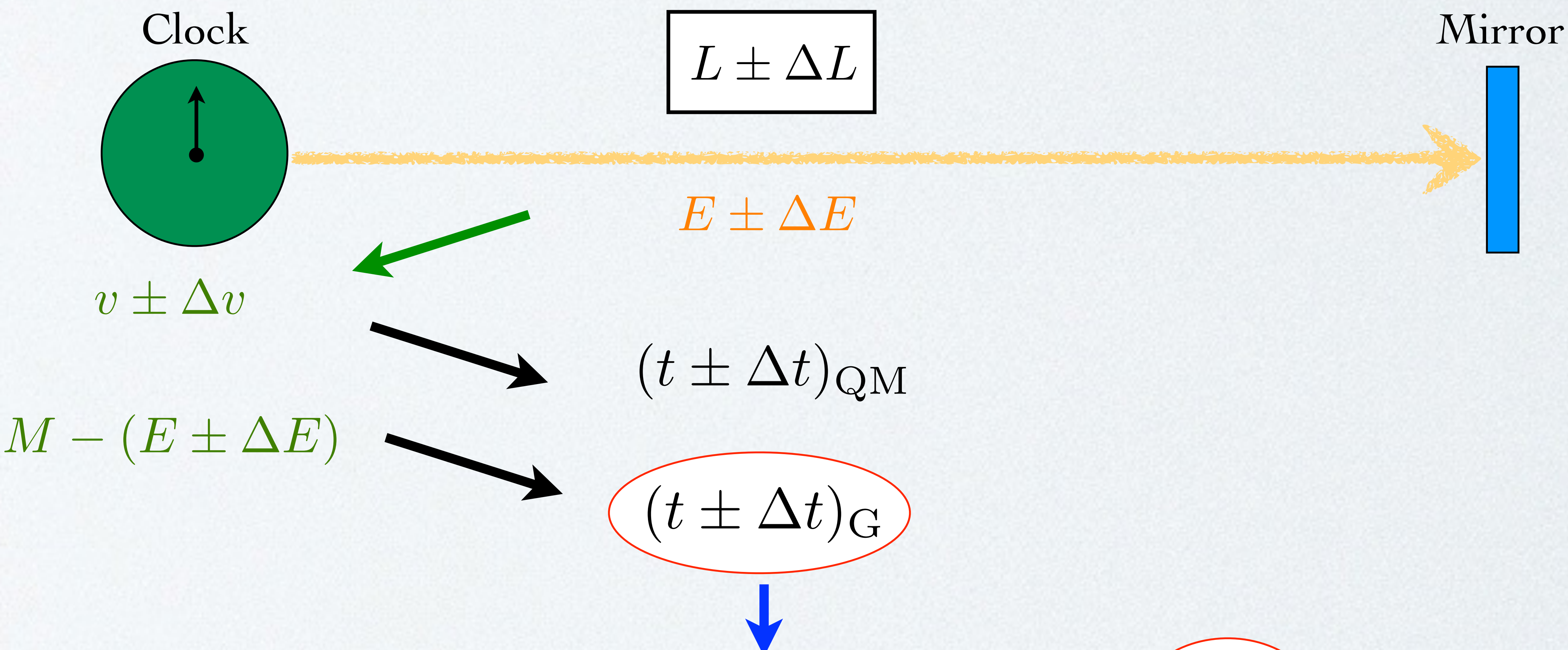




4.1) GUP in QM

[F. Scardigli, R.C., Int. J. Mod. Phys. D18 (2009) 319]

“Measuring very short lengths requires much energy: a BH is produced and precision reduces”



$$\Delta L = L_{\rm QM} + L_{\rm G} = 2\left(\ell_{\rm P}\, L\, \frac{m_{\rm P}}{M}\right)^{1/2} + \ell_{\rm P}\, \frac{M}{m_{\rm P}}$$

$$\frac{\Delta L_{\rm min}}{L} \simeq \left(\frac{\ell_{\rm P}}{L}\right)^{2/3}$$

Minimum length!

5) Horizon wave-function

[ArXiv:1305.3195]

1) Localised particle at rest: $\langle x|\psi_{\rm S}\rangle$ ~ packet

2.1) Energy (modes) of choice!

2) Spectral decomposition: $$|\psi_{\rm S}\rangle = \sum_E C(E)\,|E\rangle$$

3.1) Schwarzschild-link

$$R_{\rm H} = 2\,\ell_{\rm p}\,\frac{E}{m_{\rm p}}$$

3) Horizon wave-function: $$\langle R_{\rm H}|\psi_{\rm H}\rangle \sim C(R_{\rm H})$$

5) Horizon wave-function

[ArXiv:1305.3195]

Localised particle at rest:

Gaussian wave-function:

$$\psi_{\rm S}(r) = \frac{e^{-\frac{r^2}{2\,\ell^2}}}{\ell^{3/2}\,\pi^{3/4}}$$

Energy spectrum: $|\psi_{\rm S}\rangle = \sum_E C(E)\,|E\rangle$

Fourier transform:

$$\psi_{\rm S}(p) = \frac{e^{-\frac{p^2}{2\,\Delta^2}}}{\Delta^{3/2}\,\pi^{3/4}} \qquad \Delta = \frac{\hbar}{\ell} \sim m$$

$E^2 = p^2 + m^2$ (flat space)

Horizon wave-function:

$$R_{\rm H} = 2\,\ell_{\rm p}\,\frac{E}{m_{\rm p}}$$

$$\psi_H(R_{\rm H}) = \frac{\ell^{3/2}\,e^{-\frac{\ell^2\,R_{\rm H}^2}{8\,\ell_{\rm p}^4}}}{2^{3/2}\,\pi^{3/4}\,\ell_{\rm p}^3}$$

5) Horizon wave-function

[ArXiv:1305.3195]

