Black Holes and Qubits

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Abstract

- Quantum entanglement lies at the heart of quantum information theory, with applications to quantum computing, teleportation, cryptography and communication. In the apparently separate world of quantum gravity, the Bekenstein-Hawking entropy of black holes has also occupied center stage.
- Here we describe a correspondence between the entanglement measures of qubits in quantum information theory and black hole entropy in string theory.
- Reviewed in Borsten, Dahanayake, Duff, Ebrahim, Rubens: "Black Holes, Qubits and Octonions"
  Phys. Rep. 471:113-219,2009 arXiv:0809.4685 [hep-th].
  Duff: "Black Holes and Qubits" CERN COURIER May 2010

Repurposing string theory

- 1970s Strong nuclear interactions
- 1980s Quantum gravity; “theory of everything”
- 1990s AdS/CFT: QCD (revival of 1970s); quark-gluon plasmas
- 2000s AdS/CFT: superconductors
- 2000s Cosmic strings
- 2010s Black hole/qubit correspondence: entanglement in Quantum Information Theory
- Conclusion: May be right theory for some but not all

One qubit

- A *qubit* is any two-state quantum system. For example: spin-up/spin-down electron or left/right polarized photon.
- The one qubit system Alice (where $A = 0, 1$) is described by the state

$$|\Psi\rangle = a_A|A\rangle = a_0|0\rangle + a_1|1\rangle$$

where a_0 and a_1 are complex numbers.

Two qubits

The two qubit system Alice and Bob (where $A, B = 0, 1$) is described by the state

$$\begin{aligned}|\Psi\rangle &= a_{AB}|AB\rangle \\ &= a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.\end{aligned}$$

Entanglement

- Example, separable state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle = |0\rangle \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right)$$

  Alice measures spin up, Bob can measure either spin up or spin down. This state is not *entangled*.

- Example, Bell state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

  If Alice measures spin up, Bob has to measure spin up too! This state is entangled.

EPR paradox

- 1935 Einstein-Podolsky-Rosen "paradox": Alice might be in Falmer and Bob in Alpha Centauri. Einstein called it "spooky" action at a distance.
- 1964 John Bell describes a way of testing experimentally this quantum non-locality versus Einstein's realism. (By the way, observe the time lag between theoretical idea and falsifiable prediction. Critics of string theory take note.)
- 1982 Alain Aspect performs Bell's experiment: quantum mechanics wins out!

Entanglement measure

- The measure of the bipartite entanglement of Alice and Bob is given by the "two-tangle"

$$\tau_{AB} = 4|\det\, a_{AB}|^2 = 4|a_{00}a_{11} - a_{01}a_{10}|^2$$

or equivalently

$$\tau_{AB} = 4|\det\, \rho_A| = 4|\det\, \rho_B|$$

where ρ_A and ρ_B are the reduced density matrices

$$\rho_A = Tr_B|\Psi\rangle\langle\Psi| \qquad \rho_B = Tr_A|\Psi\rangle\langle\Psi|$$

- For normalized states

$$0 \leq \tau_{AB} \leq 1$$

Examples

- Example, separable state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

$$\tau_{AB} = 0$$

  No entanglement.

- Example, Bell state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$\tau_{AB} = 1$$

  Maximal entanglement.

Symmetries of τ_{AB}

- Under $SL(2)$ a_A transforms as a 2:

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

  where

$$\alpha\delta - \beta\gamma = 1$$

- Under $SL(2)_A \times SL(2)_B$, a_{AB} transforms as a $(2, 2)$.
- τ_{AB} is invariant under $SL(2)_A \times SL(2)_B$ and under a discrete duality that interchanges A and B.

