Scattering Amplitudes and Wilson Loops

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Ongoing work with Lionel Mason and David Skinner

See also work by Nima Arkani-Hamed, Jake Bourjaily, Freddy Cachazo, Simon Caron-Huot and Jaroslav Trnka

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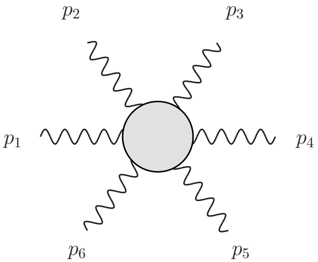
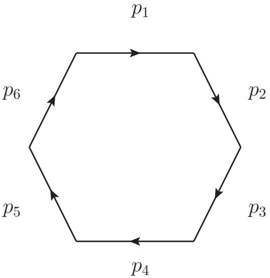
Introduction

My talk is about scattering amplitudes of gluons in (supersymmetric) gauge theories...



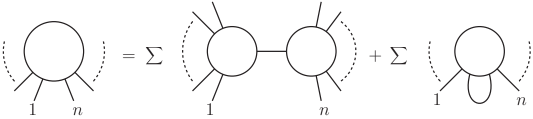


...and a remarkable duality with null polygonal Wilson loops.

Introduction

There have been remarkable developements in the last two years...

- Recursion relations for scattering amplitudes to all loop orders. Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka; Bullimore



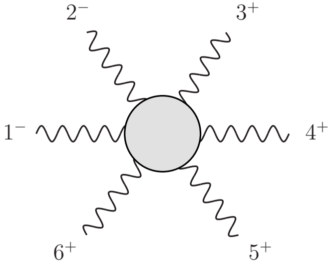
- Extension of MHV diagram formalism to all loop orders. Bullimore, Mason, Skinner

- Infinite dimensional symmetries. Drummond, Henn, Plefka

Scattering amplitudes have much hidden simplicity and structure!

Simplicity

Planar scattering of six gluons $(--++++)$ at tree-level:

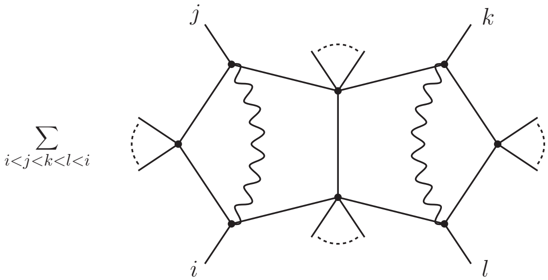


$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle 61\rangle}$$

This process requires 220 Feynman diagrams and thousands of terms at tree-level.

Simplicity

For scattering of any number of gluons $(- - + \ldots +)$ at two-loops in the theory with maximal supersymmetry: Arkani-Hamed, Bourjaily, Cachazo, Trnka



The simplicity and ease with which this result has been obtained is just spectacular!

What's wrong with Feynman diagrams?

Locality is the problem!

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Write down local theory of massless spin-one particles

$$\mathcal{L} \supset (\partial A)^2 + \bar{\psi}\,\partial\psi + \bar{\psi}\,A\,\psi + A^2\,\partial A + A^4$$

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Compute scattering amplitudes: $A^{\mu_1\cdots\mu_n}(p_1 \ldots, p_n)$ and contract with polarisation vectors $\epsilon_{\mu_1} \ldots \epsilon_{\mu_n}$.

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But since only two physical degrees of freedom

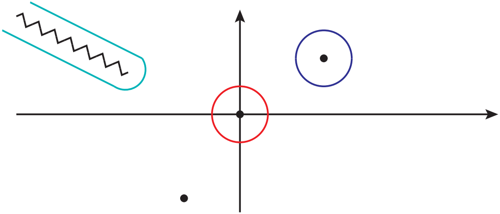
- Ward Identities $p_\mu A^{\mu\cdots}(p,\ldots) = 0$
- Gauge invariant Lagrangian $\mathcal{L}$

The Way Forward

Consider the helicity amplitudes directly

$$A^{--++++}(p_1, \ldots, p_6)$$

and compute them using their singularity structure...



...corresponding to various propagators going on-shell.

