Scaling limits of critical random graphs

Let $\mathtt{w} = (w_1, w_2, \ldots, w_n)$ be a sequence of n positive real numbers sorted in non-increasing order. Interpreting w_i as the propensity of vertex i to form edges, we set $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ to be the graph on the vertex set $\{1, 2, \ldots, n\}$ where the edges are sampled independently across pairs and

$$\mathbf{P}\big(\{i,j\} \text{ is an edge of } \boldsymbol{\mathcal{G}}_{\mathtt{w}}\big) = 1 - \exp\Big(-\frac{w_i w_j}{w_1 + \ldots + w_n}\Big).$$

This is a natural extension to the classical model of Erdős–Rényi random graphs in allowing edges to be drawn with unequal probabilities. Moreover, by choosing suitable weight sequence $\mathtt{w}$, one can adjust the degree sequence of the graph so that it exhibits desirable tail asymptotics. See Figure 1 for a simulation of the graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ in two "extremal" cases: one with constant weights and the other with polynomial weights. Note in particular the emergence of "hubs" (vertices of large degrees) in the second case.

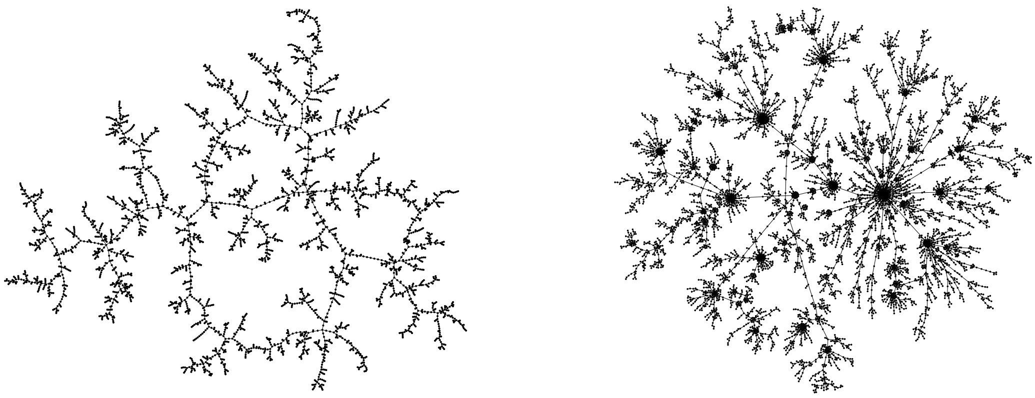

Figure 1: *Simulations of large connected components of* $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ *with different* $\mathtt{w}$. *Left:* $\mathtt{w} = (1, 1, \ldots, 1)$. *Right:* $\mathtt{w} = ((i/n)^{-\gamma})_{1\le i\le n}$, $\gamma \in (\frac{1}{3}, \frac{1}{2})$.

The random graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ shares an intimate connection with the multiplicative coalescents. This connection is the starting point of the work of Aldous & Limic [1], where they identify the entrance boundary of the multiplicative coalescents by looking at possible limit distributions for the *sizes* of the connected components found in $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$. Instead of the sizes, one can also look at the *geometry* of the $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$. This was the motivation of the work [2], where the following convergence is proven: equipping $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ with the graph distance d_{gr} as well as a measure $\mathbf{m}_{\mathtt{w}}$ by assigning the mass w_i to the vertex i, the sequence of measured metric spaces $(\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}, \varepsilon_n d_{\text{gr}}, \varepsilon'_n \mathbf{m}_{\mathtt{w}_n})_{n\ge1}$ converges weakly along suitable subsequences $(\mathtt{w}_n, \varepsilon_n, \varepsilon'_n)$ with respect to the Gromov–Hausdorff–Prokhorov topology. The scaling sequences $\varepsilon_n, \varepsilon'_n$ depend (in a rather explicit way) on the asymptotic behaviours of $\mathtt{w}_n$. The limit graphs are certain "tree-like" random fractals. The approach in [2] relies upon two key ingredients: an encoding of the graph by some Lévy process as well as an embedding of its connected components into Galton–Watson forests.

References

[1] Aldous, D., and Limic, V. The entrance boundary of the multiplicative coalescent. *Electron. J. Probab. 3* (1998), No. 3, 59 pp.

[2] Broutin, N., and Duquesne, T., and Wang, M. Limits of multiplicative inhomogeneous random graphs and Lévy trees. arXiv:1804.05871