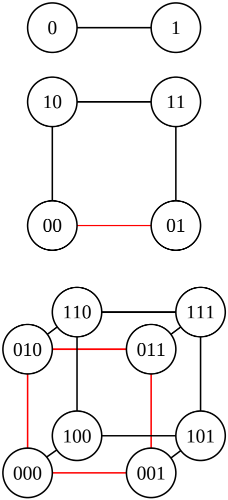
Three qubits

The three qubit system Alice, Bob and Charlie (where $A, B, C = 0, 1$) is described by the state

$$\begin{aligned}|\Psi\rangle &= a_{ABC}|ABC\rangle \\ &= a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle \\ &+ a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle.\end{aligned}$$

Hypermatrix

The 3-index quantity a_{ABC} is an example of what Cayley termed a *hypermatrix* in 1845. Its elements may be represented by the cube



Cayley's hyperdeterminant

- The tripartite entanglement of Alice, Bob and Charlie is given by the three-tangle

$$\tau_{ABC} = 4|\text{Det }a_{ABC}|,$$

Coffman et al: arXiv:quant-ph/9907047

- $\text{Det }a_{ABC}$ is Cayley's hyperdeterminant

$$\text{Det }a_{ABC} = -\frac{1}{2}\,\varepsilon^{A_1A_2}\varepsilon^{B_1B_2}\varepsilon^{C_1C_4}\varepsilon^{C_2C_3}\varepsilon^{A_3A_4}\varepsilon^{B_3B_4} \cdot a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4}$$

Miyake, Wadati: arXiv:quant-ph/0212146

- It is invariant under $SL(2)_A \times SL(2)_B \times SL(2)_C$, with a_{ABC} transforming as a $(\mathbf{2},\mathbf{2},\mathbf{2})$, and under a discrete triality that interchanges A, B and C.

Symmetry

- Explicitly

$$
\begin{aligned}
&\text{Det } a_{ABC} = \\
&a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 \\
&- 2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\
&+ a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} \\
&+ a_{001}a_{100}a_{011}a_{110} + a_{010}a_{100}a_{011}a_{101}) \\
&+ 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}).
\end{aligned}
$$

SLOCC

- In QIT the group $[SL(2)]^n$ is known as the n-qubit SLOCC equivalence group.
- SLOCC = Stochastic Local Operations and Classical Communication
- For one qubit SLOCC=$SL(2)_A$ and a_A transforms as a 2:

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

  where

$$\alpha\delta - \beta\gamma = 1$$

- For three qubits SLOCC=$SL(2)_A \times SL(2)_B \times SL(2)_C$ and a_{ABC} transforms as a $(2, 2, 2)$.

REISSNER-NORDSTRÖM SOLUTION

The most general static spherically symmetric black hole solution of Einstein-Maxwell theory is given in spherical polar coordinates (t, r, θ, ϕ) by the Reissner-Nordström line-element

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2$$

$$+r^2(d\theta^2 + sin^2\theta d\phi^2) \qquad\qquad A_t = \frac{Q}{r}$$

where M and Q are the mass and electric charge of the black hole in units $G = \hbar = c = 1$.

SURFACE GRAVITY AND THE AREA

- Besides the mass and the charge which are measured at infinity there are two other quantities, surface gravity and the area, measured on the event horizon that are given by

$$\kappa_S = \frac{\sqrt{M^2 - Q^2}}{2M(M + \sqrt{M^2 - Q^2}) - Q^2}, \quad A = 4\pi(M + \sqrt{M^2 - Q^2})^2.$$

- The R-N solution has two horizons determined by

$$r_\pm = M \pm \sqrt{M^2 - Q^2}.$$

Extremal black holes

- Cosmic censorship: If $M \geq |Q|$, the singularity at $r = 0$ is hidden behind the event horizon; otherwise there is a naked singularity.
- *Extremal* black holes have $M = |Q|$. Two horizons, r_+ and r_- coincide.
- This also allows classically stable multi-centered black hole solutions obeying a *no-force condition*: the gravitational attraction is exactly cancelled by the Coulomb repulsion.

Extremal and BPS black holes

- *Extremal* black holes obey generalised mass=charge conditions.
- A black hole that preserves some unbroken supersymmetry (admitting one or more Killing spinors) is said to be *BPS* (after Bogomol'nyi-Prasad-Sommerfield) and non-BPS otherwise.
- All BPS black holes are extremal but extremal black holes can be BPS or non-BPS.

