Probability problems from gauge theory

Introduced by C.N. Yang and R.L. Mills in 1954 [YM54], as a generalization of the Maxwell-equations for electromagnetism, the Yang-Mills equations describe the fundamental interactions between elementary particles and stand today at the core of the standard model. In a simplified Euclidean version, they can be understood as equations on 1-forms A over the vector space $\mathbb{R}^4$, valued in a Lie algebra $\mathfrak{g}$. The latter is associated to a compact Lie group G, that describes the symmetries of the type of interaction[1] involved. These 1-forms are called connexion 1-forms. The curvature of a connexion 1-form A is the two form F_A valued in $\mathfrak{g}$, whose evaluation is

$$F_A(X,Y) = dA(X,Y) + [A(X), A(Y)],$$

for any vector fields X, Y over $\mathbb{R}^4$. When $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathfrak{g}$ and $B = \sum_{1 \leq i < j \leq 4} B_{i,j} dx_i \wedge dx_j$ is a two form valued in $\mathfrak{g}$, we shall write $\|B\|^2$ for the function $\sum_{1 \leq i < j \leq 4} \|B_{i,j}\|^2$. Consider the set Ω_4 of connexion 1-forms with smooth coefficients. In their Euclidean version, the solutions to the Yang-Mills equations can be understood as critical points of the so-called Yang-Mills functional

$$\mathcal{S}(A) = \int_{\mathbb{R}^4} \|F(z)\|^2 dz. \tag{1}$$

In order to quantify it and to give a rigorous meaning to Feynman path integrals associated to $\mathcal{S}$, one approach consists in trying to construct a measure of the form

$$e^{-\frac{1}{2g} S(A)} \mathcal{D}(dA), \tag{2}$$

where $\mathcal{D}(dA)$ is a formal path integral, playing the role of a Lebesgue measure on the affine space of connexions Ω_4, and $g > 0$ is a coupling constant.

Like in the construction of the Brownian motion or of the Gaussian free field, there does not exist a natural notion of Lebesgue measure on the space Ω_4. Similarly, there is no such measure on continuous functions or on distribution of negative regularity. In contrast to these examples, when G is not abelian, the action $\mathcal{S}$ is not quadratic in A; it includes for example terms of order 4; then, the field A cannot a priori be defined as a Gaussian field associated to a Hilbert space. Giving a meaning to this object in dimensions larger than 3,

[1] for instance, the circle U(1) for the electromagnetic interaction or the special unitary group SU(3), for the strong interaction.

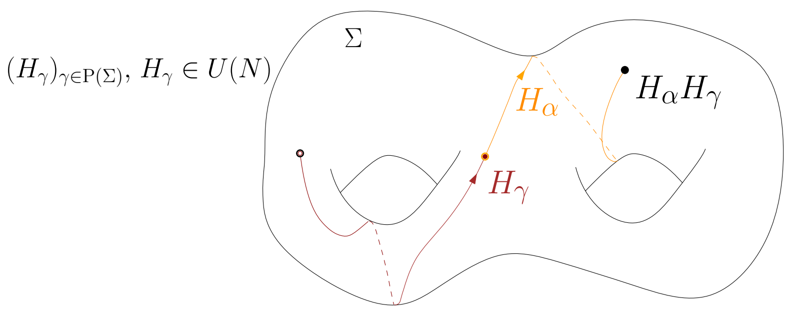
remains an open problem. Nonetheless, important progresses were achieved for instance thanks to discrete models approaches [BFS79, BFS80, BFS81], or other regularization methods [MRS93], together with Wilson's renormalization technics [BJ86]. The study of these latter models is an instance of lattice gauge theory [Sei82]; they can be understood as models of random matrices indexed by the set of edges of a lattice. Many probabilistic questions remain open in this area of research, which has been so far, mainly developed by the physics community. For a recent overview on this topic see for instance [Cha16, LS17].