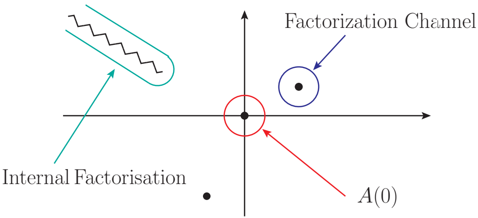
BCFW Recursion

Deform external momenta of gluons

$$p_1 \longrightarrow p_1 + z\, q \qquad\qquad p_n \rightarrow p_n - z\, q$$

and consider the contour integral

$$\oint \frac{A(z)}{z} dz = 0\,.$$

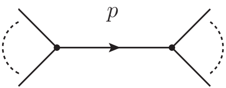


Compute scattering amplitudes recursively...

$$A_n = \sum_{L,R} A_L \frac{1}{P^2} A_R + \int \frac{d^4 l}{l^2} A_{n+2}(l, -l)$$

MHV Diagrams

Consider the MHV amplitudes $(--+\cdots+)$ as vertices for Feynman diagrams.



$$A_L(\ldots,q)\frac{1}{p^2}A_R(q,\ldots)$$

where

$$q = p - \frac{p^2}{2p\cdot q}\eta$$

and η is some reference null momentum.

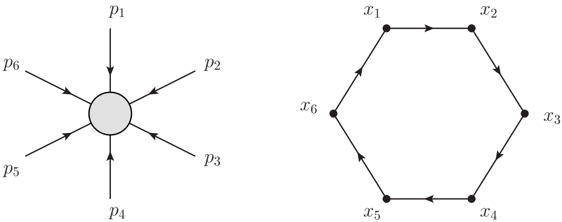
This formalism computes all scattering amplitudes in supersymmetric gauge theories.

Introduction - Wilson Loops

An explanation for the simplicity is that scattering amplitudes are Wilson loops.

For planar amplitudes, the four-momenta define a null polygon:

$$p_i = x_{i+1} - x_i$$

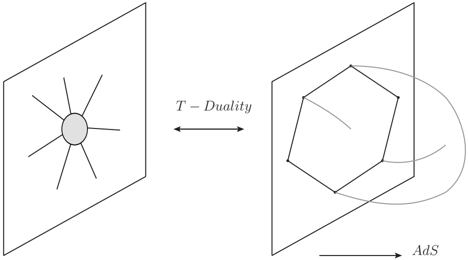


Compute the expectation value of a Wilson loop around the null polygon:

$$\left\langle \operatorname{tr} \mathrm{P} \exp \oint_C A \right\rangle$$

Introduction - Wilson Loops

At strong coupling the explanation is through T-duality and the AdS/CFT correspondence. Alday, Maldacena



$$\text{Amplitude} \sim \exp\left[-\frac{\sqrt{\lambda}}{2\pi} \times \text{Area}\right]$$

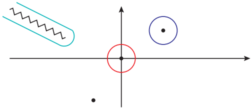
This computes the MHV amplitude $(- - + + \cdots)$ and contains the universal IR behaviour of all amplitudes.

Remarkably the equivalence was also found perturbatively at weak coupling! Drummond, Henn, Korchemsky, Sokatchev, Brandhuber, Heslop, Travaglini

Questions

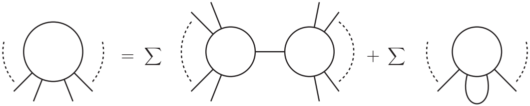
Why should Wilson loops compute amplitudes anyway?

- Can we see the singularity structure of amplitudes emerge from Wilson loops?



What about the amplitudes with other combinations of helicities?

- Is there a supersymmetric generalisation of the Wilson loop?



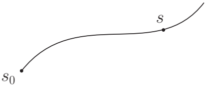
How is the hidden simplicity and structure of amplitudes apparent in the Wilson loop?

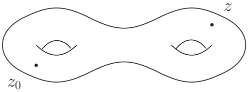
- From Wilson loops to BCFW recursion relations and MHV diagrams?

Outline of Answers

The answers are discovered through new *momentum twistor* variables.

- The planar S-matrix of $\mathcal{N} = 4$ super Yang-Mills is a *holomorphic* Wilson loop.