Probability particle is inside its own horizon:

$$P_<(r < R_{\rm H}) = P_{\rm S}(r < R_{\rm H})\, P_{\rm H}(R_{\rm H})$$

$$P_{\rm S}(r < R_{\rm H}) = 4\,\pi \int_0^{R_{\rm H}} |\psi_{\rm S}(r)|^2\, r^2\, dr$$

$$P_{\rm H}(R_{\rm H}) = 4\,\pi\, R_{\rm H}^2\, |\psi_{\rm H}(R_{\rm H})|^2$$

↓

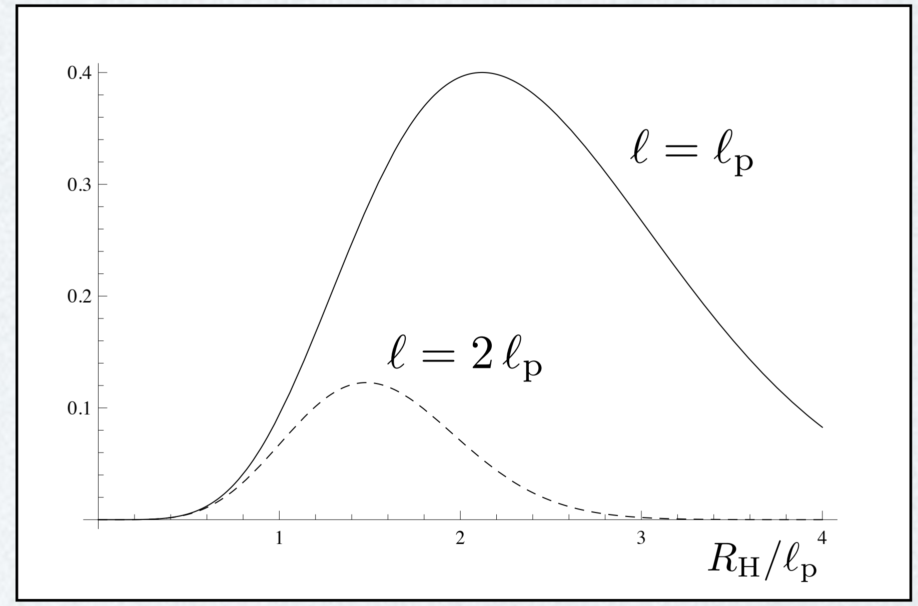
Probability particle is a Black Hole:

$$P_{\rm BH} = \int_0^\infty P_<(r < R_{\rm H})\, dR_{\rm H}$$

5) Horizon wave-function

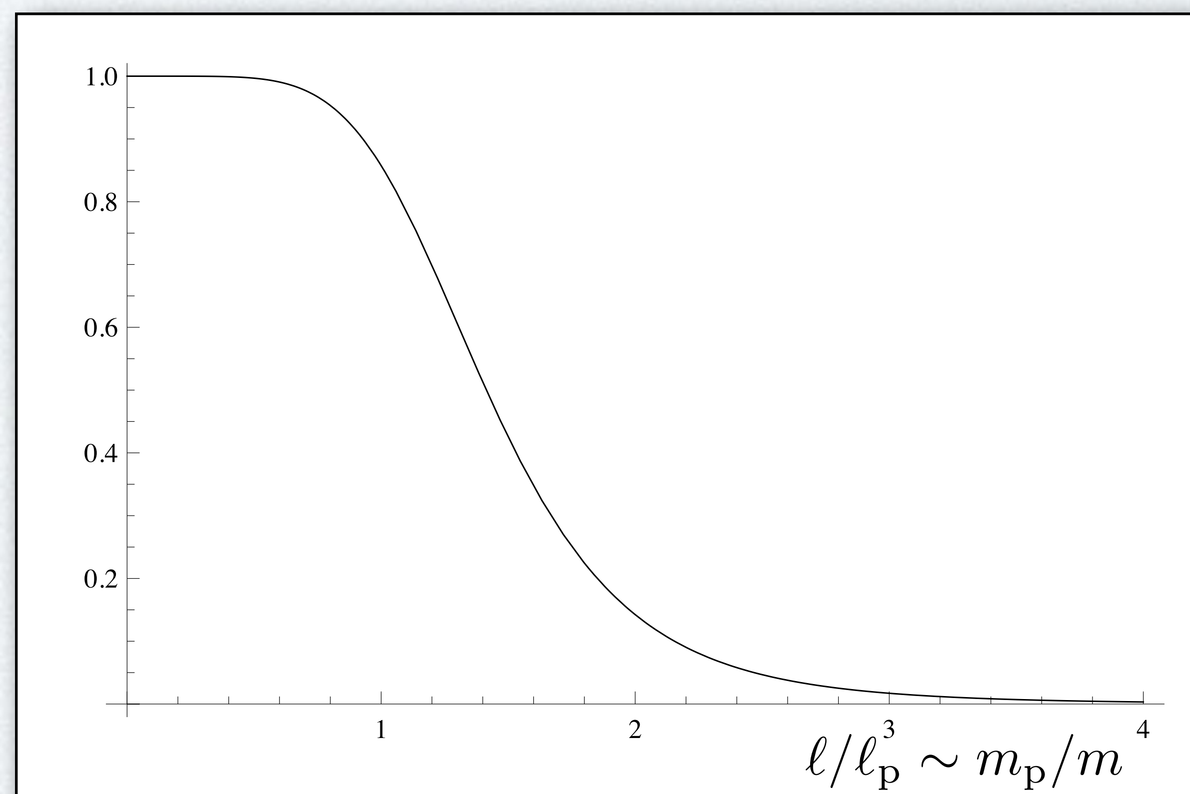
[ArXiv:1305.3195]

$$\psi_H(R_\mathrm{H}) = \frac{\ell^{3/2}\, e^{-\frac{\ell^2\, R_\mathrm{H}^2}{8\,\ell_\mathrm{p}^4}}}{2^{3/2}\,\pi^{3/4}\,\ell_\mathrm{p}^3}$$



$$P_<(r < R_\mathrm{H}) = \frac{\ell^3\, R_\mathrm{H}^2}{2\,\sqrt{\pi}\,\ell_\mathrm{p}^6}\, e^{-\frac{\ell^2\, R_\mathrm{H}^2}{4\,\ell_\mathrm{p}^4}} \left[\mathrm{Erf}\left(\frac{R_\mathrm{H}}{\ell}\right) - \frac{2\,R_\mathrm{H}}{\sqrt{\pi}\,\ell}\, e^{-\frac{R_\mathrm{H}^2}{\ell^2}}\right]$$

$$P_\mathrm{BH}(\ell) = \frac{2}{\pi}\left[\arctan\left(2\,\frac{\ell_\mathrm{p}^2}{\ell^2}\right) + 2\,\frac{\ell^2\,(4 - \ell^4/\ell_\mathrm{p}^4)}{\ell_\mathrm{p}^2\,(4 + \ell^4/\ell_\mathrm{p}^4)^2}\right]$$