Quantum black holes

- Hawking temperature

$$T_{\mathsf{H}} = \frac{\kappa_S}{2\pi}$$

- Bekenstein-Hawking entropy

$$S_{\mathsf{BH}} = \frac{A}{4}$$

ENTROPY AND TEMPERATURE

In the extremal case, the entropy is completely determined in terms of the charges

$$S_{BH} = \frac{A}{4} = \pi Q^2$$

and there is no Hawking radiation since

$$T_{\mathsf{H}} = 0$$

Magnetic charge

If magnetic monopoles are included into the theory,

$$A_\phi = P cos\theta$$

then a generalization to include magnetic charge P is obtained by replacing Q^2 by $Q^2 + P^2$ in the metric and other formulae.

Black holes in supergravity

- Supergravity incorporates bose-fermi symmetry: the spin-2 graviton can have $1 \leq \mathcal{N} \leq 8$ spin 3/2 gravitino partners.
- The supergravity theories we shall consider have more than the one photon of Einstein-Maxwell theory. The $\mathcal{N} = 2$ STU model has 4; the $\mathcal{N} = 8$ model has 28, so the black holes will carry 8 or 56 electric and magnetic charges, respectively.
- Both also involve scalar fields.

STU model

- The STU supergravity model arises in string theory. Its bosonic sector consists of gravity coupled to 4 photons and three complex scalars, denoted S, T and U.
- The equations of motion display the symmetry $SL(2)_S \times SL(2)_T \times SL(2)_U$ and a discrete triality that interchanges S, T and U.
- Duff, Liu, Rahmfeld: arXiv:hep-th/9508094

STU black hole entropy

- A general static spherically symmetric STU black hole solution depends on 8 charges denoted $q_0, q_1, q_2, q_3, p^0, p^1, p^2, p^3$.
- Black hole entropy S given by the one quarter the area of the event horizon. Hawking: 1975
- The *extremal* STU black hole entropy is a complicated function of the 8 charges :

$$(S/\pi)^2 = -(p^0q_0 + p^1q_1 + p^2q_2 + p^3q_3)^2$$
$$+4\Big[(p^1q_1)(p^2q_2) + (p^1q_1)(p^3q_3) + (p^3q_3)(p^2q_2)$$
$$+q_0p^1p^2p^2 - p^0q_1q_2q_3\Big]$$

Behrndt et al: arXiv:hep-th/9608059

Black hole/qubit correspondence

Duff: arXiv:hep-th/0601134 Identify STU with ABC and the 8 black hole charges with the 8 components of the three-qubit hypermatrix a_{ABC},

$$\begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \\ q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} a_{000} \\ -a_{001} \\ -a_{010} \\ -a_{100} \\ a_{111} \\ a_{110} \\ a_{101} \\ a_{011} \end{bmatrix}$$

Find that the black hole entropy is related to the 3-tangle as in

$$S = \pi\sqrt{|\text{Det } a_{ABC}|} = \frac{\pi}{2}\sqrt{\tau_{ABC}}$$

Turns out to be the tip of an iceberg.

Further developments

- Further papers have written a more complete dictionary, which translates a variety of phenomena in one language to those in the other, for example:
- The *attractor mechanism* on the black hole side is related to *optimal local distillation protocols* on the QI side Levay: arXiv:0708.2799 [hep-th]
- Moreover, supersymmetric and non-supersymmetric black holes corresponding to the suppression or non-suppression of bit-flip errors Levay: arXiv:0708.2799 [hep-th]
- Classification of black holes matches classification of qubit entanglement
  Kallosh, Linde: hep-th/060206
  Borsten, Dahanayake, Duff, Ebrahim, Rubens: arXiv:0809.4685 [hep-th]

Local entropy

- Besides $\mathrm{Det}\, a$, another useful quantity is the local entropy S_A, which is a measure of how entangled A is with the pair BC:

$$S_A = 4\det\ \rho_A$$

where ρ_A is the reduced density matrix

$$\rho_A = \mathrm{Tr}_{BC}|\Psi\rangle\langle\Psi|,$$

and with similar formulae for B and C.