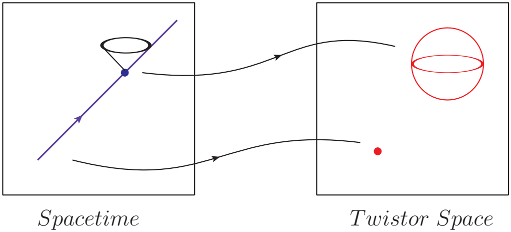




- We can see singularity structure of scattering amplitudes emerge!
- BCFW recursion relations from deforming the Wilson loop in different ways!
- MHV diagrams are Feynman diagrams for the holomorphic Wilson loop!

What is Twistor Space?

Twistor space is complex projective space $\mathbb{CP}^3$ with homogeneous coordinates $(\lambda_\alpha, \mu^{\dot{\alpha}})$ which are spacetime spinors.



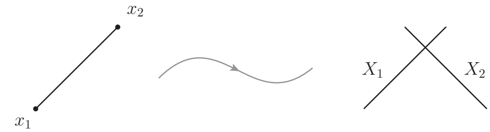
All you need to know to do twistor theory is the incidence relation

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$$

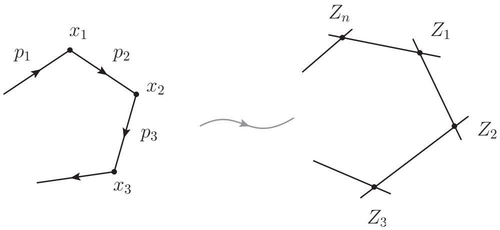
- Points in twistor space correspond to null lines in spacetime.
- Points in spacetime correspond to complex lines in twistor space.

Momentum Twistors

If two spacetime points are null separated the corresponding lines intersect.



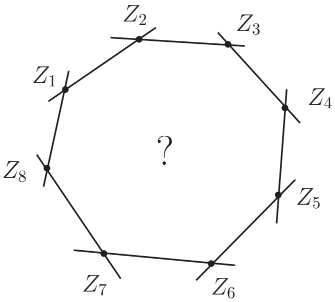
The null polygon becomes a sequence lines which intersect at points called momentum twistors. Hodges



Any choice of momentum twistors determines the data for a scattering amplitude.

What are Amplitudes in Momentum Twistor Space?

The null polygon in spacetime has become a *complex* curve in twistor space.



What computes scattering amplitudes in twistor space?

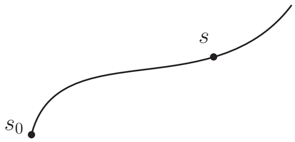
- The answer is a *holomorphic* version of a Wilson loop around this *complex* curve!

Wilson Loops - The Parallel Propagator

The parallel propagator along a real curve is defined by the differential equation

$$\left(\frac{d}{ds} - A(s) \right) U(s, s_0) = 0$$

with initial condition $U(s_0, s_0) = 1$.



The parallel propagator has the property

$$U(s, s')U(s', s'') = U(s, s'')$$

and under gauge transformations

$$U(s, s') \longrightarrow g(s)\, U(s, s')\, g^{-1}(s')$$

Wilson Loops - Perturbative Solution

To find the perturbative solution, start from the Green's function

$$\frac{d}{ds}\theta(s - s_0) = \delta(s - s_0)$$

and solve perturbatively in the gauge field $A(s)$:

$$\begin{aligned} U(s, s_0) &= 1 + \int_{s_0}^{s} ds_1\, \theta(s_1 - s_0) A(s_1) \\ &\quad + \int_{s_0}^{s} ds_2 \int_{s_0}^{s} ds_1\, \theta(s_2 - s_1)\theta(s_1 - s_0)\, A(s_2)A(s_1) + \cdots \\ &= P \exp \int_{s_0}^{s} A(s')ds' \end{aligned}$$

The gauge invariant Wilson loop is then defined for a closed curve

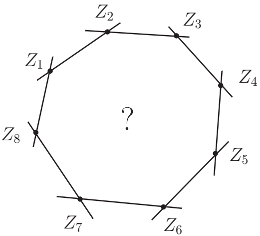
$$W(C) = tr\, P \exp \oint_C A\, .$$

Hint 3 - Holomorphic Chern-Simons Theory

Twistor space is complex and we have a *holomorphic* gauge theory:

$$S = \int \Omega \wedge (A \wedge \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A) \ + \ \text{interactions}$$

- A is a (0,1)-form gauge field on twistor space: $A = A_{\overline{z}}\, d\overline{Z}$
- Ω is a holomorphic volume form on twistor space.