"Fuzzy" minimum mass:

$$M_\mathrm{BH} \gtrsim m_\mathrm{p}$$

5.1) GUP

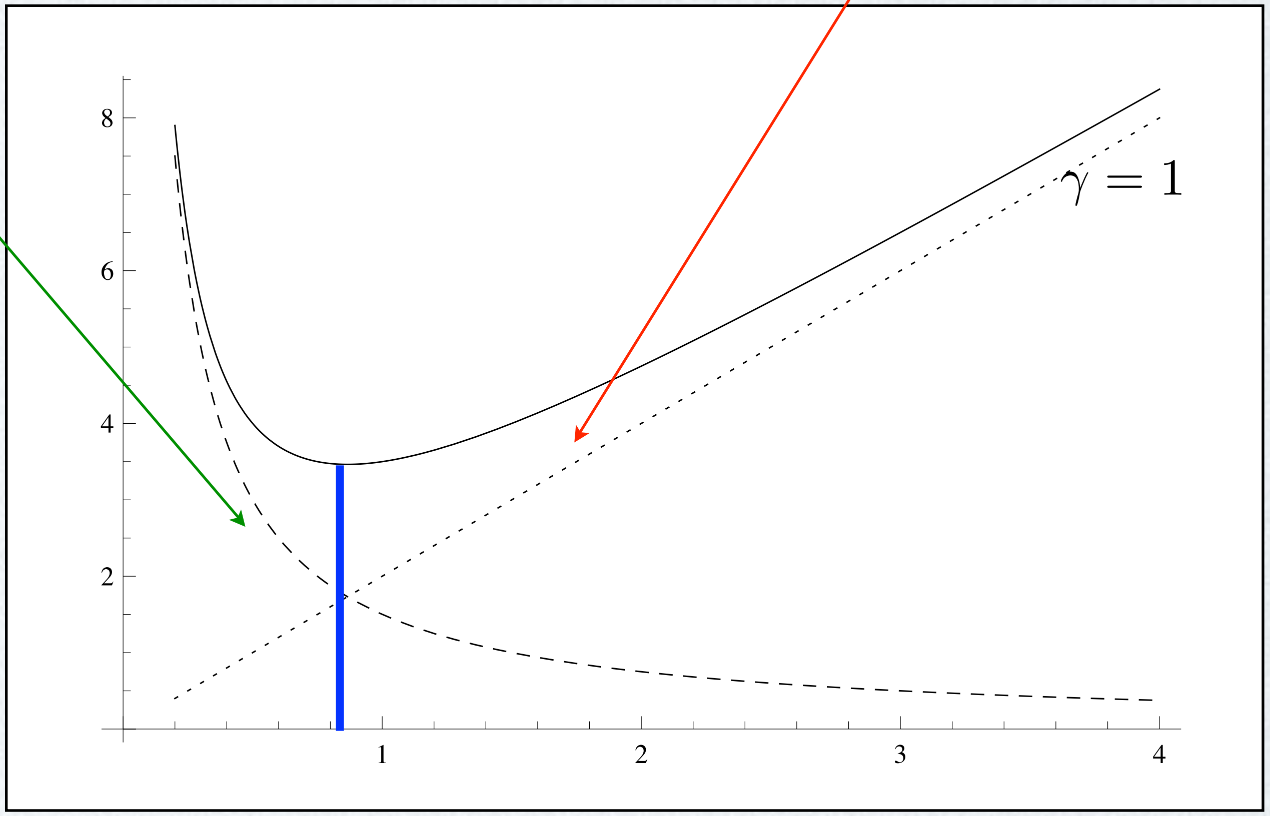
[ArXiv:1306.5298, EPJ C]

Two uncertainties:

$$\langle \Delta r^2 \rangle = \langle \psi_{\rm S} | r^2 | \psi_{\rm S} \rangle - \langle \psi_{\rm S} | r | \psi_{\rm S} \rangle^2 = \left(\frac{3\,\pi - 8}{2\,\pi} \right) \ell^2$$

$$\langle \Delta R_{\rm H}^2 \rangle = 4 \left(\frac{3\,\pi - 8}{2\,\pi} \right) \frac{\ell_{\rm p}^4}{\ell^2}$$

$$\Delta r \equiv \sqrt{\langle \Delta r^2 \rangle} + \gamma \sqrt{\langle \Delta R_{\rm H}^2 \rangle} = \left(\frac{3\,\pi - 8}{2\,\pi} \right) \ell_{\rm p} \frac{m_{\rm p}}{\Delta p} + 2\,\gamma\,\ell_{\rm p} \frac{\Delta p}{m_{\rm p}}$$



Minimum length

$$\langle \Delta p^2 \rangle = \left(\frac{3\,\pi - 8}{2\,\pi} \right) m_{\rm p}^2 \frac{\ell_{\rm p}^2}{\ell^2}$$

N.B. Uncertainty **derived** with standard canonical commutators: $[q, p] = i\,\hbar$
(gravity is more than kinematics...?)