Entanglement classes

Class	Condition				
	$\lvert\psi\rvert^2$	S_A	S_B	S_C	Det a
Zero	0	0	0	0	0
A-B-C	>0	0	0	0	0
A-BC	>0	0	>0	>0	0
B-CA	>0	>0	0	>0	0
C-AB	>0	>0	>0	0	0
W	>0	>0	>0	>0	0
GHZ	>0	>0	>0	>0	$\neq 0$

- Dur, Vidal, Cirac: arXiv:quant-ph/0005115

Representatives

- Null class: 0
- Separable class A-B-C (product states): $q_0|111\rangle$
- Biseparable class (bipartite entanglement):

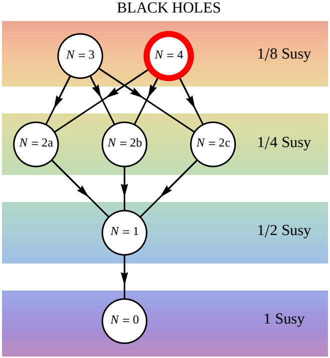
$$\begin{aligned} A\text{-}BC &: q_0|111\rangle - p^1|100\rangle \\ B\text{-}CA &: q_0|111\rangle - p^2|010\rangle \\ C\text{-}AB &: q_0|111\rangle - p^3|001\rangle \end{aligned}$$

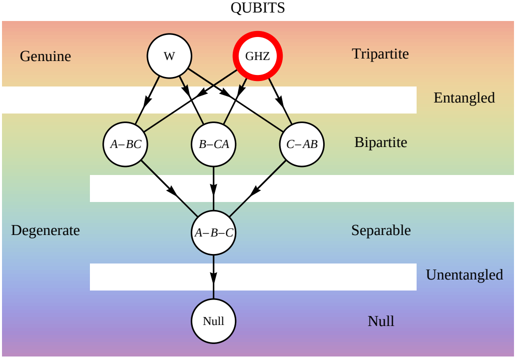
- Class W (maximizes bipartite entanglement):

$$-p^1|100\rangle - p^2|010\rangle - p^3|001\rangle$$

- Class GHZ (genuine tripartite entanglement):

$$q_0|111\rangle - p^1|100\rangle - p^2|010\rangle - p^3|001\rangle$$





- N= number of charges / number of kets

No force condition

- The 4-charge solution with just q_0, p^1, p^2, p^3 switched on obeys the no-force condition and may be regarded as a bound state of four individual black holes with charges q_0, p^1, p^2, p^3, with zero binding energy.
- This translates into the special GHZ (or Mermin) state

$$|\Psi\rangle = -p^3|001\rangle - p^2|010\rangle - p^1|100\rangle + q_0|111\rangle.$$

- Flipping the sign of q_0 flips the sign of $\mathrm{Det}\, a_{ABC}$ and corresponds to going from $1/8$ susy (BPS) to 0 susy (non-BPS) black hole.
- Similarly GHZ state

$$|\Psi\rangle = p^0|000\rangle + q_0|111\rangle.$$

corresponds to non-BPS black hole.

16 PARAMETERS

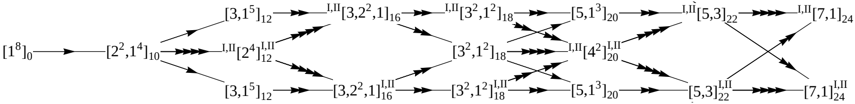
- In addition to the 4 electric and 4 magnetic charges, an STU black hole is also specified by its mass, its NUT charge (gravity analog of magnetic charge) and the values of the 6 real scalars at infinity, making 16 parameters in all.
- Suggests a correspondence with 4 qubits

$$|\Psi\rangle = a_{ABCD}|ABCD\rangle$$

- Levay: arXiv:1004.3639 [hep-th]