1 Yang-Mills measure in dimension 2

When the space $\mathbb{R}^4$ is replaced by a compact Riemann surface Σ, or simply $\mathbb{R}^2$, it is possible to rigorously define (2) thanks to a continuous model [Dri89, Sen97, GKS89] or thanks to a family of discrete models [Lév10], which have the miraculous property of being compatible. The Yang-Mills measure on Σ is defined as a process $(H_\gamma)_{\gamma\in P(\Sigma)}$ of matrices indexed by the set $P(\Sigma)$ of paths of finite length in Σ. This model captures from its geometric origin a *multiplicative* property: almost surely, for all concatenations $\gamma_1\gamma_2$ of a path γ_1 followed by another γ_2,

$$H_{\gamma_1\gamma_2} = H_{\gamma_2}H_{\gamma_1}.$$

A unitary holonomy field



The matrix associated to a path γ, is called its *holonomy*, since it should be thought of, as the holonomy of a random connexion 1-form. It features also an important invariance property. When the group is abelian, the action depends on a connexion one-form only through its differential: it is invariant when translated by any closed one form. *Gauge invariance* is an analog invariance, valid for any compact Lie group, that satisfies the Yang-Mills functional. A satisfying notion of Yang-Mills measure must feature this invariance. The invariance of the Yang-Mills functional also implies that gauge invariant observables are physically more relevant. In this setting, a gauge transformation can be understood as the data of a function $j : \Sigma \to G$. Given a multiplicative process $(H_\gamma)_{\gamma\in P(\Sigma)}$, setting for any path $\gamma \in P(\Sigma)$, starting from $\underline{\gamma}$ and ending at $\overline{\gamma}$,

$$j(H)_\gamma = j(\overline{\gamma})^{-1}H_\gamma j(\underline{\gamma})$$

defines another multiplicative process. Gauge invariance of the Yang-Mills measure can be stated this way: under the Yang-Mills measure, for any gauge transformation j, the process $(j(H)_\gamma)_{\gamma\in P(\Sigma)}$ has same law as $(H_\gamma)_{\gamma\in P(\Sigma)}$. The gauge invariant random variables can then be described as follows. Given a character $\chi : G \to \mathbb{C}$ of G and $\ell \in P(\Sigma)$ a loop, since χ is invariant by conjugation,

$$\chi(H_\ell) = \chi(j(\underline{\ell})^{-1} H_\ell j(\underline{\ell})) = \chi(j(H)_\ell)$$

for any gauge transformation $j : \Sigma \to G$. In other words, this random variable is gauge invariant. Such random variables are called **Wilson loops**. For a large family of groups G, including for instance unitary, orthogonal groups, the sigma field generated by the Wilson loops

$$W_\ell = \mathrm{Tr}(H_\ell), \qquad \ell \in L(\Sigma)$$

is exactly (see for instance [Lév04]) the one generated by gauge invariant random variables. For this reason, the most relevant information about the random holonomy under the Yang- Mills measure is captured by Wilson loops. The articles [Dah16, CDG17, DN] studied these random variables in the following framework close the one of random matrix theory and free probability.

2 Yang-Mills measure, large N-limits

In a pioneering work, trying to understand quark confinement ['t 74], the physicist t'Hooft discovered that several observables have a simpler behavior when replacing the structure group $SU(3)$ for the strong interaction, by $SU(N)$, with $N \to \infty$. It raised a vivid interest in the physics literature, see for instance [Kaz81, KK81, MM79, Pol80, GG95] where this regime is considered and the limit described, highlighting the existence of a *master field*. Based on these works and focusing on the two dimensional setting, the mathematician I.M. Singer proposed several conjectures in [Sin95], giving a mathematical framework to understand them. One of them can be expressed as follows. Consider the set $\mathrm{L}(\Sigma)$ of finite length loops of Σ and define a metric on $\mathfrak{su}(N)$, setting for any $X, Y \in \mathfrak{su}(N)$,