5.2) Hawking radiation

Uncertainty principle for horizon wave-function:

$$\Delta R_{\rm H}\, \Delta P_{\rm H} \sim \ell_{\rm p}\, m_{\rm p}$$

$$\Delta R_{\rm H} \sim \frac{\ell_{\rm p}^2}{\ell} \sim \ell_{\rm p}\, \frac{M}{m_{\rm p}}$$

$$\Downarrow$$

$$\Delta P_{\rm H} \sim \frac{m_{\rm p}^2}{M}$$

Conjugate momentum to horizon position:

$$\Delta P_{\rm H} \sim M\, \dot{R}_{\rm H} \sim M\, \ell_{\rm p}\, \frac{\dot{M}}{m_{\rm p}}$$

Hawking flux:

$$\dot{M} \sim \frac{m_{\rm p}^2\, \ell_{\rm p}\, m_{\rm p}}{\ell_{\rm p}^2\, M} = \frac{\hbar}{G_{\rm N}^2\, M^2}$$

5.3) Decay

[ArXiv:1306.5298, EPJ C]

"Amount of particle's energy" outside its horizon:

$$\Delta m \simeq m\, P_{\rm T} \simeq a\, m + \mathcal{O}(m - m_{\rm p})$$

$$P_{\rm T}(m) = 1 - P_{\rm BH}(m) \simeq a - b\,\frac{m - m_{\rm p}}{m_{\rm p}}$$

Typical "emission time" from Heisenberg UP:

$$\Delta t \simeq \frac{\ell_{\rm p}\, m_{\rm p}}{\Delta m} \simeq \frac{\ell_{\rm p}^2}{\Delta R_{\rm H}} \simeq \ell$$

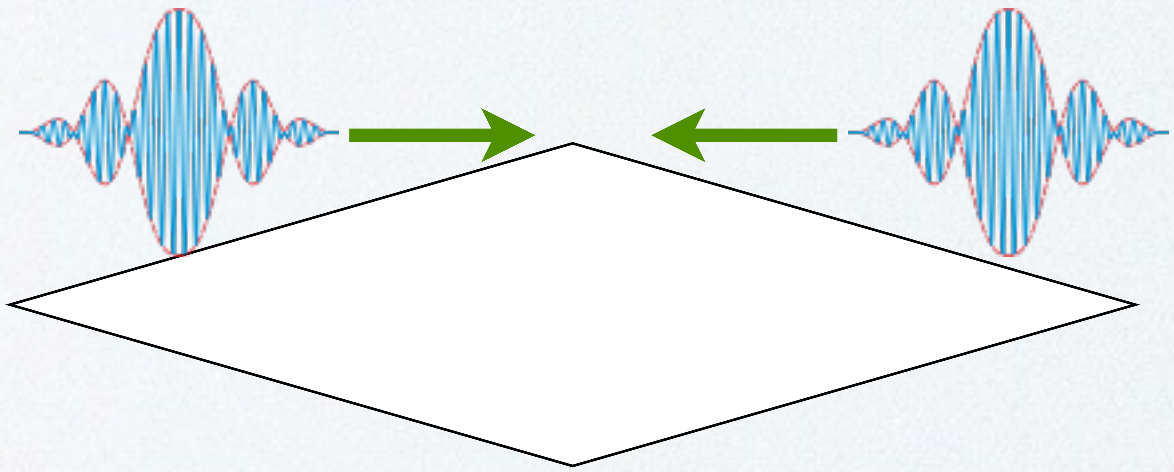
"Emission rate" for Planck-size black hole (same as from GUP):

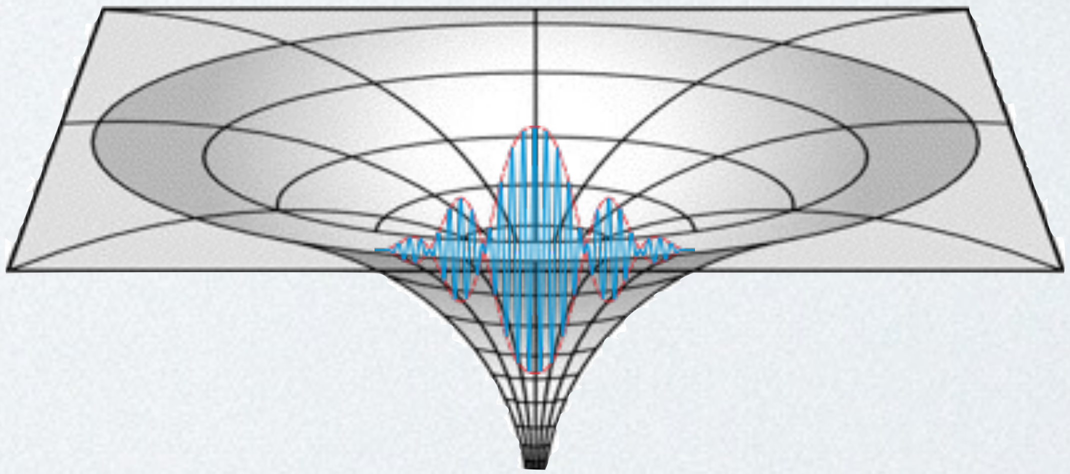
$$-\frac{\Delta m}{\Delta t} \simeq a\,\frac{m^2}{m_{\rm p}\,\ell_{\rm p}} + \mathcal{O}(m - m_{\rm p}) \qquad \not\simeq \frac{m_{\rm p}^3}{\ell_{\rm p}\, m^2} \quad \text{Hawking}$$

6) Collisions

We recovered GUP: $\Delta x \gtrsim \ell_{\rm p}\, \frac{m_{\rm p}}{\Delta p} + \alpha\, \ell_{\rm p}\, \frac{\Delta p}{m_{\rm p}}$

We can study collisions:



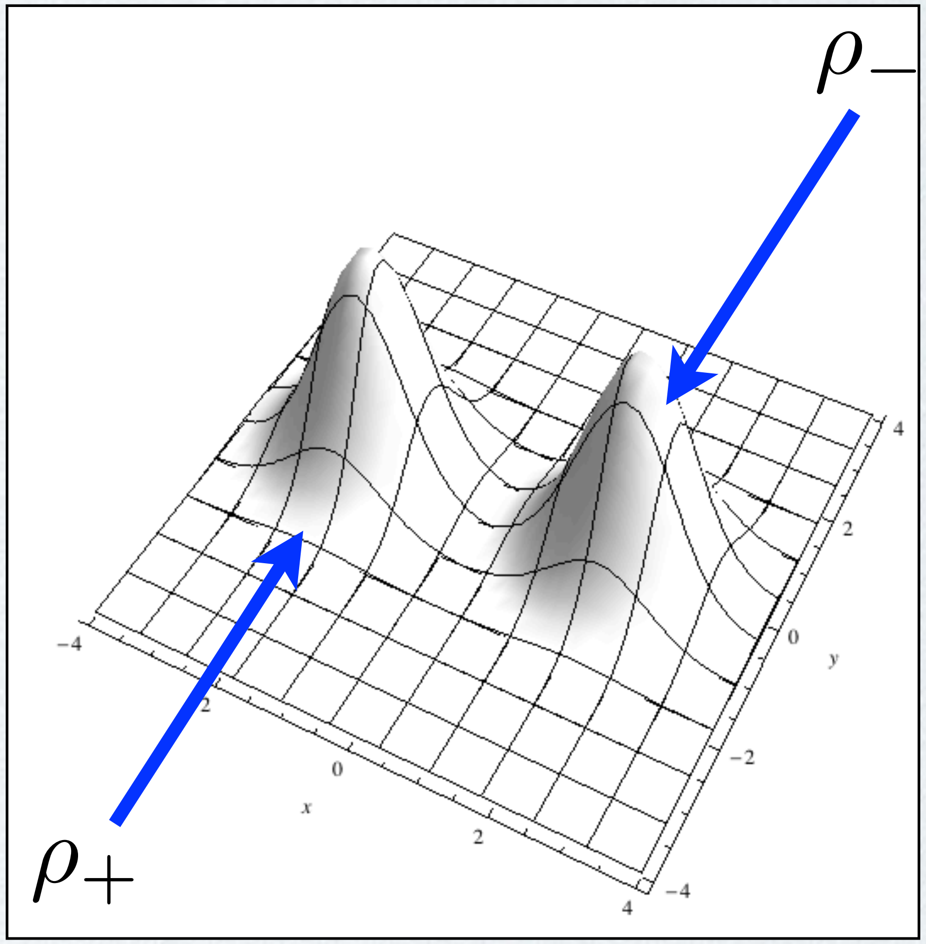


6) Collisions

[R.C., O.Micu, A.Orlandi, arXiv:1205.6303, EPJ C]

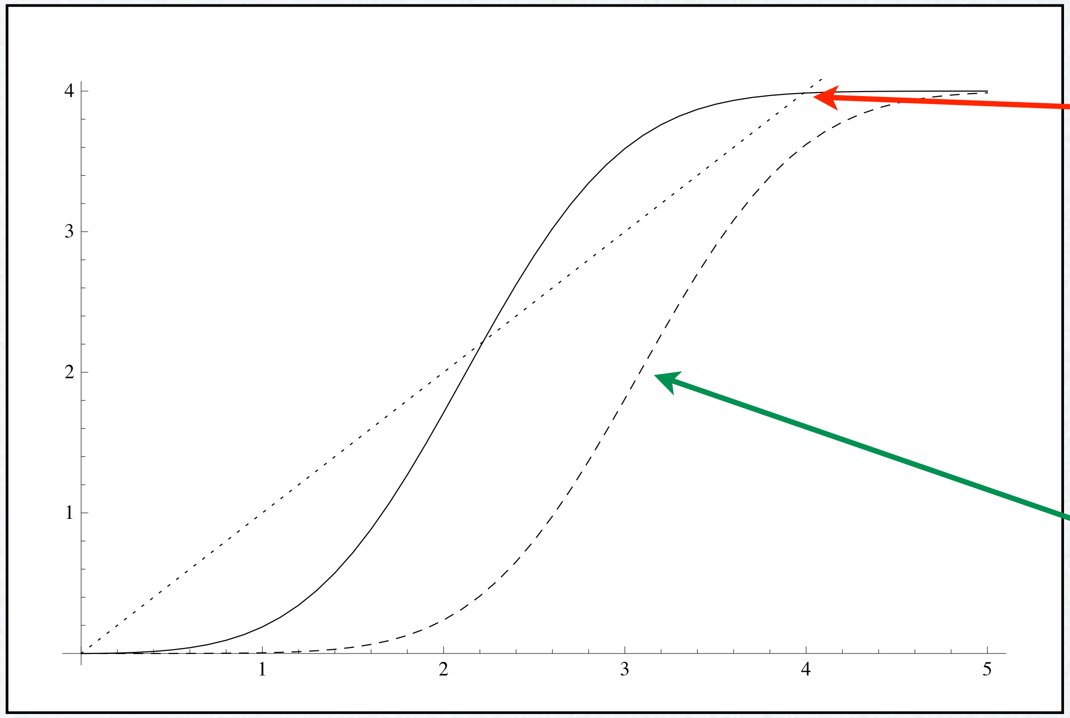
Example: classical Gaussian packets

$$\rho_\pm(x,y) = \frac{\rho_0}{\pi\,\ell^2}\,\exp\left\{-\frac{(x\pm b)^2+y^2}{\ell^2}\right\}$$
$$= \frac{\rho_0}{\pi\,\ell^2}\,\exp\left\{-\frac{r^2\pm 2\,b\,r\,\cos(\theta)+b^2}{\ell^2}\right\} = \rho_\pm(r,\theta)$$



(Spherically symmetric) mass function:

$$M(r) = \frac{4\,\pi}{3}\int_0^r \rho(t,\bar r)\,\bar r^2\,d\bar r$$



$$2\,M(r) = r$$

(Outer) horizon!

$$r = 2\,\ell_p\,\frac{M(r)}{M_p}$$

No BH!