Families/Classes

- Under this finer classification there are 31 families of black hole



- Bergshoeff et al: arXiv:0902.4438 [hep-th]
  Bossard, Michel, Pioline: arXiv:0902.4438 [hep-th]
- Suggests a way to classify 4 qubit entanglement
  Borsten, Dahanayake, Duff, Marrani, Rubens
  Phys. Rev. Lett. 105:100507,2010 arXiv:1002.4223 [hep-th]

Families/Classes

- Four-qubit literature is confusing
- Classes: vanishing or not of SLOCC covariants/invariants
- Families: normal forms parameterized by SLOCC invariants e.g.

$$\begin{aligned} G_{abcd} = &\frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) \\ &+\frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|1001\rangle + |0110\rangle). \end{aligned} \quad (1)$$

- Example of difference: the separable EPR-EPR state $(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$, obtained by setting $b = c = d = 0$, belongs to the G_{abcd} family, whereas in the covariant approach it forms its own class.

- Four-qubit literature is contradictory

Paradigm	Author	Year	result mod perms		result incl. perms	
classes	Wallach	2005	?		90	
	Lamata et al,	2006	8 genuine,	5 degenerate	16 genuine,	18 degenerate
	Cao et al	2007	8 genuine,	4 degenerate	8 genuine,	15 degenerate
	Li et al	2007	?		≥ 31 genuine,	18 degenerate
	Akhtarshenas et al	2010	?		11 genuine,	6 degenerate
families	Verstraete et al	2002	9		?	
	Chretrentahl et al	2007	9		?	
	String theory	2010	9		31	

- String theory lends itself to the families approach

Extremal black hole / 4 qubit correspondence

Extremal black holes classification of STU model

$\updownarrow$

31 real nilpotent orbits of $SO(4,4)$ acting on the $\mathbf{28}$

$\updownarrow$

Kostant-Sekiguchi Correspondence

$\updownarrow$

31 complex nilpotent orbits of $SL(2)^4$ acting on the $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$

$\updownarrow$

4 qubits entanglement classification

Borsten, Dahanayake, Duff, Marrani, Rubens

Phys. Rev. Lett. 105:100507,2010 arXiv:1002.4223 [hep-th]

STU black holes	perms	nilpotent rep		family
trivial	1	0	$\in$	G_{abcd}
doubly-critical $\frac{1}{2}$BPS	6	$\|0110\rangle$	$\in$	L_{abc_2}
critical, $\frac{1}{2}$BPS and non-BPS	4	$\|0110\rangle + \|0011\rangle$	$\in$	$L_{a_2b_2}$
lightlike $\frac{1}{2}$BPS and non-BPS	1	$\|0110\rangle + \|0101\rangle + \|0011\rangle$	$\in$	$L_{a_2 0_{3\oplus\bar{1}}}$
large non-BPS $z_H \neq 0$	1	$\frac{i}{\sqrt{2}}(\|0001\rangle + \|0010\rangle - \|0111\rangle - \|1011\rangle)$	$\in$	L_{ab_3}
“extremal”	6	$i\|0001\rangle + \|0110\rangle - i\|1011\rangle$	$\in$	L_{a_4}

STU black holes	perms	nilpotent rep		family
large $\frac{1}{2}$BPS and non-BPS $z_H = 0$	4	$\|0000\rangle + \|0111\rangle$	$\in$	$L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$
"extremal"	4	$\|0000\rangle + \|0101\rangle + \|1000\rangle + \|1110\rangle$	$\in$	$L_{0_{5\oplus\bar{3}}}$
"extremal"	4	$\|0000\rangle + \|1011\rangle + \|1101\rangle + \|1110\rangle$	$\in$	$L_{0_{7\oplus\bar{1}}}$

- Total number of families without permutations = 9
- Total number of families including permutations = 31
- NB Trivially permuting the 9 yields many more than 31; still need to check equivalence

Falsifiable predictions

- Previous result 2006:
  STU black holes imply 5 ways to entangle three qubits
  Already known in QI; verified experimentally
- New result 2010:
  STU black holes imply 31 ways to entangle four qubits
  Not already known in QI: in principle testable in the laboratory