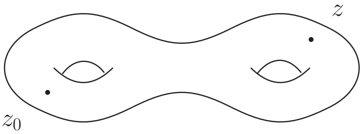
Is there a *holomorphic* analogue of the Wilson loop defined along a *complex* curve?

Holomorphic Wilson Loops - The Parallel Propagator

The parallel propagator is now defined on a Riemann surface by

$$\left(\overline{\partial} - A(z)\right) U(z, z_0) = 0$$

with initial condition $U(z_0, z_0) = 1$.



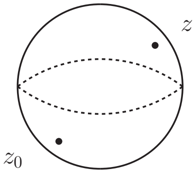
The parallel propagator has the properties

$$U(z, z')U(z', z'') = U(z, z'')$$

$$U(z, z') \longrightarrow g(z)\, U(z, z')\, g^{-1}(z')\,.$$

Holomorphic Wilson Loops - Green's function

In order to construct a perturbative solution we need the Green's function for $\bar{\partial}$ on the Riemann surface Σ.



For scattering amplitudes only the genus zero case is required

$$\bar{\partial}\, \frac{dz}{z - z_0} = \delta^2(z - z_0)$$

and hence the Green's function for $\bar{\partial}$ is

$$\omega(z) = \frac{dz(z_\infty - z_0)}{(z_\infty - z)(z - z_0)}\,.$$

Holomorphic Wilson Loops - Perturbative Solution

Now solve the differential equation pertubatively in the partial connection $A(z)$:

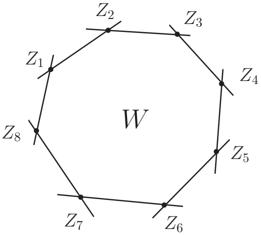
$$\begin{aligned} U(z, z_0) &= 1 + \int_\Sigma \frac{dz_1(z-z_0)}{(z-z_1)(z_1-z_0)} A(z_1) \\ &\quad + \int_\Sigma \frac{dz_2 dz_1 (z-z_0)}{(z-z_2)(z_2-z_1)(z_1-z_0)} A(z_2)A(z_1) + \cdots \\ &= 1 + \sum_{m=1}^{\infty} \int \prod_{i=1}^{m} \omega(z_i) \wedge A(z_i) \end{aligned}$$

This is the analogue of path-ordered exponential

$$U(z, z_0) = P \exp \int_\Sigma \omega \wedge A$$

The Holomorphic Wilson Loop

For amplitudes we only need a degenerate curve with genus zero components:



Construct a holomorphic Wilson loop by propagating from node to node and taking the trace:

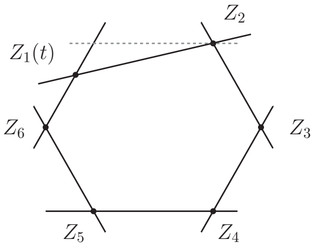
$$\begin{aligned} W(Z_1, \ldots, Z_n) &= tr\,(\,U(Z_1, Z_n) \ldots U(Z_2, Z_1)\,) \\ &= tr\ P\ exp \int_\Sigma \omega \wedge A \end{aligned}$$

How do we see that this behaves like a scattering amplitude?

Deforming the Curve

Examine how the Wilson loop behaves under deformations of the curve!

An important example is the BCFW deformation:



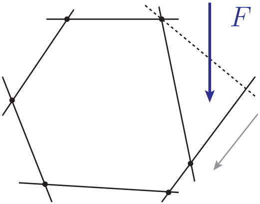
The Wilson loop becomes a meromorphic function of the deformation parameter

$$W(t) = W(Z_1 - tZ_n, Z_2, \ldots, Z_n)$$

Deforming the Curve - Classical Theory

Deformations of the curve involve only the $(0,2)$ component of the curvature:

$$\overline{\partial}_t W \sim \text{flux of } F^{(0,2)}$$

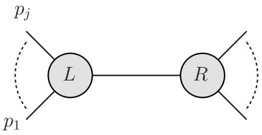


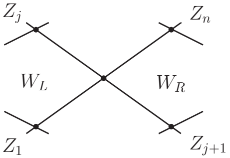
$$F^{(0,2)} = 0$$

The classical equations of motion imply that the Wilson loop changes holomorphically!