6) Collisions

[ArXiv:1311.5698]

1) Two localised particles: $\psi_{\mathrm{S}}(x_1, x_2) = \psi_{\mathrm{S}}(x_1)\, \psi_{\mathrm{S}}(x_2)$

$$\psi_{\mathrm{S}}(x_i) = e^{-i\,\frac{P_i\,x_i}{\hbar}}\, \frac{e^{-\frac{(x_i - X_i)^2}{2\,\ell_i}}}{\sqrt{\pi^{1/2}\,\ell_i}}$$

$$\Delta_i = \hbar/\ell_i$$

2) Two particles in momentum space: $\psi_{\mathrm{S}}(p_i) = e^{-i\,\frac{p_i\,X_i}{\hbar}}\, \dfrac{e^{-\frac{(p_i - P_i)^2}{2\,\Delta_i}}}{\sqrt{\pi^{1/2}\,\Delta_i}}$

$$|\psi_{\mathrm{S}}^{(1,2)}\rangle = \prod_{i=1}^{2} \left[\int_{-\infty}^{+\infty} dp_i\, \psi_{\mathrm{S}}(p_i, t)\, |p_i\rangle \right]$$

X_i

P_i

6) Collisions

[ArXiv:1311.5698]

3) Unnormalised horizon wave-function:

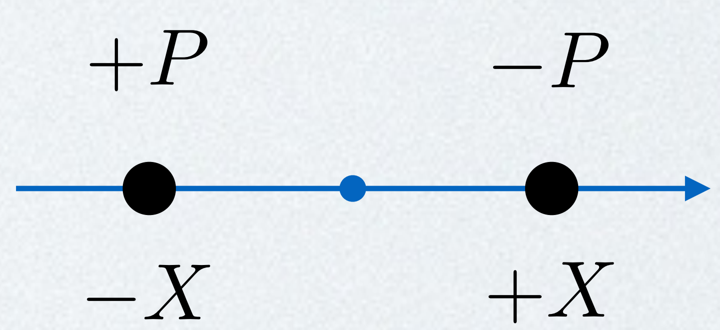
$$|\psi_{\rm S}\rangle = \sum_E C(E)\,|E\rangle \quad \longrightarrow \quad C(E) = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \psi_{\rm S}(p_1)\,\psi_{\rm S}(p_2)\,\delta(E - E_1 - E_2)\,dp_1\,dp_2$$

4) Centre-mass and relativistic limit:

$$\ell_i = \frac{\hbar}{\sqrt{P_i^2 + m_i^2}} \simeq \frac{\ell_{\rm p}\, m_{\rm p}}{|P_i|} \qquad \Delta_i \simeq |P_i|$$