$$\langle X, Y\rangle_N = N\mathrm{Tr}(X^*Y).$$

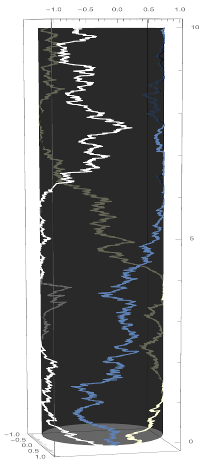
For any two dimensional compact manifold Σ, under YM_Σ with structure group $SU(N)$ and fixed coupling constant $g > 0$, the random variables $(\frac{1}{N}\mathrm{Tr}(H_\ell))_{\ell\in\mathrm{L}(\Sigma)}$ converge in probability towards a deterministic function

$$\tau : \mathrm{L}(\Sigma) \to [-1, 1].$$

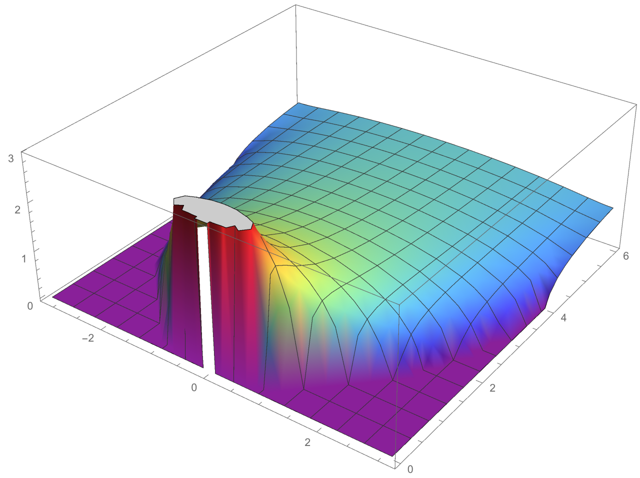
The limit field τ should satisfy some positivity conditions and give *invariants for rectifiable loops on Riemann surfaces* under the action of area-preserving diffeomorphisms. A full answer to this question remains largely open. In the plane and the sphere this conjecture has been proven in [Xu97, AS12, LÍ7] and [DN].

These models enjoy two very nice features which make them sometimes amenable to analysis.

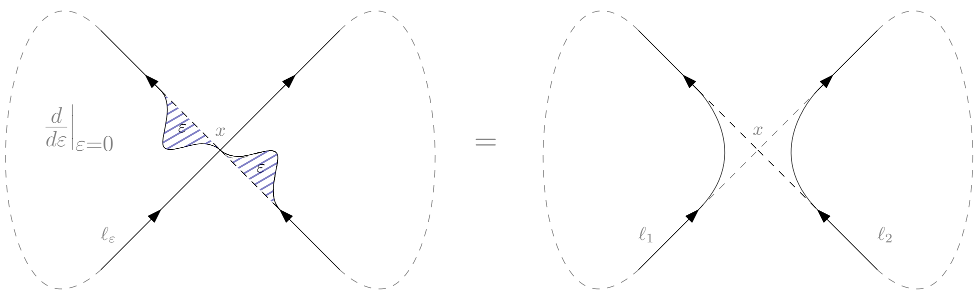
- For each new surface corresponds a new model of random matrices that admits specific tools. For instance, for the plane, holonomies of simple loops are given by Brownian motion on the $SU(N)$ starting from the identity at a time given by the area enclosed by the loop. When the plane is replace with a sphere, the Brownian motion must conditioned to go back to the identity when the time equals to total area of the sphere. The study of the Yang-Mills measure is therefore closely tied up with understanding of Brownian bridges on compact Lie group and master fields with their large N-limits and the understanding of their eigenvalues. From instance, the eigenvalues of a Brownian motion of $U(N)$ are known to follow the law of N Brownian motion on the circle conditioned never to collide. A picture when $N = 3$:



  Considering the empirical measure of the angles, when $N \to \infty$, the limit has been first proved to exist by Philippe Biane [Bia97] and is related to the free Brownian motion. It has a density over $(-\pi, \pi]$ which evolves like this over the interval of time $[0, 6)$:



- Non-simple loops can often be deformed into iterated simple loops. When doing these deformations, the expected Wilson loops and the master fields follow very nice equations first guessed by Makeenko and Migdal. The equations can be represented by the following schematic picture:



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