Deforming the Curve - Quantum Anomaly

There is a quantum anomaly when components of the curve intersect:





$$\overline{\partial}_t \langle W \rangle \sim \langle W_L \rangle \langle W_R \rangle$$

and there is a simple pole in the deformation parameter.

$$A(p_1, \ldots, p_n) \longrightarrow A_L(p_1, \ldots, p_j) \frac{1}{(p_1 + \ldots + p_j)^2} A_R(p_{j+1}, \ldots, p_n)$$

- This is precisely how scattering amplitudes factorise!

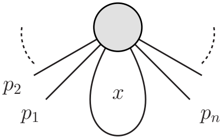
Deforming the Wilson Loop - Forward Terms

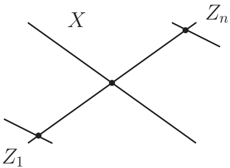
There are additional interactions from the anti-self dual sector of the theory. Mason, Skinner, Boels

$$S_2 = \int d^{4|8}X \log \det(\overline{\partial} + A)|_X$$

- The integral d^4X is a spacetime loop integration!

Additional contributions to the anomaly arise when the line X intersects the curve.

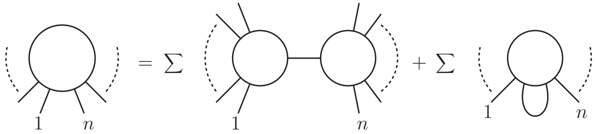




This is the forward limit of a scattering amplitude!

All-loop BCFW Recursion

Integrate over the deformation parameter to find BCFW recursion relations.

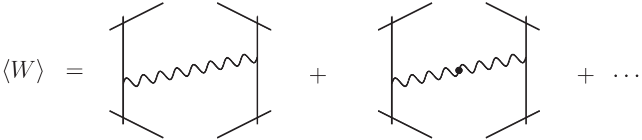


$$A_n = A_{n-1} + \sum_{L,R} A_L \frac{1}{P^2} A_R + \int \frac{d^4 l}{l^2} A_{n+2}(l, -l)$$

- This makes clear an infinite dimensional symmetry algebra of the amplitudes.
- Proof of Amplitude - Wilson Loop duality for the complete planar S-matrix!

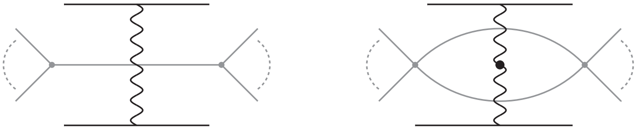
MHV Diagrams as Feynman Diagrams

We can compute the expectation value $\langle W \rangle$ perturbatively:



The Feynman diagrams for $\langle W \rangle$ are dual to MHV diagrams for scattering amplitudes.

Mason and Skinner



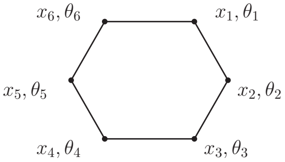
The Spacetime Wilson Loop

For MHV amplitudes $(--++++)$ we recover the standard spacetime Wilson loop:

$$\left\langle \operatorname{tr} \exp \oint_C A \right\rangle$$

For general helicites we have a supersymmetric Wilson loop on chiral superspace:

$$\left\langle \operatorname{tr} \exp \oint_C A_{\alpha\dot{\alpha}}(x,\theta) dx^{\alpha\dot{\alpha}} + \Gamma_{\alpha A}(x,\theta) d\theta^{\alpha A} \right\rangle .$$



Computing with the space-time formula is a complete mess!

- Scattering amplitudes naturally live in momentum twistor space!

Summary and Furture Directions

A holomorphic Wilson Loop computes planar S-matrix of $\mathcal{N} = 4$ super Yang-Mills.

- Extension of the Amplitude-Wilson Loop duality to all amplitudes.

Loop equations reveal singularity structure of scattering amplitudes.

- BCFW Recursion
- MHV diagram expansion

Questions:

- Regularisation and Strong Coupling?
- New methods for computing amplitudes?
- Uncover the deep relationship to AdS/CFT?
- Theories with less supersymmetry?