$$P_1 = -P_2 \equiv P > 0$$

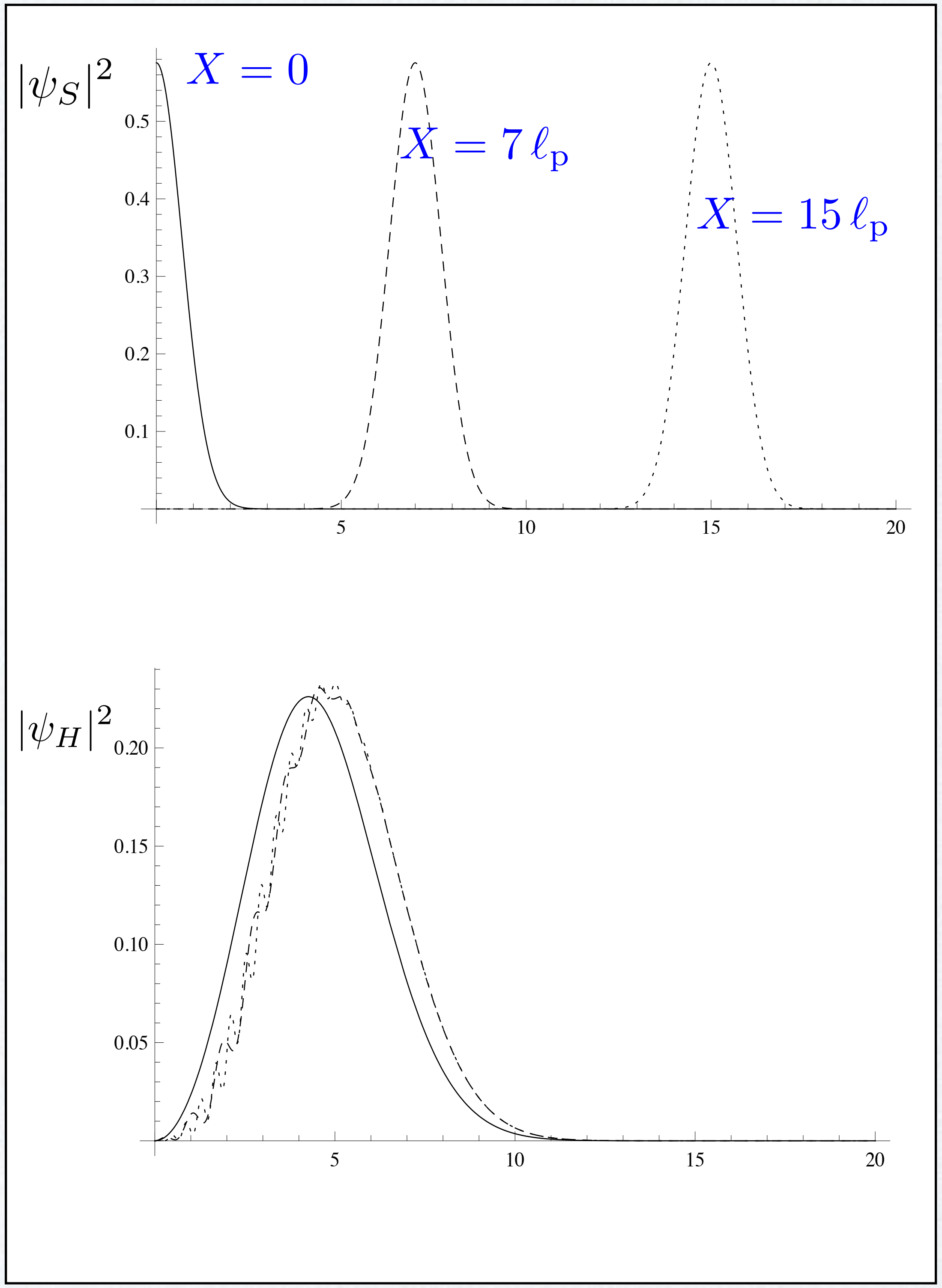
$$X_1 \simeq -X_2 \equiv X > 0$$



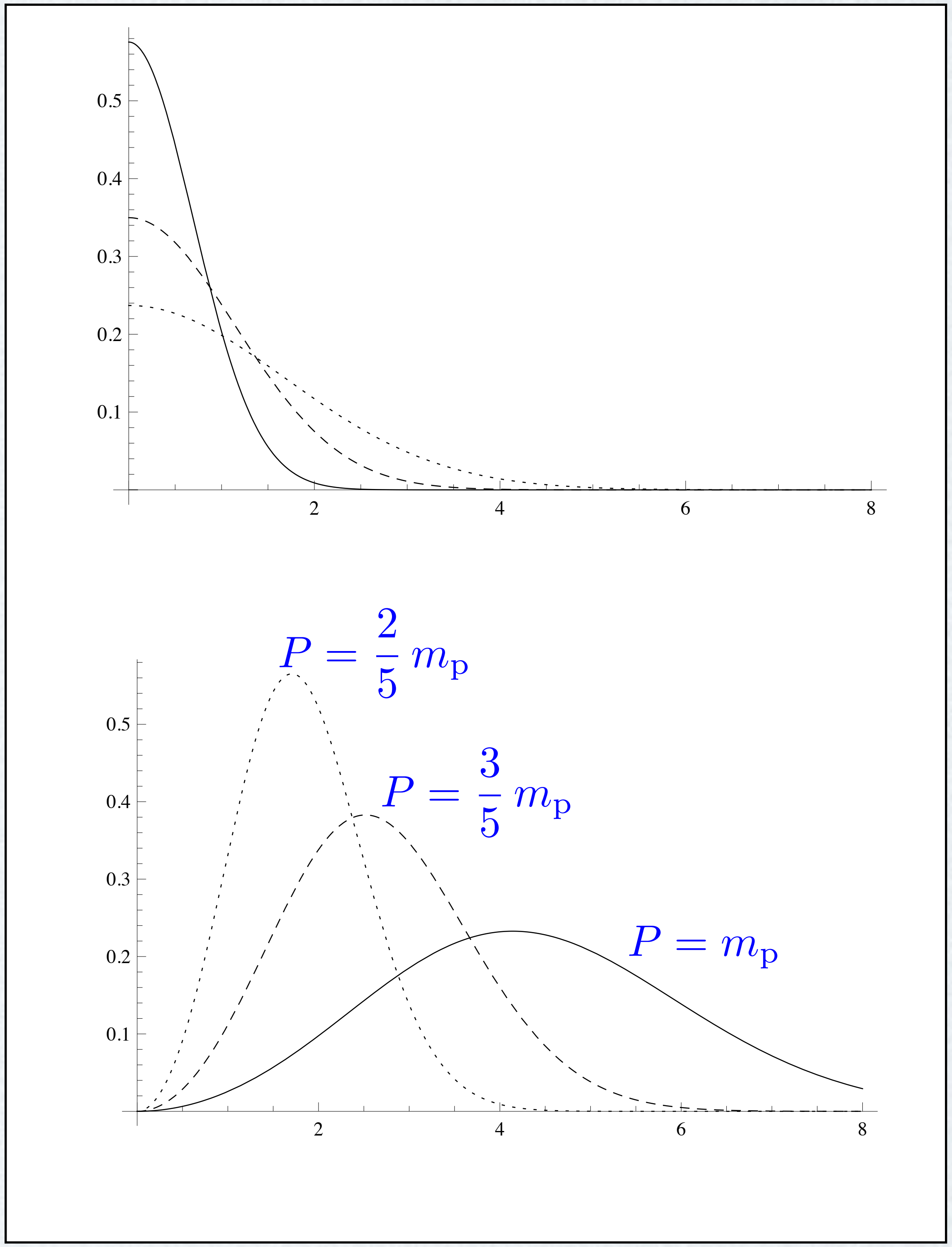
6) Collisions

[ArXiv:1311.5698]

5) Horizon wave-function:



$P = m_p$

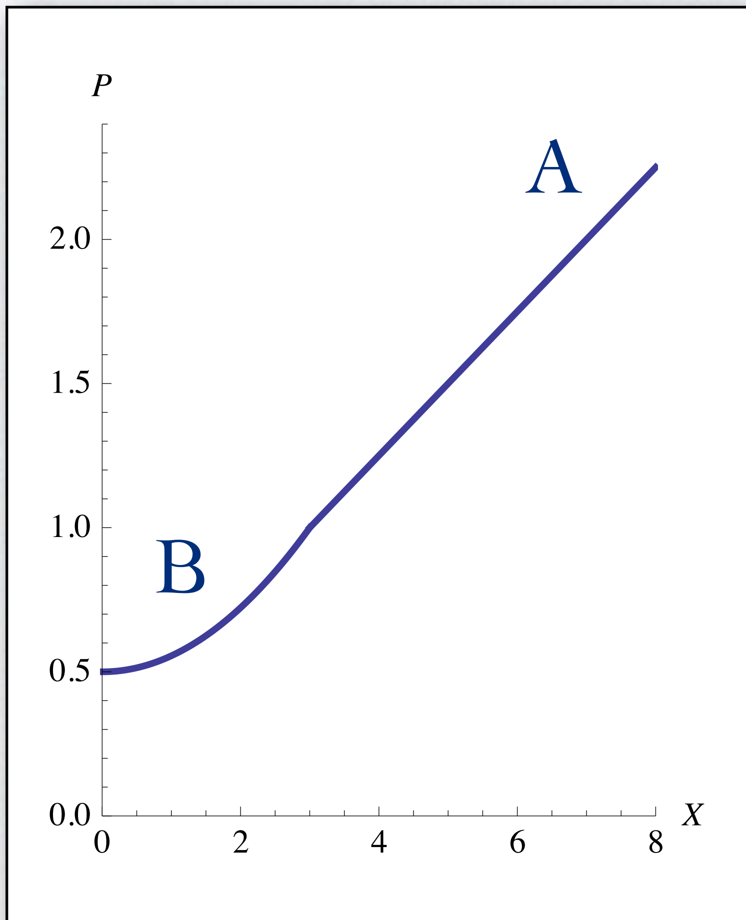


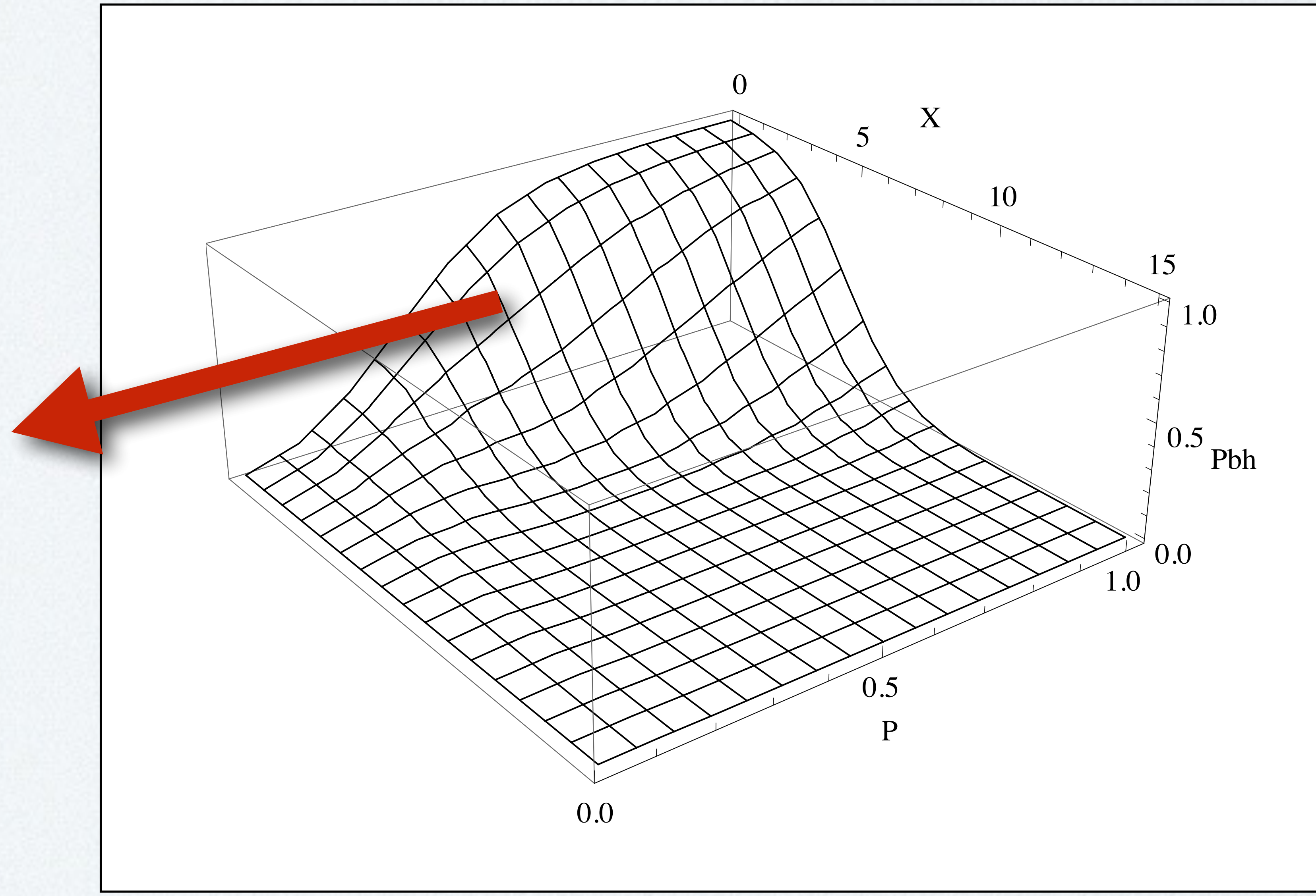
$X = 0$

6) Collisions

[ArXiv:1311.5698]

6) Hoop conjecture:





A) classical

$$P_{\mathrm{BH}}(X, 2P \gtrsim 2\, m_{\mathrm{p}}) \gtrsim 80\%$$

$$X \lesssim 2\,\ell_{\mathrm{p}}\,(2P/m_{\mathrm{p}}) - \ell_{\mathrm{p}} \simeq R_{\mathrm{H}}(2P)$$

B) quantum

$$P_{\mathrm{BH}}(X, 2P \lesssim 2\, m_{\mathrm{p}}) \gtrsim 80\%$$

$$2P - m_{\mathrm{p}} \gtrsim \frac{m_{\mathrm{p}}\, X^2}{9\,\ell_{\mathrm{p}}}$$

Summary and outlook

1. Horizon wave-function in flat space describes spherical particle/black hole + **GUP**
2. Horizon wave-function yields **quantum hoop conjecture** for 2-particle collisions in flat 1+1 dimensions
3. Account for particle(s) self-gravity (refine spectral decomposition - work in progress)
4. Analyse more spherical systems (simple models of gravitational collapse - work in progress)
5. Generalise to non-spherical systems (and spin)
6. Analyse (2-)particle collisions with angular momentum+spin
7. (Hope for?) quantum description of gravitational collapse